Mathematisches Forschungsinstitut Oberwolfach

Report No. 12/2026

DOI: 10.4171/OWR/2026/12

Higher Structures from Symmetries in Quantum Field Theory

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15 March – 20 March 2026

Abstract. Symmetries in the form of actions of groups and Lie algebras have been an important tool for the analysis of physical systems for a very long time. It has been realized only relatively recently that quantum field theories admit symmetries that are described by more general mathematical structures: fusion categories and their higher-categorical generalizations.

The understanding of such higher categories and the construction of concrete examples raise many mathematical challenges which are at present addressed with a wide variety of mathematical tools, including homotopy theory, the theory of C^*-algebras, factorization algebras and operads.

An improved mathematical understanding of higher categories has an immediate impact on the classification and characterization of quantum systems which can realize these categorical structures as symmetries. Particularly note-worthy applications of these symmetries include integrable systems with fusion category symmetries, defects in topological and conformal field theories, the characterization of three-dimensional topological order and systems in quantum information theory realizing quantum codes.

The workshop *Higher Structures from Symmetries in Quantum Field Theory*, brought a wide range of mathematical researchers together, working on a broad range of topics, from category theory to condensed matter physics, and provided an extremely fruitful interaction across disciplines. For some leading researchers from neighbouring fields, this was their first visit to Oberwolfach.

Mathematics Subject Classification (2020): 18M20, 81T40, 81T45, 81T25, 81P65, 57R56.

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Introduction by the Organizers

The workshop *Higher Structures from Symmetries in Quantum Field Theory*, organized by David Reutter (Hamburg), Sakura Schafer-Nameki (Oxford), Christoph Schweigert (Hamburg) has brought together 48 participants with broad representation – both scientifically, with participants from pure mathematics (topology, category theory) to condensend matter physicists – as well as geographically (from Germany, Europe, the United States and Japan) and demographically (from PhD students to senior academics).

Higher categorical structures, such as higher fusion categories, have seen an increasing relevance in quantum physics. The connection is made through the identification of topological defects in a quantum theory – which can be a lattice system, or a continuum quantum field theory – with symmetries. Such topological defects encode symmetries that go beyond standard group-like symmetries, which admit inverses, and extends these, in the simplest instances, to fusion categorical symmetries, also called *non-invertible symmetries*. Moreover, defects in field theories can have various dimensions and can be contained in higher-dimensional defects which leads directly to more layers of structure and thus to higher categories. For symmetries generated by finitely many simple objects and obeying (generalized) semisimplicity conditions, much progress has been made in mathematics and in physics in the past years in the understanding of their structure.

There were 19 contributed talks to the workshop, of a length of 45 minutes (plus 15 minutes discussion) each. All talks were presented at the blackboard. On Monday evening, young participants had the opportunity to give a short presentation of three minutes to introduce themselves to the participants of the workshop. Eight of the regular workshop talks were given by young participants (without tenured positions). For many of them and many participants with a background in theoretical or mathematical physics, this was their first contribution to a workshop in Oberwolfach.

On Tuesday and Wednesday evening, we organized two discussion sessions. The idea to have such a discussion was brought up by young researchers before the workshop. Each session had three discussion leaders; four out of the six discussions leaders were young scientists. Under the lead of Thomas Bartsch, Lukas Müller and Luuk Stehouwer, physical applications of dagger structures on higher categories were discussed on Tuesday evening. On Wednesday evening, Peter Teichner, Kevin Walker and Matthew Yu lead a discussion on the applications of topology in algebra and physics. The two discussion sessions were truly cross-disciplinary, with a vivid participation of the audience which helped to identify and highlight important open problems.

In the remaining part of this introduction, we briefly describe the interrelations of the contributions.

The structure theory of higher categories were the subject of several talks: Décoppet gave an overview over his classification results of fusion 2-categories while Nikshych's talk focused on aspects of braided tensor categories that have

direct implications for the understanding of their associated fusion 2-categories. Ayala demonstrated in his talk how tools and concepts inspired by the theory of factorization algebras improve our understanding of rigid braided categories. Johnson-Freyd introduced the idea of *firm* categories, a class of theories that should allow to address larger classes of quantum field theories.

Another important theme of several talks was the construction of a so-called Symmetry Topological Field Theory (SymTFT), which captures the key aspects of generalized symmetries: the symmetry structure, the generalized charges (or representations), anomalies, and Morita equivalences. Brunner's talk explained physical applications of symmetry TFTs to systems with Yang-Mills-type equations of motion. Scheimbauer explained the role of dualizability for symmetry TFTs and Jordan combined ideas inspired by symmetry TFTs with Hamiltonian reduction to understand structures in representation theory. Meusburger showed how systems of categories with a higher-categorical flavour can be used to construct new invariants of four-dimensional manifolds.

Continuous symmetries and continuous structures more generally played an important role in the contributions of Henriques and Teschner which both studied specific classes of two-dimensional conformal field theories. Woike's talk in contrast explained operadic techniques that allow to address two-dimensional conformal field theories with chiral symmetries which are not even rigid. Carqueville reviewed in his talk how defects in topological field theories allow to relate different topological field theories by gauging them.

Another exciting area that crystallized out from this meeting, where mathematicians and physicists can find a fruitful interaction is that of applications to topological orders, phases, and more recently quantum computing. Fendley's talk explained applications of topological defects to lattice systems from the point of view of the practitioner. The emergence of topological orders in Hamiltonian lattice systems was the topic of Ogata's talk. The realization of non-invertible symmetries on lattice systems was the topic of Inamura's and Ranard's contributions.

Warman showed applications in the realm of quantum computing and Tantivasadakarn explained applications to quantum cellular automata. Komatsu finally explained how non-invertible symmetries constrain in a highly surprising way scattering amplitudes of a quantum field theory, which are quantities in close relation to experiments.

Numerous discussions among the participants, in particular those involving both physicists and mathematicians, have contributed to the workshop in an essential way. Thus the workshop provided an ideal opportunity for exchange and interaction. We would like to thank in particular the administration and the staff of MFO for their highly efficient and friendly support.

Workshop: Higher Structures from Symmetries in Quantum Field Theory

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Abstracts

Orthogonal symmetries of higher categories

David Ayala
(joint work with John Francis)

Taking opposites in each categorical tier defines an action by the group $\mathsf{O}(1)^n$ on the ∞-category $\mathsf{Cat}_{(\infty,n)}(\mathcal{V})$ of (∞,n)-categories that are enriched over a presentably symmetric monoidal ∞-category $\mathcal{V}$. (Taking $n=1$ and $\mathcal{V}=\mathsf{Vect}_{\mathbb{C}}$ recovers $\mathbb{C}$-linear categories.) Evidently invariant with respect to this action is the full ∞-subcategory $\mathsf{Cat}^{\mathsf{rigid}}_{(\infty,n)}(\mathcal{V})\subset\mathsf{Cat}_{(\infty,n)}(\mathcal{V})$ consisting of the *rigid* (∞,n)-categories (ie, those (∞,n)-categories in which, for each $0<k<n$, each k-morphism has a left and right adjoint).

Conjecturally, this $\mathsf{O}(1)^n$-action on $\mathsf{Cat}^{\mathsf{rigid}}_{(\infty,n)}(\mathcal{V})$ extends as an action by the orthogonal group $\mathsf{O}(n)$. Ongoing joint work with Francis, Masuda, Reutter seeks to prove this conjecture. I will state a special case of this conjecture, which is the main result of [2], which is joint work with John Francis: There is an action by $\mathsf{O}(n)$ on $\mathsf{Cat}^{\mathsf{rig},\geq n-1}_{(\infty,n)}(\mathcal{V})$, the ∞-category of rigid $\mathcal{V}$-enriched (∞,n)-categories that are concentrated in degrees $\geq n-1$. This result implies the folk-lore result, that ribbon categories are the $\mathsf{SO}(3)$-fixed rigid braided-monoidal $\mathbb{C}$-linear categories.

This conjecture is logically related to, but not implied by, the cobordism hypothesis, as I will indicate. Generally, this conjecture is expected to be true based on heuristics of topological defects/operators in a quantum field theory on $\mathbb{R}^n$.

The first case concerns framed n-dimensional TQFTs (whose value on n-manifolds is a (possibly derived) vector space) that are determined by their topological point/local operators. Topological point-operators possess the structure of a $\mathcal{E}_n$-algebra. Better, as established in [4], an $\mathcal{E}_n$-algebra determines a (certain formulation of a) framed n-dimensional TQFT (ie, a topologically invariant QFT) via (alpha-version) factorization homology, and the TQFTs determined in this way are precisely those that are determined by their point-operators. Meanwhile, Dunn's additivity implies the n-fold de-loop functor identifies $\mathcal{E}_n$-algebras as (pointed) $\mathcal{V}$-enriched (∞,n)-categories that are concentrated in degree n:

$$\mathsf{Cat}^{\mathsf{rig},\geq n}_{(\infty,n)}(\mathcal{V})\xleftarrow{\simeq}\mathsf{Alg}_n(\mathcal{V})\xrightarrow{\int^{\alpha}} n\mathsf{TQFT}^{\mathsf{fr}}\ .$$

Here, the left term admits a canonical $\mathsf{O}(1)^n$-action, the middle an $\mathsf{O}(n)$-action, and the right an $\mathsf{O}(n)$-action.

The next case concerns framed n-dimensional TQFTs (whose value on n-manifolds is a (possibly derived) vector space) that are determined by their topological point and line operators. Topological point- and line-operators organize as the structure of a rigid $\mathcal{E}_{n-1}$-monoidal $\mathcal{V}$-enriched $(\infty,1)$-category. As established in forthcoming joint work with John Francis, following our joint work [1], a rigid $\mathcal{E}_{n-1}$-monoidal $\mathcal{V}$-enriched $(\infty,1)$-category determines a (certain formulation of

a) framed n-dimensional TQFT (ie, a topologically invariant QFT) via (beta-1-version) factorization homology, and the TQFTs determined in this way are precisely those that are determined by their point- and line-operators. Meanwhile, Dunn's additivity implies the $(n-1)$-fold de-loop functor identifies rigid $\mathcal{E}_{n-1}$-monoidal $\mathcal{V}$-enriched $(\infty,1)$-categories as (pointed) $\mathcal{V}$-enriched (∞,n)-categories that are concentrated in degree $n-1$:

$$\mathsf{Cat}^{\mathsf{rig},\geq \mathsf{n}-1}_{(\infty,n)}(\mathcal{V}) \xleftarrow{\simeq} \mathsf{Alg}^{\mathsf{rig}}_{n-1}(\mathsf{Cat}_{(\infty,1)}(\mathcal{V})) \xrightarrow{\int^{\beta_1}} n\mathsf{TQFT}^{\mathsf{fr}} \ .$$

Here, the left term admits a canonical $\mathsf{O}(1)^n$-action, the middle an $\mathsf{O}(n-1)\times\mathsf{O}(1)$-action, and the right an $\mathsf{O}(n)$-action.

As conjectured in [3], this pattern continues, trading a monoidal direction for a categorical direction, on the one hand, captures more TQFTs (those determined by deeper operators) while, on the other hand, obscures $\mathsf{O}(n)$-equivariance: for each $0 \leq k \leq n$,

$$\mathsf{Cat}^{\mathsf{rig},\geq \mathsf{k}}_{(\infty,n)}(\mathcal{V}) \xleftarrow{\simeq} \mathsf{Alg}^{\mathsf{rig}}_{k}(\mathsf{Cat}^{\mathsf{rig}}_{(\infty,n-k)}(\mathcal{V})) \xrightarrow{\int^{\beta_k}} n\mathsf{TQFT}^{\mathsf{fr}} \ .$$

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Symmetry Theories from the Universal Bulk

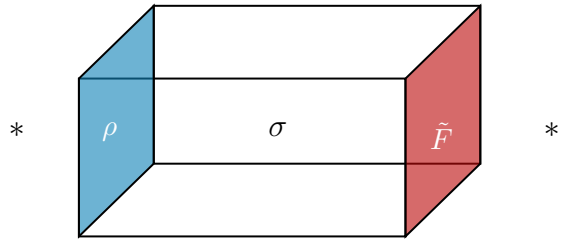
Ilka Brunner

(joint work with Severin König, Ingmar Saberi)

This talk, based on [1], brings together various approaches on symmetries and their realization in semi-classical field theories in n dimensions. Our starting point is the recent proposal, originating in work of Pulmann–Ševera–Valach [2] and Freed–Moore–Teleman [3], to formulate abstract symmetry data for such a field theory F in terms of an $n+1$-dimensional topological field theory σ together with a topological boundary condition ρ. Its physical realization, or representation, is given in terms of a second boundary $\tilde{F}$ that may or may not be topological. The n-dimensional physical theory F is recovered by 'merging' the two boundaries ρ and $\tilde{F}$ over the squeezed-in TFT σ. The above figure summarizes this setting.

Many examples of this 'sandwich' setup have been studied, largely working with discrete global symmetries or higher generalizations of these. However, finding a precise technique for reformulating various standard examples of symmetries (continuous symmetries, spacetime symmetries) in terms of such data has largely remained an open question.

In our work, we attempt to close this gap by providing such a technique. To any pair consisting of a perturbative field theory F and a local infinitesimal symmetry



$\mathscr{G}$, we associate a tuple $(\rho, \sigma, \tilde{F})$ as above, thus providing a 'sandwich' realization of the input data. We work in the Batalin–Vilkovisky formalism as axiomatized in [4], and our treatment is enormously indebted to the pioneering work of Butson–Yoo [7], of which it makes essential use. While our treatment is at the semiclassical level, it dovetails with Rabinovich's construction of classical and quantum factorization algebras for bulk-boundary systems [8], so that discussions of perturbative BV quantization of our data are possible.

The field spaces of a perturbative theory in the BV formalism are formal moduli spaces, modelling 'on-shell field configurations near the vacuum, up to gauge equivalence.' Derived deformation theory shows that any such space is the Maurer–Cartan space of an L_∞ algebra; correspondingly, Costello and Gwilliam work with *local L_∞ algebras*. A theory described by a nondegenerate action functional is further equipped with a local symplectic structure of cohomological degree -1. Butson and Yoo generalize this to include *degenerate* BV theories; the field space of a degenerate theory is equipped with a (+1-shifted) Poisson structure. The reader should be reminded of classical mechanics, which in a Hamiltonian approach is described by ordinary symplectic geometry. As in that setting, boundary conditions are described in terms of Lagrangian subspaces of boundary fields, and their composition ('merging') is given by fiber products. Butson and Yoo reformulate degenerate theories as relative theories: to any Poisson BV theory, they associate a *universal bulk theory* for which it is naturally a boundary condition.

Suitable formulations of Lagrangians in derived geometry were studied by many authors, including [5, 6]. We emphasize that the derived setting allows one to realize a nondegenerate n-dimensional theory as a boundary condition of the empty theory. It is thus tailor-made to discuss extra degrees of freedom at boundaries and defects, as is required by physics applications.

In perturbation theory, a local symmetry is encoded at the level of its Lie algebra; in a BV setting, this is again a local L_∞-algebra $\mathscr{G}$ over space-time, and the corresponding moduli space consists of 'background fields.' Examples include $\mathscr{G} = \Omega^\bullet \otimes \mathfrak{g}$ (infinesimal flat $\mathfrak{g}$-connections) or $\mathscr{G} = (\Omega^0 \to \Omega^1) \otimes \mathfrak{g}$ (generic infinitesimal $\mathfrak{g}$-connections). A symmetry action on a BV theory $\mathscr{F}$ is specified by a semi-direct product $\mathscr{F} \rtimes \mathscr{G}$ of L_∞ algebras [4]. The corresponding moduli space describes the theory as a family over the background fields; it is equipped with a fiberwise Poisson structure that is usually degenerate.

Our perspective is that the *'equivariant' universal bulk theory* for $\mathscr{F} \rtimes \mathscr{G}$ plays the role of σ in the above diagram. By construction, $\mathscr{F} \rtimes \mathscr{G}$ defines the boundary

condition $\tilde{F}$; its Lagrangian structure can be thought of as a version of the moment map [6], and is related to the factorization Noether map [4].

If $\mathscr{G}$ acts without anomalies on a nondegenerate theory, then the universal bulk is the $n+1$ dimensional BF theory $T^*[-1](\mathscr{G} \otimes \Omega^\bullet(\mathbb{R})[1])$. (The nondegeneracy of the symplectic structure on $\mathscr{F}$ means that the corresponding fields can be 'integrated out' from the universal bulk.) The space of boundary fields in BF theory is given by $T^*(\mathscr{G}[1]) \cong \mathscr{G}[1] \oplus \mathscr{G}^![-1]$. The two summands provide two canonical Lagrangians, corresponding to the zero section and the fiber over the basepoint, and hence two possible topological boundary conditions. Physically, they correspond to gauging and 'forgetting' (or coupling to a trivial background), respectively; these operations are recovered by a derived fiber product as anticipated.

If $\mathscr{G}$ is anomalous, the anomaly determines a class in the degree-one local BRST cohomology of $\mathscr{G}$, which in turn determines a deformation of the BF theory. The Lagrangian corresponding to 'gauging' the symmetry is obstructed and does not persist in the deformation, nicely capturing the idea of an anomaly as an 'obstruction to gauging;' the 'forgetful' boundary condition makes sense even in the presence of the anomaly, as one expects.

We would like to highlight a few results of our upcoming work:

(1) Depending on $\mathscr{G}$, the BF theory may or may not be topological. It is however topological in one direction, the one transverse to the boundary. It is hence natural to relax the condition that σ be topological, and allow for theories topological transverse to the boundary.
(2) It is likewise useful to relax the condition on the topological boundary ρ, and to allow for non-topological boundary conditions as well. This in particular allows to implement (non-topological) equations of motions on gauge fields, such as Yang-Mills equations. As such, the choice of '*how* to gauge the symmetry'—what dynamics to take for the gauge sector—is packaged into the choice of ρ in the above picture.
(3) As our construction makes no assumptions about the local Lie algebra $\mathscr{G}$, the universal bulk may not be a 'theory' in the usual sense, e.g. the moduli problem of all $\mathfrak{g}$ connections is not elliptic.
(4) As our construction makes no assumptions about the local Lie algebra $\mathscr{G}$, space time symmetries naturally fit in this framework.
(5) Recalling that an anomaly is related to a projective action of the symmetry, one may reformulate the above obstruction by considering the action of a local central extension of the symmetry algebra, and allowing for curvings. This is connected to the 'anomaly theory' of [3].
(6) The equivariant universal bulk can be formulated for any (possibly degenerate) BV theory, and just reduces to the normal universal bulk. For a nondegenerate theory, the universal bulk is empty. In other examples, though, it is interesting: for chiral WZW models, it is Chern–Simons theory. Taking the perspective that the anomaly theory measures the "failure" of the equivariant universal bulk to agree with BF theory, one

may want to interpret Chern–Simons theory as an anomaly theory in this setting, despite the fact that it is not invertible.

(7) As emphasized above, derived Lagrangians perfectly encapsulate ubiquitous constructions in the physics of defects. We revisit boundary conditions for the B-model with superpotential defined by matrix factorizations from this new perspective. We show how matrix factorizations can be viewed as Lagrangian maps, how this reconstructs a semiclassical version of the category of matrix factorizations, and how the bulk theory can be reinterpreted, at least in certain cases, as the equivariant universal bulk for an anomalous symmetry acting on a system of free boundary fermions.

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Gauging Symmetries of TQFTs

Nils Carqueville

(joint work with Benjamin Haake)

It is expected that there is an n-category $\mathcal{D}$ whose objects are n-dimensional quantum field theories, and whose k-cells are $(n-k)$-dimensional topological defects. Then an action of a group G on $Q \in \mathcal{D}$ is an n-functor $\rho\colon \mathrm{B}\underline{G} \to \mathcal{D}$ with $\rho(*) = Q$, where $\underline{G}$ is the discrete monoidal $(n-1)$-category obtained from G, and $\mathrm{B}(-)$ denotes delooping. This parallels the fact that a G-action on a vector space V is a functor $\rho\colon \mathrm{B}G \to \mathrm{Vect}$ with $\rho(*) = V$.

Given a monoidal $(n-1)$-category $\mathcal{S}$, an *$\mathcal{S}$-action* is an n-functor $\rho\colon \mathrm{B}\mathcal{S} \to \mathcal{D}$; the special cases $\mathcal{S} = \mathrm{B}^p\mathcal{H}$ for sufficiently braided monoidal (higher) categories $\mathcal{H}$ are called *p-form symmetries*. Below we focus on the case of topological quantum field theories (TQFTs) in $n = 3$ dimensions, and we answer the question of when and how $\mathcal{S}$-actions can be gauged in this setting.

An n-dimensional *defect TQFT* was defined in [6] to be a symmetric monoidal functor $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \to \mathrm{Vect}$ out of a category of oriented stratified bordisms labelled by chosen *defect data* $\mathbb{D}$ that involve label sets D_j for all j-strata, $j \leqslant n$.

(Codomains other than Vect are possible.) Locally a TQFT $\mathcal{Z}$ is expected to be algebraically encoded in a "pivotal n-category" $\mathcal{D}_{\mathcal{Z}}$. This is established as a theorem for $n \leqslant 3$ in [9, 5]: objects and k-cells for $k \leqslant n-1$ are obtained from $\mathbb{D}$, while n-cells are defined by evaluating $\mathcal{Z}$ on defect $(n-1)$-spheres and n-balls.

For example, the pivotal 2-category associated to the "trivial" 2-dimensional defect TQFT $\mathcal{Z}_2^{\mathrm{triv}}$ (which is trivial on 2-strata, while interpreting 1- and 0-strata as string diagrams in vect) is $\mathcal{D}_{\mathcal{Z}_2^{\mathrm{triv}}} \cong \mathrm{Bvect}$. For 2-dimensional state sum models $\mathcal{Z}_2^{\mathrm{ss}}$ we have $\mathcal{D}_{\mathcal{Z}_2^{\mathrm{ss}}} \cong \mathrm{ssFrobAlg}$, the pivotal 2-category of separable symmetric Frobenius algebras and their bimodule categories. Similarly one finds [8, 13, 3, 1] $\mathcal{D}_{\mathcal{Z}_3^{\mathrm{triv}}} \cong \mathrm{BssFrobAlg}$, $\mathcal{D}_{\mathcal{Z}_3^{\mathrm{ss}}}$ is the 3-category of spherical fusion categories, and for Reshetikhin–Turaev theories $\mathcal{Z}_{\mathcal{C}}^{\mathrm{RT}}$ based on a modular fusion category $\mathcal{C}$ we have

$$\mathcal{D}_{\mathcal{C}} := \mathrm{BssFrobAlg}(\mathcal{C}) \quad \subset \quad \mathcal{D}_{\mathcal{Z}_{\mathcal{C}}^{\mathrm{RT}}} .$$

See [12, 7, 11, 3] for details and the full structure of $\mathcal{D}_{\mathcal{Z}_{\mathcal{C}}^{\mathrm{RT}}}$.

An *orbifold datum* $\mathcal{A}$ for $\mathcal{Z}$ consists of elements $\mathcal{A}_j \in D_j$, $j \leqslant n$, such that (1) $\mathcal{A}_j$ can label j-strata dual to oriented $(n-j)$-simplices of any (unstratified) bordism M, giving rise to a defect bordism $M_{\mathcal{A},t}$ for any oriented triangulation t of M; (2) $\mathcal{Z}(M_{\mathcal{A},t})$ is independent of the choice of t in the interior of M. The latter constraint amounts to finitely many conditions coming from Pachner moves, which are solved by separable symmetric Frobenius algebras and spherical fusion categories *internal* to $\mathcal{D}_{\mathcal{Z}}$ for $n = 2$ and $n = 3$, respectively, see [6, 8]. For example, for any finite group G, a functor $\rho\colon \mathrm{B}\underline{G} \to \mathrm{BVect}$ gives rise to a separable symmetric Frobenius algebra structure on $\bigoplus_{g\in G} \rho(g)$ with multiplication $\bigoplus_{g,g'\in G} \rho^2_{g,g'}$; it is symmetric (and hence an orbifold datum for $\mathcal{Z}_2^{\mathrm{triv}}$) iff ρ is pivotal.

By construction, a symmetry of a defect TQFT $\mathcal{Z}$ can be *gauged* if it assembles into an orbifold datum $\mathcal{A}$ for $\mathcal{Z}$. Then, as shown in [6], by triangulating, labelling dual stratifications with $\mathcal{A}$, evaluating with $\mathcal{Z}$, and taking the colimit over all triangulations, one obtains the (*gauged* or) *orbifold TQFT*

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}^{\mathrm{or}}_{n,n-1} \to \mathrm{Vect} .$$

Examples include n-dimensional state sum models $(\mathcal{Z}_n^{\mathrm{triv}})_{\mathcal{A}}$ where $\mathcal{A}$ is a separable symmetric Frobenius algebra for $n = 2$, a spherical fusion category for $n = 3$, and a spherical fusion 2-category for $n = 4$, see the survey [1] and references therein for details as well as examples of orbifold constructions from non-semisimple data.

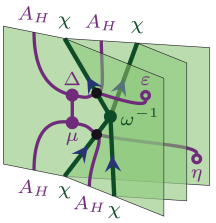
We now fix $n = 3$ and a finite 2-group $\mathcal{G}$. Using the equivalence between 2-groups and crossed modules consisting of groups G, H, a group homomorphism $t\colon H \to G$ and a compatible G-action on H, we can assume that the objects of $\mathcal{G}$ are the elements of G, while $\mathrm{Hom}_{\mathcal{G}}(g, g') = \{(g, h) \,|\, t(h)g = g'\}$. It follows that any $\mathcal{G}$-action $\rho\colon \mathrm{B}\underline{\mathcal{G}} \to \mathcal{D}$ involves 1-cells $\rho(g)\colon \rho(*) \to \rho(*)$, 2-cells $\rho(g, h)\colon \rho(g) \to \rho(t(h)g)$, 2-cells $\chi_{g,h}\colon \rho(g) \,\square\, \rho(g') \to \rho(gg')$, and 3-cells $\rho^2_{g,h,h'}\colon \rho(t(h)g, h') \otimes \rho(g, h) \to \rho(g, hh')$. Assuming that $\mathcal{D}$ is a Gray category with duals whose Hom 2-categories have direct sums and are Vect-enriched, we find:

Theorem 1 ([2])**.** For $\rho\colon \mathrm{B}\underline{\mathcal{G}} \to \mathcal{D}$ with pivotal 2-functor components, set

$$A_G := \bigoplus_{g\in G} \rho(g)\,,\ A_H := \bigoplus_{\substack{g\in G\\ h\in H}} \rho(g,h)\,,\ \mu := \bigoplus_{\substack{g\in G\\ h,h'\in H}} \rho^2_{g,h,h'}\,,\ \Delta := \frac{1}{|H|} \bigoplus_{\substack{g\in G\\ h,h'\in H}} \left(\rho^2_{g,h,h'}\right)^{-1}.$$

Then (A_H, μ, Δ) is a separable symmetric Frobenius algebra, and

$$\mathcal{A}_\rho := \left(\rho(*),\ (A_G, A_H),\ \bigoplus_{g,g'\in G} \Big(\bigoplus_{h\in H} \rho(gg', h) \Big) \otimes \chi_{g,g'},\right.$$



$$\left.\vphantom{\bigoplus_{g,g'\in G}}\right)$$

is a weakly associative algebra in $\mathcal{D}_{\text{Hom-orb}}$, the 3-category obtained from $\mathcal{D}$ by orbifold completing its Hom 2-categories in the sense reviewed in [1].

Note that this result only holds if we "gauge the H-symmetry", i.e. in $\mathcal{D}_{\text{Hom-orb}}$.

The question about gaugeability of the $\mathcal{G}$-action ρ now becomes the question whether $\mathcal{A}_\rho$ is an orbifold datum in $\mathcal{D}_{\text{Hom-orb}}$. We expect in general that a symmetry is gaugeable if the associated n-functor is "pivotal" (where all morphisms have compatible adjoints whose squares are isomorphic to the identity, in a sense that we do not know rigorously for $n > 3$), see [2, Rem. 3.7]. We prove this in the case of Reshetikhin–Turaev theories:

Recall from [14] that from any modular fusion category $\mathcal{C}$ and any orbifold datum $\mathcal{A}$ in $\mathcal{D}_{\mathcal{C}}$ one obtains a new modular fusion category $\mathcal{C}_{\mathcal{A}}$, and from [4] that $(\mathcal{Z}^{\mathrm{RT}}_{\mathcal{C}})_{\mathcal{A}} \cong \mathcal{Z}^{\mathrm{RT}}_{\mathcal{C}_{\mathcal{A}}}$. Any 0-form symmetry $\rho_G\colon \mathrm{B}\underline{G} \to \mathcal{D}_{\mathcal{C}}$ comes from a G-crossed braided fusion category $\mathcal{C}^\times_G = \bigoplus_{g\in G} \mathcal{C}_g$ with $\mathcal{C}_e = \mathcal{C}$, and gauging ρ_G amounts to G-equivariantisation $(-)^G$:

Theorem 2 ([2, 10])**.** $\mathcal{A}_{\rho_G}$ is an orbifold datum in $(\mathcal{D}_{\mathcal{C}})_{\text{Hom-orb}}$, and $\mathcal{C}_{\mathcal{A}_{\rho_G}} \cong (\mathcal{C}^\times_G)^G$.

Hence every 0-form symmetry can be gauged, and we have $(\mathcal{Z}^{\mathrm{RT}}_{\mathcal{C}})_{\mathcal{A}_{\rho_G}} \cong \mathcal{Z}^{\mathrm{RT}}_{(\mathcal{C}^\times_G)^G}$.

Finally, one finds that the theory obtained by gauging the 0-form symmetry ρ_G inherits a 1-form symmetry $\widehat{\rho}_G\colon \mathrm{B}^2\mathrm{Rep}(G) \to \mathcal{D}_{(\mathcal{C}^\times_G)^G}$. This can also be gauged, undoing the gauging of ρ_G:

Theorem 3 ([2])**.** $\mathcal{A}_{\widehat{\rho}_G}$ is an orbifold datum in $\mathcal{D}_{(\mathcal{C}^\times_G)^G}$, and $(\mathcal{C}_{\mathcal{A}_{\rho_G}})_{\mathcal{A}_{\widehat{\rho}_G}} \cong \mathcal{C}$.

In the special case when G is abelian, $\widehat{\rho}_G$ can be interpreted as a 1-form symmetry $\mathrm{B}^2\underline{\widehat{G}} \to \mathcal{D}_{(\mathcal{C}^\times_G)^G}$ of the Pontryagin dual $\widehat{G} = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{U}(1))$.

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The Classification of Fusion 2-Categories

Thibault D. Décoppet

(joint work with Peter Huston, Theo Johnson-Freyd, Dmitri Nikshych, David Penneys, Julia Plavnik, David Reutter and Matthew Yu)

Fusion 2-categories, first introduced in [10], categorify the notion of fusion 1-category, and have found relevance in the study of TQFTs via a state-sum construction [10], braided fusion 1-category theory [12], high energy physics [3], and condensed matter physics [9].

One point of physical interest in the theory of (higher) fusion categories is that many relevant physical theories enjoy symmetries given by not necessarily invertible topological operators. Such operators assemble naturally into a (higher) category. Most prominently, systems in (1+1)-spacetime dimensions, such as the critical Ising model, do enjoy a fusion 1-category of symmetries, which describes the anyons, or 1-spacetime-dimensional operators. As for fusion 2-categories, they describe the fusion of 2-spacetime-dimensional surface operators, which are prevalent for many quantum field theories, especially those in higher dimensions.

Our classification shows that a fusion 2-category is completely described by the braided fusion 1-category of endomorphisms of its unit together with certain group theoretical and cohomological data. This is in sharp contrast to the situation with fusion 1-categories: While it is sometimes possibly to classify special examples and classes of fusion categories, there are many "exotic" examples constructed from quantum groups at roots of unity [2] and examples arising from the theory of subfactors [1]. In particular, while not every fusion 1-category is "group-theoretical" in the sense of [11], our main result can be interpreted as the statement that every fusion 2-category is "braided fusion 1-category plus group" theoretical.

A fusion 2-category is by definition a finite semisimple rigid monoidal 2-category with simple monoidal unit. The classification of such structures naturally splits off into two cases. For any fusion 2-category $\mathfrak{C}$, there is a braided fusion 1-category $\Omega\mathfrak{C}$ of endomorphisms of the monoidal unit. Its *symmetric center* (also called Müger center) $\mathcal{Z}_2(\Omega\mathfrak{C})$ is a symmetric fusion 1-category. A fusion 2-category $\mathfrak{C}$ is *bosonic* if $\mathcal{Z}_2(\Omega\mathfrak{C})$ is Tannakian, i.e. is equivalent to the symmetric monoidal category $\mathbf{Rep}(H)$ of finite-dimensional representations of a finite group H.

Theorem 1 ([7]). *Bosonic fusion 2-categories are parameterised by:*

- *A nondegenerate braided fusion 1-category $\mathcal{A}$;*
- *An inclusion of finite groups $\iota : H \hookrightarrow G$;*
- *A monoidal functor $\rho : H \to \mathcal{A}ut^{\mathrm{br}}(\mathcal{A})$, where $\mathcal{A}ut^{\mathrm{br}}(\mathcal{A})$ denotes the 2-group of braided autoequivalences of $\mathcal{A}$;*
- *A class $\pi \in \mathrm{H}^4(\mathrm{B}G, \mathbb{C}^\times)$; and*
- *A homotopy between the anomaly of ρ and $\pi|_H$.*

Given a bosonic fusion 2-category $\mathfrak{C}$, the data appearing above may be extracted as follows:

- Let $\mathcal{Z}(\mathfrak{C})$ denote the (2-categorical) Drinfeld center of $\mathfrak{C}$. We show that the canonical symmetric monoidal functor $\Omega\mathcal{Z}(\mathfrak{C}) \to \mathcal{Z}_2(\Omega\mathfrak{C})$ is dominant. In particular, $\Omega\mathcal{Z}(\mathfrak{C})$ is Tannakian if and only if $\mathcal{Z}_2(\Omega\mathfrak{C})$ is, so that, by appealing to [5], the functor $\Omega\mathcal{Z}(\mathfrak{C}) \to \mathcal{Z}_2(\Omega\mathfrak{C})$ is equivalent to the restriction functor $\mathbf{Rep}(G) \to \mathbf{Rep}(H)$ along an inclusion of finite groups $H \hookrightarrow G$.
- The braided fusion 1-category $\Omega\mathfrak{C}$ has symmetric center $\mathcal{Z}_2(\Omega\mathfrak{C}) = \mathbf{Rep}(H)$ and hence is by the (de-)equivariantization theorem of [6] classified by the nondegenerate braided fusion category $\mathcal{A} = \Omega\mathfrak{C} \boxtimes_{\mathcal{Z}_2(\Omega\mathfrak{C})} \mathbf{Vect}$ together with an action of H by braided monoidal equivalences (and $\Omega\mathfrak{C}$ can be recovered as the fixed points of that action on $\mathcal{A}$). Thus, the first three bullet points above fully determine the braided fusion 1-category $\Omega\mathfrak{C}$ and the symmetric functor $\Omega\mathcal{Z}(\mathfrak{C}) \to \mathcal{Z}_2(\Omega\mathfrak{C})$.
- The cohomology class π, along with $\mathcal{A}$, determine the Morita equivalence class of $\mathfrak{C}$. This observation is intimately related to the fact, established in [4], that every fusion 2-category is Morita equivalent to a connected fusion 2-category, that is, one arising from a braided fusion 1-category.
- The last bullet point records more subtle cohomological information pertaining to the higher coherences of $\mathfrak{C}$.

With regard to the second half of the classification, recall from [5] that any symmetric fusion 1-category that is not Tannakian is *super-Tannakian*, i.e. of the form $\mathbf{Rep}(H, z)$ for a finite supergroup (H, z) with z nontrivial. The simplest example of such a super-Tannakian symmetric fusion 1-category is $\mathbf{sVect}$, the symmetric fusion 1-category of super-vector spaces, corresponding to the supergroup $(H, z) = (\mathbb{Z}/2, -1)$. A fusion 2-category $\mathfrak{C}$ is said to be *fermionic* if $\mathcal{Z}_2(\Omega\mathfrak{C})$ is super-Tannakian.

Theorem 2. *Fermionic fusion 2-categories are parameterized by:*

- *An* $\mathbf{SVect}$*-nondegenerate braided fusion* 1*-category* $\mathcal{A}$*, i.e. a braided fusion 1-category* $\mathcal{A}$ *equipped with a symmetric equivalence* $\mathbf{sVect} \to \mathcal{Z}_2(\mathcal{A})$*;*
- *An inclusion* $\iota : (H, z) \hookrightarrow (G, z)$ *of finite supergroups with* z *nontrivial;*
- *A super monoidal functor* $\rho : (H, z) \to (\mathcal{A}ut^{\mathrm{br}}_{\mathbf{SVect}}(\mathcal{A}), (-1)^f)$*;*
- *A class* ϖ *in a torsor for the supercohomology group* $\mathrm{SH}^4(\mathrm{B}(G, z))$*; and*
- *A homotopy between the anomaly of* ρ *and* $\varpi|_{(H,z)}$*.*

Our classification results immediately yield various profits. Specifically, we obtain rank finiteness as well as Ocneanu rigidity for fusion 2-categories, that is to say, fusion 2-categories admit no nontrivial deformations.

It remains to make precise sense in which the above data "parametrizes" all fusion 2-categories. For the sake of brevity, we will only do so in the bosonic case. This makes use of two spaces: Firstly, we write $\mathrm{BMF1C}^{\mathrm{ndeg}}(\mathbf{Vect})$ for the 2-groupoid of nondegenerate braided fusion 1-categories and braided monoidal equivalences. Secondly, we write $\mathcal{W}itt(\mathbf{Vect})$ for the 4-groupoid of nondegenerate braided fusion 1-categories and Witt equivalences as introduced in [8]. The next result provides a homotopy coherent version of the classification of bosonic fusion 2-categories.

Theorem 3 ([7])**.** *For any inclusion of finite groups* $\iota : H \hookrightarrow G$*, there is an equivalence of* 3*-groupoids between the* 3*-groupoid*

$$\left\{ \begin{array}{c} \text{Fusion 2-categories } \mathfrak{C} \text{ equipped with an identification} \\ [\Omega\mathcal{Z}(\mathfrak{C}) \to \mathcal{Z}_2(\Omega\mathfrak{C})] \cong [\mathbf{Rep}(G) \twoheadrightarrow \mathbf{Rep}(H)] \end{array} \right\}$$

and the space of commuting squares of (unpointed!) spaces

$$\begin{array}{ccc} \mathrm{B}H & \dashrightarrow & \mathrm{BMF1C}^{\mathrm{ndeg}}(\mathbf{Vect}) \\ {\scriptstyle\iota}\downarrow & \overset{\simeq}{\swarrow} & \downarrow{\scriptstyle[-]} \\ \mathrm{B}G & \xrightarrow[\pi]{} & \mathcal{W}itt(\mathbf{Vect})\,. \end{array} \tag{4}$$

To conclude, let us mention that the above statement, in addition to admitting a fermionic counterpart, can be extended to account for multifusion 2-categories, that is, finite semisimple rigid monoidal 2-categories for which the monoidal unit is not necessarily simple.

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My two favourite uses of topological defects

Paul Fendley

(joint work with D. Aasen, R.S.K. Mong; and Y. Ueda, V. Vander Linden, L. Lootens, J. Haegemans, F. Verstraete)

Statistical mechanics and mathematics have always been closely linked. A particularly beautiful connection arose via Temperley and Lieb's work on the former and Jones's work on the latter. Namely, the former authors showed how the transfer matrices of a host of two-dimensional classical lattice statistical models could be written in terms of generators of an algebra that now bears their name [1]. This algebra played a central role in the construction of the Jones polynomial [2], and this connection resulted in a host of new knot invariants being discovered; for a review see e.g. [3].

After a great string of successes, this cross-fertilisation went quiet for quite a while. However, there was progress in the mathematical world developing the structure of fusion categories, placing algebras like that of Temperley and Lieb into a broader context. The close connections re-emerged with the pioneering construction of Ref. [4], where a certain Temperley-Lieb quantum spin chain (now known as the "golden chain") was rewritten in terms of a fusion category. Namely, one can view the degrees of freedom as being the fusion channels of collection of anyons. The utility of this rewriting is that the Hamiltonian commutes with an unusual non-local symmetry that the authors dubbed "topological symmetry". Subsequently it was shown how such symmetries occur automatically whenever a quantum Hamiltonian or transfer matrix is defined via the idempotents built from a fusion category [5].

One key property the category builds into the model is a set of *topological defects*. The partition function in the presence of such defects remains invariant under the deformation of their paths, even if the defect separates regions described by different theories. These topological defects thus give a precise method for defining non-invertible mappings and symmetries. Moreover, the defects themselves satisfy the fusion rules and the F moves of the underlying category [5]. The canonical example of such a defect is that implementing the Kramers-Wannier duality of the Ising model. In [6], we showed how to utilise the fusion-category structure in Ising, deriving new results for this venerable model.

These topological defects provide a myriad of interesting applications, two of which were described in my talk. The first is to find the effect of the Dehn twist on the partition function of two-dimensional classical lattice models in the presence of defects [5]. The fusion category data allow the construction of defect junctions in these models and give the F moves that relate different configurations. One then can choose some basis for these configurations on a torus, and Dehn twists give a linear map between them. The eigenvalues of this linear map in all known examples are roots of unity. If this lattice model has a continuum limit described by a conformal field theory, the exponent of this root of unity is related to the chiral dimension of of a conformal primary field. For example, one finds eigenvalue $e^{\pi i/8}$ for the Dehn twist in the presence of a Kramers-Wannier duality defect in the Ising model, and $e^{4\pi i/5}$ for the analogous defect in the Fibonacci category. Thus if a continuum limit of such a lattice model exists and is described by a CFT, it must possess an operator of chiral dimension $1/16$ and $2/5$ respectively, up to half-integer shifts. The Ising and tricritical Ising CFTs respectively do possess operators with these chiral dimensions. This I believe is the absolute simplest way to see how this famous $1/16$ arises.

The other nice application I described greatly illuminates an old problem in quantum field theories with monopoles. Scattering a particle off such a monopole seems paradoxical in that there is no apparent outgoing state. The resolution is that the outgoing states are created by non-local operators, so that while their energy is localised, they have a "string" attached; see [7] and references therein. We showed [8] that this seemingly mysterious phenomenon occurs very naturally in theories with a topological duality defect. Because the spectrum on both sides of the defect must be the same, particles are perfectly transmitted. The string occurs because of the duality; a local excitation on one side becomes non-local in the dual. One elegant example of this phenomenon occurs in the Ising quantum chain, where the string is the spln-flip defect. Other examples are not integrable, such as the Kennedy-Tasaki duality defect separating the spin-1 Heisenberg antiferromagnet in the Haldane/SPT phase from an ordinary ferromagnetic phase. By direct calculation we also found an example in a 2d quantum lattice model, but here the connection with categories remains an open problem. Could a 2-category be involved?

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Construction of fully extended 2d CFTs via conformal net methods

André Henriques
(joint work with James Tener)

Let $\mathrm{Bord}^{\mathrm{Conf}}$ be the fully extended conformal bordism bicategory, and vNAlg the bicategory of von Neumann algebras and bimodules between them. These are symmetric monoidal bicateogories with respective monoidal structures given by disjoint union and spatial tensor product. Associated to every conformal net $\mathcal{A}$, we construct a fully extended functorial CFT in the form of a symmetric monoidal functor

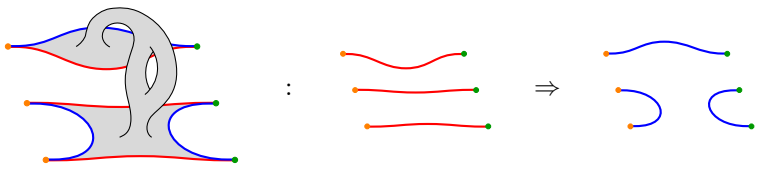
$$\mathrm{Bord}^{\mathrm{Conf}} \to \mathrm{vNAlg}.$$

Such a CFT assigns a von Neumann algebra to the point, bimodules to 1-dimensional bordisms, and bounded bimodule maps to 2-dimensional conformal bordisms. When the conformal net $\mathcal{A}$ is rational, the associated CFT is defined on the entire conformal bordism bicategory. When it is not rational, the CFT is only defined on the sub-category $\mathrm{Bord}^{\circ,\mathrm{Conf}} \subset \mathrm{Bord}^{\mathrm{Conf}}$ with same 0- and 1-morphism, but whose 2-morphisms are disjoint unions of connected bordisms with non-empty incoming boundary and non-empty outgoing boundary (in other words, the inclusions of the source and target of a 2-dimensional bordism must be π_0-surjective).

We conjecture that, when applied to the Virasoro conformal net, our construction produces Liouville theory. And in the case of rational conformal nets, our construction should recover the CFTs with diagonal modular invariant.

1. The conformal bordism bicategory

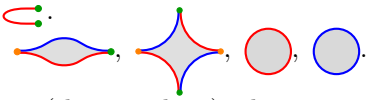
The objects of the fully extended conformal bordism category are finite collections of points. The 1-morphisms are smooth 1-dimensional bordisms equipped with germs of local coordinates at each boundary point. Finally, the 2-morphisms are conformal 2-dimensional bordism between 1-dimensional bordisms. No orientations are required, at any level. Here is an example of a conformal bordism



which is a 2-morphism in $\mathrm{Bord}^{\mathrm{Conf}}$ between 1-morphisms .

Formally, a conformal bordism is a locally ringed space modelled on closed subspaces of the complex plane of the form $\{x+iy \in \mathbb{C} : f(x) \leq y \leq g(x), 0 \leq x\}$, where $f, g \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ satisfy $f \leq g$ and vanish identically in a neighbourhood of zero. Such closed subspaces are equipped with the sheaf of $\mathbb{C}$-valued functions that are continuous, C^∞ when restricted to the graphs of f and of g, and holomorphic when restricted to the interior. The structure sheaf is considered here as a sheaf of mere $\mathbb{R}$-algebras: in that way, oriented and unoriented surfaces may be treated on the same footing.

The symmetric monoidal bicategory $\mathrm{Bord}^{\mathrm{Conf}}$ admits the following set of generators:

– Degree 0: the point .
– Degree 1: the intervals , and .
– Degree 2: bigons, 4-pointed stars, and discs:  .

If we remove the two last degree two generators (the two discs), the remaining morphisms generate $\mathrm{Bord}^{\circ,\mathrm{Conf}}$.

2. von Neumann algebras and conformal nets

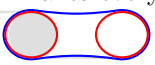
von Neumann algebras form the objects of the bicategory vNAlg. The 1-morphisms are bimodules, and the 2-morphisms are the bounded linear maps which are at the same time bimodule homomorphisms. The unit 1-morphism on $A \in \mathrm{vNAlg}$ is denoted ${}_AL^2A_A$ (for any faithful state $\varphi : A \to \mathbb{C}$, this is canonically unitarily isomorphic to the GNS representation $L^2(A,\varphi)$), and the composition of bimodules is provided by Connes fusion:

$$ {}_AH \boxtimes_B K_C \;=\; {}_A\big(\mathrm{Hom}_B(L^2B, H) \otimes_B K\big)_C $$

(where the tensor product in the right is suitably completed to a Hilbert space). Replacing a von Neumann algebra by its opposite does not affect the L^2 space construction in the sense that there is a canonical unitary $L^2A \cong L^2A^{\mathrm{op}}$, compatible with the two actions of A.

A conformal net consists of a Hilbert space H_0, a projective positive energy representation of $\mathrm{Diff}(S^1)$ on H_0, a collection of von Neumann algebras $\mathcal{A}(I)$ indexed by the poset of subintervals $I \subset S^1$, and a cyclic vector $\Omega \in H_0$. The assignment $I \mapsto \mathcal{A}(I)$ is compatible with the action of $\mathrm{Diff}(S^1)$, and the algebras $\mathcal{A}(I)$ satisfy $(I \subset J) \Rightarrow (\mathcal{A}(I) \subset \mathcal{A}(J))$ and $(\mathring{I} \cap \mathring{J} = \emptyset) \Rightarrow ([\mathcal{A}(I), \mathcal{A}(J)] = 0)$. Given a conformal net, we get:

- A von Neumann algebra $\mathcal{A}(I)$ associated to every oriented interval I (not necessarily embedded in S^1). Changing the orientation amounts to replacing the von Neumann algebra by its opposite: $\mathcal{A}(\bar{I}) = \mathcal{A}(I)^{\mathrm{op}}$.
- A Hilbert space $H_0(D)$ called vacuum sector associated to every complex disc D, along with a vacuum vector $\Omega_D \in H_0(D)$, and actions of $\mathcal{A}(I)$ for every $I \subsetneq \partial D$.
- Given two discs D_1, D_2 and an orientation preserving diffeomorphism between ∂D_1 and ∂D_2, there exists a unitary $H_0(D_1) \to H_0(D_2)$ intertwining the actions of the algebras $\mathcal{A}(I)$. This unitary is unique up to phase, but otherwise non-canonical. (It typically does not preserve the vacuum vectors.)
- Given two complex discs D_1, D_2 and an orientation reversing diffeomorphism $\partial D_1 \to \partial D_2$, there is a unique bounded bilinear pairing $b : H_0(D_-) \times H_0(D_+)$ compatible with the actions of the algebras $\mathcal{A}(I)$, and normalised so as to have $b(\Omega_-, \Omega_+) = 1$.
- (Bisognano-Wichmann theorem) If a complex disc D is equipped with an antilinear involution $\vartheta : D \to D$ that exchanges the two complementary halves $\partial_+ D$ and $\partial_- D$ of its boundary, then there is a canonical unitary isomorphism between $H_0(D)$ and $L^2(\mathcal{A}(I_+))$, compatible with the two actions of $\mathcal{A}(I_+)$.

The new technical result that James Tener and I were recently able to establish, and which makes everything work is the following. Consider a bordism like this one: $\Sigma =$  , and pick an embedding $\varphi : \Sigma \hookrightarrow \mathbb{C}$. Such an embedding defines two disjoint discs $D_1, D_2 \subset \mathbb{C}$ (bound by the red curves), along with a bigger one $D_3 \subset \mathbb{C}$ (bound by the blue curve) that contains $D_1 \sqcup D_2$. Let $\Omega_i \in H_0(D_i)$ be the respective vacuum vectors, and let $I := \partial D_1 \cap \partial D_3$ and $J := \partial D_2 \cap \partial D_3$. We assume that I and J are connected and have non-empty interior.

Proposition (proof to appear in [1])**.**

(1) In the above situation, there exists a unique bounded linear map

$$Y : H_0(D_1) \otimes H_0(D_2) \to H_0(D_3)$$

which satisfies $Y(\Omega_1 \otimes \Omega_2) = \Omega_3$ and which is linear for the algebras $\mathcal{A}(I)$ and $\mathcal{A}(J)$.

(2) Let $\varphi' : \Sigma \hookrightarrow \mathbb{C}$ be another embedding, with corresponding discs D_1', D_2', D_3', and let $u_i : H_0(D_i) \to H_0(D_i')$ be the unitaries (well defined up to phase) afforded by the third bullet above. Then the corresponding map $Y' : H_0(D_1') \otimes$

$H_0(D_2') \to H_0(D_3')$ makes the following diagram commute up to phase:

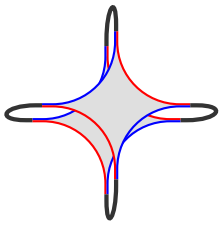
$$\begin{array}{ccc} H_0(D_1) \otimes H_0(D_2) & \xrightarrow{Y} & H_0(D_3) \\ \downarrow{\scriptstyle u_1 \otimes u_2} & & \downarrow{\scriptstyle u_3} \\ H_0(D_1') \otimes H_0(D_2') & \xrightarrow{Y'} & H_0(D_3') \end{array}$$

3. The fully extended CFT: value on the generators

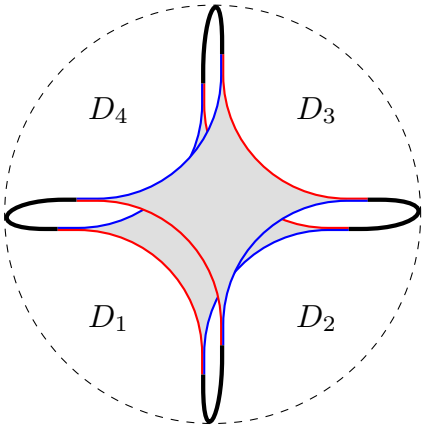
Given a conformal net $\mathcal{A}$, the corresponding fully extended CFT assigns the value $A^{\mathrm{CFT}}(pt) := \mathcal{A}([-1,1])$ to the point.

To an interval I, we assign the Hilbert space $H^{\mathrm{CFT}}(I) := L^2(\mathcal{A}([0,1] \cup I \cup [0,1]))$, after an arbitrary choice of an orientation of I. The above Hilbert space is isomorphic to $H_0(I \times [-1,1])$ (provided $I \times [-1,1]$ is equipped with a suitably chosen complex structure), and the latter description makes it evident that the algebras $\mathcal{A}(\partial_- I \times [-1,1]) \cong A^{\mathrm{CFT}}(pt)$ and $\mathcal{A}(\partial_+ I \times [-1,1]) \cong A^{\mathrm{CFT}}(pt)$ act on $H^{\mathrm{CFT}}(I)$.

The most important part of the construction is what one assigns to 4-pointed stars. Given a 4-pointed star, consider its oriented double cover (which is a disjoint union of two copies of the 4-pointed star). Then connect the four pairs of vertices with copies of $[-1,1]$ (in black in the following picture):



Pick an embedding of the above space into $\mathbb{C}P^1$ intertwining its obvious $\mathbb{Z}/2$ symmetry with the complex conjugation involution on $\mathbb{C}P^1$. Such an embedding defines four complex discs $D_1, \dots, D_4 \subset \mathbb{C}P^1$:



Let $\hat{D}_1$ be the disc which is the union of D_1, D_3, and the top 4-pointed star. And let $\hat{D}_2$ be the union of D_2, D_4, and the bottom 4-pointed star. Then $\hat{D}_1$ and $\hat{D}_2$ cover $\mathbb{C}P^1$ and share the same boundary (the boundary's shape is reminiscent of

the seam of a baseball). By Proposition 2, we have bounded linear maps

$$H_0(D_1) \otimes H_0(D_3) \to H_0(\hat{D}_1) \qquad H_0(D_2) \otimes H_0(D_4) \to H_0(\hat{D}_2).$$

And by the fourth bullet above, we have a bounded bilinear map

$$H_0(\hat{D}_1) \times H_0(\hat{D}_2) \to \mathbb{C}.$$

Composing these maps and then taking adjoints (and using the fact that the Hilbert spaces $H_0(\hat{D}_i)$ are canonically isomorphic to their complex conjugates), we get a bounded linear map

$$H_0(D_1) \otimes H_0(D_3) \to H_0(D_2) \otimes H_0(D_4).$$

That's the map associated to the 4-pointed star by the CFT.

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Non-invertible symmetries on a tensor product Hilbert space in 1+1 dimensions

Kansei Inamura

We discuss general constraints on non-invertible symmetries realized in 1+1d lattice models with a tensor product Hilbert space. Recently, it was shown in [1] that a finite non-invertible symmetry described by a unitary fusion category $\mathcal{C}$ can be realized on a tensor product Hilbert space, or more precisely on a quasi-local tensor product algebra, if and only if $\mathcal{C}$ is integral. This result implies that the fusion rules of finitely many symmetry operators on a tensor product Hilbert space can agree only with those of integral fusion categories. On the other hand, in the physics community, it is known that there exist non-invertible symmetries whose fusion rules agree with those of non-integral fusion categories up to lattice translations. A prototypical example of such a symmetry is the Kramers-Wannier self-duality of the critical Ising model [2], where the symmetry operators obey the fusion rules of a $\mathbb{Z}_2$ Tambara-Yamagami category up to lattice translations [3]. Based on this example, it was conjectured in [4, 5] that fusion category symmetries realized on a tensor product Hilbert space up to lattice translations must be weakly integral. In [6], we provided a conditional proof of this conjecture based on several physical assumptions on symmetry operators. In particular, we pointed out that the above conjecture follows from the yet-to-be-established index theory of non-invertible symmetry operators on the lattice. More evidence of this conjecture was also provided in recent papers by other authors [7, 8].

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Rigid Firm Categories

Theo Johnson-Freyd

(joint work with Jake McNamara and David Reutter)

Quillen [1] calls a nonunital algebra A *firm* when the multiplication map $A \otimes_A A \to A$ is an isomorphism; an A-module M is firm when $A \otimes_A M \to M$ is an iso. Every unital algebra is firm, and a module for a unital algebra is unital (in the sense that the unit acts as a unit) if and only if it is firm. The many-object generalization of firm algebras to (enriched, ∞-)categories: a nonunital category C is *firm* when, for every x, z, the composition map $\mathrm{coend}_{y \in C} C(x,y) \times C(y,z) \to C(x,z)$ is an isomorphism; a presheaf f on C is firm when $\mathrm{coend}_{y \in C} C(x,y) \times f(y) \to f(x)$ is an isomorphism; a functor $f : C \to D$ of firm categories is a *firm functor* when $D(f(-), f(-))$ is firm in each variable. These notions they have been independently discovered multiple times. For example, [2] uses the term "H-unital algebra" for a firm algebra in Sp, and [3] uses the term "taxonomy" for a firm category enriched in Set.

Given a firm category C, enriched in some ambient presentably symmetric category $\mathcal{V}$, write $P_{\mathcal{V}}(C)$ for its presentable $\mathcal{V}$-category of firm $\mathcal{V}$-enriched presheaves. These categories have an Eilenberg–Watts theorem: colimit-preserving $\mathcal{V}$-functors between $P_{\mathcal{V}}(C)$ and $P_{\mathcal{V}}(D)$ are precisely the firm C-D-profunctors. One reason to care about firm categories is the following amazing result of Ramzi, which says that "firm Morita theory" fully captures the theory of dualizable presentable $\mathcal{V}$-categories. (Ramzi states the theorem differently, but this is essentially Theorem B of [4].)

Theorem ([4])**.** *For any firm $\mathcal{V}$-category C, $P_{\mathcal{V}}(C)$ is 1-dualizable in the 2-category $\mathsf{Pr}_{\mathcal{V}}$ of presentable $\mathcal{V}$-categories. Moreover, every 1-dualizable presentable*

$\mathcal{V}$-category is $P_{\mathcal{V}}(C)$ for some firm category C. Moreover, given a 1-dualizable presentable $\mathcal{V}$-category $\mathcal{M}$, a firm category C presenting it can be constructed as follows: choose any subspace $X \subset \mathrm{ob}(\mathcal{M})$ which is "spanning" in the sense that its full image is dense; take just the "atomic" morphisms between objects of X; then X withjust the atomic morphisms is a firm category presenting $\mathcal{M}$.

Note that by taking a large enough object, in fact every dualizable presentable $\mathcal{V}$-category is the firm modules for a firm $\mathcal{V}$-algebra. The precise definition of "atomic morphism" is in [4]. Essentially, the atomic morphisms with domain $x \in \mathcal{M}$ are what would be the answer if x were an atomic object. Here is a valuable false etymology of the word "atomic": in Ban, the category of Banach spaces and short maps, the atomic morphisms are precisely the nuclear kernels.

One can easily talk about monoidal firm categories (and even firm-monoidal firm categories, where both the categorical composition and the monoidal product are firm rather than unital). Without units morphisms, it becomes more interesting to say the word "rigid": one cannot ask for a zig-zag composite to be an identity. Suppose that C is monoidal firm, and consider the associativity square at the level of dualizable presentable categories:

$$\begin{array}{ccc} P(C) \boxtimes P(C) \boxtimes P(C) & \xrightarrow{P(C)\boxtimes m} & P(C) \boxtimes P(C) \\ \downarrow{\scriptstyle m\boxtimes P(C)} & \circlearrowright & \downarrow{\scriptstyle m} \\ P(C) \boxtimes P(C) & \xrightarrow{\quad m \quad} & P(C) \end{array}$$

The sides of this square are all internal left adjoints in $\mathsf{Pr}_{\mathcal{V}}$. C is called *Gaitsgory rigid* if this square is adjointable aka Beck–Chevalley in both directions, i.e. if both mates of this square strongly commute. It is called *rigid* if it is Gaitsgory rigid and moreover the induced firm profunctors $(-)^*, {}^*(-) : C \nrightarrow C^{\mathrm{op}}$ are realized by functors.

One can moreover talk about dagger structures on firm categories. A *unitary dual functor* on a rigid monoidal dagger (firm) category is a choice of compatibility between $(-)^*$ and $(-)^\dagger$. A UDF selects a cyclic trace $\mathrm{Tr}_c : \mathrm{End}(c) \to \Omega C := \mathrm{End}(1_C)$.

The most important example of a rigid firm monoidal dagger category with UDF is a geometric bordism category ($\mathsf{Bord}^{\mathrm{Riem}}_{n,n-1}$ and its cousins). The firm representations of $\mathsf{Bord}^{\mathrm{Riem}}_{n,n-1}$ are the ones that Kontsevich and Segal [5] call *continuous*.

Let B be any rigid firm symmetric monoidal dagger category with UDF, and let $\mathbb{C}B$ denote its $\mathbb{C}$-linearization. Write $\Omega\mathbb{C}B = \mathbb{C}\Omega B$ for the $*$-algebra of endomorphisms of the unit object. The *positive cone* in $\Omega\mathbb{C}B$ is the set of all elements of the form $\mathrm{Tr}_b(f^\dagger f)$ for any $f : b \to c$ in $\mathbb{C}B$.

Definition. *A* reflection-positive partition function on B *is a positive tracial state on $\mathbb{C}B$: a $*$-monoid homomorphism $z : \Omega B \to \mathbb{C}$ whose free linearization sends positives in $\Omega\mathbb{C}B$ to positives.*

Any positive tracial state begs you to run a GNS construction. The many-object version builds, for each object $b \in B$, a Ban-valued (in fact, Hilb-valued, but I want to keep with presentable categories) firm presheaf $H_{z,b}$ on C; the traciality says that $H_{z,b}$ depends functorially on b, so that we have built a functor $H_z : B \to P_{\mathsf{Ban}}(B)$. In fact, it is better than landing just in $P_{\mathsf{Ban}}(B)$: it lands among the atomic morphisms. Let $\tilde{B}_z$ denote the "full atomic image" of $H_z : B \to P_{\mathsf{Ban}}(B)$. Then $\tilde{B}_z$ is again firm, now Ban-enriched, and again rigid symmetric monoidal dagger with UDF.

When is a firm category C Morita-equivalent to a unital one? In other words, when is $P(C)$ atomically generated? The atomic objects in $P(C)$ are the *Karoubi completion* $\mathrm{Kar}(C)$ of C: the (unital!) category whose objects the idempotents in C, and with $\mathrm{Kar}(C)(p,q) = pCq$. In the unital world, C and $\mathrm{Kar}(C)$ are always Morita equivalent; in the firm world, this happens if and only if $P(C)$ is atomically generated. Consider the case when C is Vec-enriched and has one object, i.e. when C is a firm algebra A. Build the ideal $K(A) \subset A$ generated by generated by all idempotent elements. As an idempotent ideal in a firm algebra, $K(A)$ is again firm, and by construction $K(A)$ is Morita equivalent (as a firm category) to $\mathrm{Kar}(A)$. In the unital world, $K(A) = A$, whereas in the firm world, $K(A)$ can be anywhere between A and 0. Thus one sees that A is "Morita unital" if it "has enough idempotents" — if $K(A) = A$.

Colafranceschi, Dong, Marolf, and Wang have studied in [6] precisely this question for the firm Ban-category $\tilde{B}_z$. Their main theorem is not stated this way (indeed, their paper is written in the language of quantum gravity, not mathematics, and so they do not even distinguish a "main theorem," let alone talk in term of firm Ban-categories), but their argument applies at the level of generality herein:

Theorem ([6]). *$\tilde{B}_z$ has enough idempotents: it is Morita equivalent to its Karoubi completion. Moreover, all endomorphism algebras in $\mathrm{Kar}(\tilde{B}_z)$ are finite-dimensional W^*-algebras.*

But $\tilde{B}_z$ is Gaitsgory-rigid and hence $\mathrm{Kar}(\tilde{B}_z)$ is as well (since Gaitsgory-rigidity is a Morita-invariant notion) and hence $\mathrm{Kar}(\tilde{B}_z)$ is rigid (since it is Karoubi complete). By construction, $\Omega\tilde{B}_z = \mathbb{C}$ (and $\Omega B \to \Omega\tilde{B}_z$ is z). The famous reconstruction result of Doplicher and Roberts [7] thus says that there is a unique (up to non-unique isomorphism) super fibre functor $\mathrm{Kar}(\tilde{B}_z) \to \mathsf{sHilb}^{\mathrm{fd}}$. Recompile via the equivalences $P_{\mathsf{Ban}}(\mathrm{Kar}(\tilde{B}_z)) \simeq P_{\mathsf{Ban}}(\tilde{B}_z)$ and $P_{\mathsf{Ban}}(\mathsf{sHilb}^{\mathrm{fd}}) = \mathsf{sBan}$, and note that the presentably-dagger-with-UDF structure on sBan is dagger-with-UDF-in-the-small-category-sense precisely on $\mathsf{sHilb} \subset \mathsf{sBan}$. One ends up with the following amazing result reconstruction result of McNamara and Wang, which will appear in upcoming work and which is what inspired this project:

Theorem (McNamara and Wang). *Let B be any rigid symmetric monoidal firm category with dagger and UDF. Let $z : \Omega B \to \mathbb{C}$ be any reflection positive partition function on B. Then z extends, uniquely up to nonunique isomorphism, to a symmetric monoidal dagger-with-UDF functor $Z : B \to \mathsf{sHilb}$.*

In other words:

Unitary functorial field theories, including the geometric ones, are determined by their partition functions.

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Global symmetries from Hamiltonian automorphisms

David Jordan

(joint work with Ibou Bah and Nitu Kitchloo)

In this talk I explain some ongoing work with Ibrahima Bah and Nitu Kitchloo, aimed at understanding how to deduce discrete topological symmetries of quantum field theories coupled to continuous background fields.

A typical setup in physics is that we have some dynamical quantum field theory T_d in dimension d, which is coupled to a bulk topological field theory B_{d+1} in dimension $d+1$. Here B_{d+1} describes the allowed background gauge fields for a global symmetry, and also encodes any anomaly attached to that symmetry.

A special case of this setup which is well-understood mathematically is when the bulk theory B_{d+1} is governed by a homotopy type, especially when it is a finite homotopy type. However, what typically happens in physical examples is that the bulk fields are not purely homotopical, but have a discrete and a continuous flavour. This can be captured by saying that the background fields form a sheaf on the site of smooth manifolds, but that this sheaf is not (in general) representable by a topological space.

In this work, we consider various classes of examples of this latter setup with continuous background fields. This includes including BF theories, Chern-Simons theories, and super-gravity theories as appear in holography and in geometric engineering. In all of these examples, the bulk TQFT B_{d+1} is captured by abelian gauge fields, modeled via a stack of differential cohomology.

The key observation in our work, which will be explained in the talk, is that the bulk topological theory B_{d+1} itself has a suitable automorphism group, which

we call the Hamiltonian automorphisms H_{d+2}. The theory H_{d+2} is itself a TQFT in dimension $d+2$, having B_{d+1} as a boundary condition, and the original theory T_d as a corner.

In all such examples considered the symmetry group we construct is homotopical. Moreover, it is often straightforward to compute H_{d+2} using elementary homotopy theory. By construction these computation are all compatible with dimensional reduction. We examine a number of physical phenomena through this lens, including flux non-commutativity of Freed, Moore and Segal, and the computation of 2-form symmetries in geometric engineering.

Non-Invertible Symmetries and Scattering Amplitudes

Shota Komatsu
(joint work with Christian Copetti and Lucía Córdova)

Scattering amplitudes are one of the basic physical observables in quantum field theory. They compute the probability of a given process to happen when we collide particles, and are of direct relevance to experiments in colliders, such as Large Hadron Collider (LHC) at CERN. On the theoretical side, they have been intensively studied since the early days of quantum field theory—the main tool being the famous Feynman diagrams—and significant effort has been put into developing techniques for evaluating relevant Feynman diagrams. At the same time, they were also studied from more formal viewpoints, such as deriving general analytic properties of scattering amplitudes from axiomatic quantum field theory.

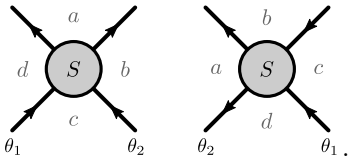
One of the remarkable properties of scattering amplitudes, which was first discovered through the computation of numerous Feynman diagrams, and later derived from the axiomatic viewpoint in specialized setups, is *crossing symmetry*. To describe its meaning, let us consider the so-called "$2 \to 2$" amplitudes; a process in which two in-coming particles collide and produce two out-going particles. (For simplicity, below we assume that all the particles have equal mass.) A convenient way to characterize this process is to go to the center of mass frame. In the center of mass frame, the two in-coming particles approach each other from the opposite directions with the same velocity. Using the rotational invariance, this can be fully characterized by the total energy of the in-coming particles. Due to the energy- and momentum-conservations, the two out-going particles move away with the same velocity in the opposite directions, but the direction from which the in-coming particles approach may not coincide with the direction in which the out-going particles move away. Therefore, the angle between the two directions provide the (Lorentz-)invariant information characterizing the scattering amplitudes.

These two invariant information can be packaged into the so-called Mandelstam variables s and t, and the $2 \to 2$ scattering amplitudes S can be viewed as functions of these variables; $S(s,t)$. Roughly speaking, s corresponds to the total energy while t corresponds to the angle. Crossing symmetry states that, if we exchange the two Mandelstam variables s and t, the resulting amplitude $S(t,s)$ now computes the scattering amplitude of a completely different process, in which one of the

in-coming particles becomes out-going while one of the out-going particle becomes in-coming:

$$S^{ab}_{dc}(s,t) = S^{bc}_{ad}(t,s)\,. \tag{1}$$

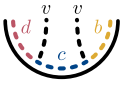
Pictorially, the two sides of the crossing symmetry equation can be represented as



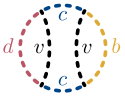
Remarkably, we found in [1] and [2] that the crossing symmetry is modified in some theories with non-invertible symmetries in $1+1$ dimensions. The detailed setups—unitary minimal model CFTs perturbed by $\phi_{1,3}$ operators—are described in the papers cited above, but the key reason for the modification is rather simple to state: when the theory is described in infrared by nontrivial TQFT, one of the basic assumptions (*asymptotic state assumption*, to be described below), needed for crossing symmetry, is violated.

In the standard analysis of scattering amplitudes, one assumes that the incoming state in the far past or the out-going state in the far future (called *asymptotic states*) can be described in terms of a collection of free non-interacting particles. Normally, this is physically a reasonable assumption since, in the infinite past or future, the particles are far away from each other and the interactions between them are negligible. Hence one expresses the in-coming and out-going states by a simple tensor product of free particle Fock states. However, this assumption is violated when the infrared is described by TQFT since there can be nontrivial "topological" correlation between particles even when the particles are far apart from each other. Because of this, the two-particle state is no longer a simple tensor product of two one-particle states. In particular, the norms of the two-particle states receive correction from TQFT dynamics even if one properly normalizes the one-particle states.

In the examples analyzed in the papers, this correction of the norm can be computed using the fusion category data. Concretely, using the fact that the particle worldlines can be viewed as topological defect from the TQFT viewpoint, one can represent the two-particle state as a network of defects as follows:

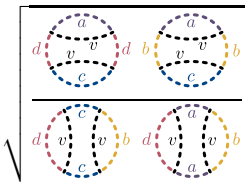


We can then compute the norm of the incoming two-particle states by evaluating the following diagram:



Therefore, to obtain scattering amplitudes that satisfy unitarity, we need to additionally divide scattering amplitudes by these "topological norms". Now, the crucial point is that these topological norms are different between the two sides of the equation (1). Hence, this induces a modification of the crossing symmetry in the following manner:

$$S^{ab}_{dc}(s,t) = \sqrt{\frac{\quad}{\quad}}\, S^{bc}_{ad}(t,s)\,. \tag{2}$$



These diagrams can be evaluated in terms of the quantum dimensions of topological defects d_a's and the final formula is

$$S^{ab}_{dc}(s,t) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(t,s)\,. \tag{3}$$

Although we only considered theories in $1+1$ dimensions, we expect similar violations of crossing symmetry in more general theories in which the infrared dynamics contains nontrivial TQFT. Possible candidates are

- Chern-Simons theory in $2+1$ dimensions coupled to matter; for special cases of $SU(N)$ gauge group, the modification of crossing symmetry is discussed in [3]. It would be interesting to reformulate their result in terms of "categorical" languages and generalize other TQFT in 2+1 dimensions coupled to matter.
- It has been argued that, in the presence of electron and monopole, the crossing symmetry receives corrections. This theory is not described by TQFT in infrared, but it has a common feature that the particles are not completely independent from each other no matter how far they are separated. (In this case, the correlation comes from a nonzero angular momentum produced by the Poynting vector $\propto E \times B$ stored in electromagnetic field between the particles.)
- More generally, we expect similar effects to be present in the presence of long-range forces, such as the Coulomb interaction or gravitational interaction. These theories are harder to deal with than TQFT since the theories are gapless and the modes that give rise to long-range correlations are continuously connected to non-topological degrees of freedom. However, it is believed that certain generalizations of symmetries (called asymptotic symmetries) play an important role in this context and it would be interesting if one can reformulate them from the viewpoints of categorical symmetries.

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Trisection invariants of four-manifolds

Catherine Meusburger
(joint work with Vincentas Mulevičius and Fiona Torzewska)

In [3] Gay and Kirby introduced a new description of (smooth oriented closed) four-manifolds in terms of trisections and trisection diagrams. Trisection diagrams of four-manifolds resemble Heegaard diagrams for three-manifolds. They involve curves of three different colours on a surface of genus $g \geq 1$ such that curves of any two colours form a Heegaard diagram for $\#_k(S^1 \times S^2)$.

Gay and Kirby showed in [3] that every four-manifold has a trisection and hence a trisection diagram and identified a set of moves that relate distinct trisection diagrams of diffeomorphic four-manifolds. They are diffeomorphisms of surfaces, two-point and three-point moves, handle slides and (de)stabilisations. A quantity assigned to a trisection diagram that is invariant under these moves defines a four-manifold invariant.

Chaidez, Cotler and Cui used this description in [2] to define trisection invariants in the spirit of Kuperberg invariants of three-manifolds. The algebraic data is a Hopf triplet, three finite-dimensional semisimple Hopf algebras that are related by compatible skew pairings. They show that the resulting four-manifold invariant reproduces the four-manifold invariant of Bärenz and Barrett [1] in those cases, where the data of the latter arises from representations of Hopf algebras.

We define a categorical counterpart and generalisation of the Hopf algebraic trisection invariants in [2] and relate them to diagrammatic calculi used in defect TQFTs [5]. The algebraic data are three spherical fusion categories, assigned to the three colours of curves in the trisection diagram, a finite semisimple indecomposable bimodule category over two of them, equipped with a bimodule trace, and a pivotal functor from the third spherical fusion category into the spherical fusion category of their bimodule endofunctors and natural transformations. This allows the three spherical fusion categories to interact.

We then interpret trisection diagrams as a diagrammatic calculus for this algebraic data following [4]. The three spherical fusion categories are assigned to the three colours of the trisection diagrams, and crossings of curves of different colours are labeled canonically with coherence isomorphisms. By summing over all labellings of curves and regions in the diagram with simple objects we assign to each trisection diagram an evaluation. It is already invariant under diffeomorphisms of surfaces, two-point and three-point moves and handle slides. After a rescaling that ensures invariance under (de)stabilisations, we obtain a four-manifold invariant.

The Hopf algebraic trisection invariants by Chaidez, Cotler and Cui [2] coincide with those cases of our invariant, where the underlying bimodule category is the category of finite-dimensional vector spaces. This situation provides fiber functors for the three spherical fusion categories in our formalisms and yields a Hopf triplet. Our construction suggests that there should be a generalisation of the formalism in [2] to *weak* Hopf triplets.

The four-manifold invariant by Bärenz and Barrett [1] for a pivotal functor from a spherical into a modular fusion category also arises as a special case of our construction. Here, two of the three spherical fusion categories and the corresponding bimodule category are identified with the modular fusion category, and the coherence isomorphisms assigned to crossings become braidings.

Open Questions

- It is unknown to us, if our trisection invariants of 4-manifolds can be extended to a 4d TQFT. Trisection diagrams can be defined for four-manifolds with boundaries [3], but their gluing behaviour is delicate. It is also not clear to us, if their data fits into the framework of fusion 2-categories.
- Although our invariants are at least as strong as the invariants of Chaidez, Cotler and Cui [2] for a Hopf triplet and the invariant of Bärenz and Barrett [1] for a pivotal functor from a spherical fusion category into a modular fusion category, we do not have a clear idea how strong they are in general. In particular, do they detect smooth structures? It was suggested by Peter Teichner and Christopher Schommer-Pries to investigate this by studying trisections of $S^2 \times S^2$ for appropriate algebraic data.
- Our algebraic data, spherical fusion categories and finite semisimple bimodule categories with bimodule traces, is semisimple. This raises the question if more sophisticated techniques such as modified traces or diagrammatic calculi for the non-semisimple setting can be used to generalise these invariants. One would need a suitable replacement for the Kirby colour, i. e. for the sums over simple objects, and control over the evaluation of the trisection diagram that governs the (de)stabilisation move.

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Higher Grothendieck rings and braided lax endofunctors

Dmitri Nikshych

For a non-degenerate braided fusion category $\mathcal{B}$, we establish a monoidal equivalence between the 1-categorical truncation of the fusion 2-category $\mathbf{Mod}(\mathcal{B})$ of $\mathcal{B}$-module categories and the category of certain braided lax endofunctors of $\mathcal{B}$. This result generalizes the well-known equivalence [2] between the categorical Picard group of $\mathcal{B}$ and the group of braided autoequivalences of $\mathcal{B}$. In addition, it yields a categorical interpretation of the theory of modular invariants arising from $\mathcal{B}$-module categories [3]. As is well known, indecomposable $\mathcal{B}$-module categories are in bijection with Lagrangian algebras in the center of $\mathcal{B}$. We explain that under this bijection the tensor product of module categories corresponds to a certain convolution of Lagrangian algebras. The latter can be understood as a categorical analogue of the composition of orthogonal Lagrangian correspondences, providing a conceptual bridge between tensor-categorical and classical geometric constructions.

We use these ideas to study the the higher Grothendieck ring $K(\mathcal{B}) := K_0(\mathbf{Mod}(\mathcal{B}))$ formed by equivalences classes of $\mathcal{B}$-module categories. It is somewhat surprising that the complexified algebra $K(\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{C}$ of this ring is, in general, non-semisimple (in contrast with the situation of the usual Grothendieck ring of a fusion 1-category). In fact, $K(\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{C}$ typically has many two-sided annihilating elements. It would be interesting to describe its irreducible representations and the Jacobson radical in terms of the category $\mathcal{B}$ and to relate these structures to intrinsic categorical invariants of $\mathcal{B}$.

We introduce a natural grading of $K(\mathcal{B})$ and the corresponding algebra by the canonical partially ordered set of localizations of $\mathcal{B}$. This rise to a corresponding filtration by ideals of $K(\mathcal{B})$. We study the *intervals* associated with this grading, i.e., the quotients I'/I coming from consecutive ideals I' and I. In the simplest example, when $\mathcal{B}$ is the center of the category of vector spaces graded by an elementary Abelian p-group, these ideals admit an elegant description in terms of the combinatorics of polar graphs [1] constructed from orthogonal Grassmannians (this result is obtained in a joint work with Riley Chinburg).

Finally, we discuss the convolution of étale algebras (not just Lagrangian ones) in the center of $\mathcal{B}$. It corresponds to the tensor product of *representations* of $\mathcal{B}$, i.e, dominant tensor functors from $\mathcal{B}$ to multifusion categories. This appears to be a new higher monoidal structure naturally associated with a braided fusion category, extending that of the fusion 2-category $\mathbf{Mod}(\mathcal{B})$.

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Topological Order: The Operator-Algebraic Approach

Yoshiko Ogata

A state on a quantum spin system is a positive, normalized linear functional. A state is said to be pure if it is an extreme point of the set of all states. Otherwise, it is mixed. By Powers, it is known that the set of all pure states is homogeneous in the sense that any pure state can be mapped to another by an automorphism. However, the point of view of quantum many-body physics suggests another classification: classification of gapped ground states via quasi-local automorphisms. Under the condition of Haag duality, DHR theory allows us to derive a braided tensor category, from representations satisfying the superselection criterion. This braided tensor category turns out to be an invariant of the classification[1]. This braided tensor category is physically considered as anyon, as it coincides with what is called anyons in known exactly solvable models. These anyons, if the braiding is non-trivial, can work as an index in the classification of mixed state topological order, where we ask if there is a finite-depth quantum channel to map one state to the other[2].

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Classifying 2D topological phases on the lattice up to invertible phases

Daniel Ranard

(joint work with Isaac Kim, Alexei Kitaev and Milo Moses)

Two gapped Hamiltonians on a quantum lattice system are said to be in the same phase if they can be continuously connected without closing the gap. Equivalently, two gapped ground states are said to be in the same phase if they are related by some quasi-local evolution [1]. It is conjectured that in two spatial dimensions, for some suitable choice of definitions, gapped phases are classified by pairs $(\mathcal{C}, c_-)$ where $\mathcal{C}$ is a unitary modular tensor category (UMTC) describing the anyons and $c_- \in \mathbb{Q}$ is the chiral central charge [2]. This belief is related to the classification of three-dimensional topological field theories.

Not all gapped ground states have purely topological properties: without translation invariance, states may have domain walls and other inhomogeneities; even with translation invariance, there exist fracton models in higher dimensions. Thus to classify topological phases on the lattice, we must define some suitable subclass of "topological" states among all gapped ground states. To start with a more tractable problem, we may consider states with zero correlation beyond some finite range, classified modulo finite-depth circuits with strict locality. Even then, one must impose conditions to guarantee a homogeneous topological phase.

The entanglement bootstrap program [3] initiated such an approach, considering states satisfying certain conditions on their entanglement entropy. However, these conditions are not invariant under constant-depth circuits and can fail even for states in the trivial phase. In [4], we introduce a weaker condition—*local extendibility*—that is circuit-invariant. This condition may also be formulated to hold with a small error; we expect the approximate condition to allow all familiar topological phases, providing a candidate definition for a "topological" state. Below we discuss only the strict version with zero error; the approximate setting is the subject of ongoing work.

We consider lattices of spins in $\mathbb{R}^d$. Our starting point is *local data*: mutually consistent density matrices $\{\sigma_K\}$ on balls K of constant radius, with no global state assumed. For disjoint regions B, C, D, E contained in such a ball, we say the local data is (B, C, D, E)-*extendible* if there exists a quantum channel $R_{BE\to BC}$ such that

$$(\mathrm{id}_A \otimes R_{BE\to BC})[\bar{\sigma}_{ABE}] = \bar{\sigma}_{ABC}$$

for any purification $\bar{\sigma}_{ABCDE}$ of σ_{BCDE} using ancilla A. The channel extends the state onto C while discarding E. We require the above condition for a certain topology of configurations B, C, D, E, for a finite number of configurations in each patch of local data.

Under these assumptions, local data determines a unique global state. Local extendibility also reproduces itself at larger scales, and it is preserved (with a shifted scale) by constant-depth circuits. The main structural result of [4] is an isomorphism theorem: for sufficiently smooth embedded or immersed regions Y, the information convex set $\Sigma(Y)$ (the set of states on Y consistent with the local data [3]) depends only on the topology of Y, up to canonical isomorphism.

We refer to states satisfying local extendibility along with two closely related conditions as *topological states*. Such states have gapped parent Hamiltonians. In two spatial dimensions, we can assign a UMTC $\mathcal{C}$ to a topological state; the category $\mathcal{C}$ only depends on local data and is invariant under constant-depth circuits. To discuss classification results, it is convenient (but not necessary) to work in infinite volume. Then we show [5]:

Two 2D topological states with the same UMTC are related by a finite-depth circuit of local quantum channels. Equivalently, they are related by a finite-depth unitary circuit and stacking with an invertible state.

We do not claim that every UMTC is realized by a topological state satisfying our exact axioms, only that the UMTC uniquely determines the phase up to stacking with invertible states. The Levin-Wen string-net states furnish examples for all Drinfeld centers. It is plausible that these are the only UMTCs compatible with the exact axioms, since chiral phases may require nonzero correlation length. This possible restriction disappears in the approximate setting, where we expect a generalization of the above theorem to still hold.

We have discussed classifying modulo invertible phases; one outstanding problem is the classification of 2D invertible phases themselves. An index for such

phases has been constructed by Sopenko [6], but the completeness of this invariant remains to be understood.

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Dualizability towards SymTFTs

Claudia Scheimbauer

(joint work with Pelle Steffens and Will Stewart)

For understanding symmetries of (topological) field theories in the functorial field theory setting, recently the quiche or sandwich picture was proposed in [2] following long-standing previous ideas on defects and boundaries as describing symmetries. A mathematically rigorous way to implement this picture is to use twisted *aka* relative field theories following Stolz–Teichner [13] and Freed–Teleman [3], for which a mathematical framework was provided in [6].

A popular target of fully extended topological field theories is the higher Morita category of E_n-algebras as constructed in [10, 5, 6]. Objects therein are E_n-algebras, 1-morphisms are E_{n-1}-algebras which are bimodules, 2-morphisms are bimodules of bimodules etc. The different models implement this in different ways: Haugseng's model is operadic and combinatorial, whereas the one in my thesis uses a geometric variant using factorization algebras.

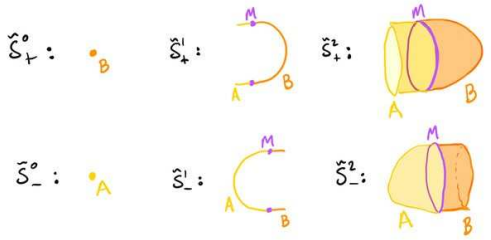
In this talk I explained ingredients to find symmetry data for the quiche/sandwich TFT picture in these higher Morita categories. The main result presented is a proof of a conjecture by Lurie [9, Remark 4.1.27] (see also [1, Conjecture 1.8]) characterizing $(n+1)$-dualizability in the higher Morita categories $\mathrm{Alg}_n(\mathcal{C})$. I also explained a relative version thereof, which, assuming the Cobordism Hypothesis reads as follows:

Theorem. Let $M : A \to B$ be a 1-morphism in $\mathrm{Alg}_n(\mathcal{C})$. Then M determines an n-dimensional fully extended framed oplax twisted topological field theory if and only if the object M is right dualizable as a right module over the factorization homologies

$$\int_{\widetilde{S}^{1w}_{+}} M_{A\to B}$$

where w ranges over binary words of length $n-1$.

Here $\mathcal{C}$ is a symmetric monoidal (∞, N)-category satisfying certain technical assumptions and $\widetilde{S}_+^{1w}$ is an n-sphere with stratification depending on the word w. For instance,



The proofs of these theorems build upon previous work by Gwilliam and myself on n-dualizability [4] in the factorization model for the higher Morita category. The crucial new ingredient to lift these results is Karlsson's thesis results [7, 8] on a higher Morita category defined using *pointless* factorization algebras, which is essential to obtain non-trivial $(n+1)$-dualizable objects. The final steps are a lifting lemma for adjoints in this factorization setting and a geometric argument.

These results have several applications. First, an immediate corollary is the proof on an invertibility conjecture by Brochier–Jordan–Safronov–Snyder [1, Conjecture 1.9]. We obtain examples of Dirichlet and Neumann boundary conditions. Finally, we construct the SymTFT attached to $\operatorname{Rep} G$ for any compact Lie group G in any dimension. For this last example, a crucial ingredient is the correct functional analytic setting, namely using Steffens' aggregated vector spaces [12].

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Quantum Cellular Automata from TQFT

Nathanan Tantivasadakarn
(joint work with Meng Sun, Bowen Yang, Zongyuan Wang and Yu-An Chen)

By now, we have a pretty good mathematical understanding of symmetries in quantum field theory in terms of category theory. However, symmetries in discrete lattice models are much less understood. To make matters worse, the mapping between symmetries on the lattice into symmetries in the continuum is neither injective nor surjective. For example, on the lattice, a symmetry which has no continuum counterpart is lattice translation.

In order to classify symmetries on the lattice, we need to first give a definition of what we want to classify. Here, we restrict ourselves to invertible symmetries acting on a tensor product of finite-dimensional Hilbert spaces embedded in some space of a fixed dimension so that there is a notion of geometric locality. We define a *Quantum Cellular Automata* (QCA) as a bounded spread automorphism on this Hilbert space. QCAs naturally form a group. In simple terms, QCAs are locality-preserving unitaries. An example of QCAs are given by local unitary circuits of constant depth, which forms a normal subgroup of the group of QCAs.

One can ask what is the quotient, meaning what are QCAs that cannot be implemented by constant depth circuits. In one spatial dimension, [2] showed that this quotient group is exactly generated by lattice translations of various dimensional Hilbert spaces to the left or the right, which can be classified by a rational index. Indeed, lattice translation can at best be implemented by a circuit of linear depth, by a sequential application of swaps.

A natural question is whether there are anything else interesting in the quotient other than lattice translation in higher dimensions. [3] constructed the first example of a QCA in 3 spatial dimensions. In particular, they made a surprising connection to TQFTs by showing that the QCA they constructed prepares the ground state of the so-called the Walker-Wang (WW) model [4] which is a commuting projector Hamiltonian constructed from an input braided fusion category.

To give a pedagogical example, we pick the category to be a pointed modular tensor category, whose data is specified by an Abelian group A and a non-degenerate quadratic form $q : A \to \mathbb{Q}/\mathbb{Z}$. The particular modular tensor category we will discuss is the topological lines of $SU(3)_1$ Chern-Simons theory, where the simple objects are labeled $\{1, a, \bar{a}\}$ with the obvious $\mathbb{Z}_3$ fusion rules and $q(a) = a^2/3$.

The TQFT describing Abelian Walker-Wang models can be described by a 2-form gauge theory, whose action heuristically corresponds to the quadratic form

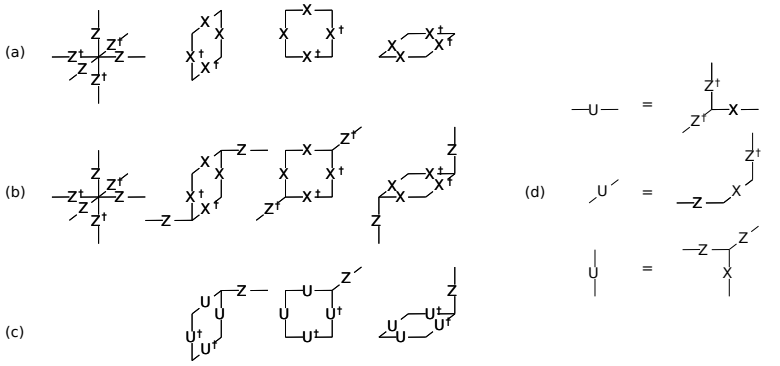

FIGURE 1. (a): Commuting Pauli operators corresponding to (a) the $\mathbb{Z}_3$ gauge theory ($\mathrm{Vec}_{\mathbb{Z}_3}$ WW) model, and (b) the $SU(3)_1$ WW. The generating set in (b) is equivalent to those in (c). The QCA sends Z on each edge to Z attached with a closed loop of U, the hopping operator for a $\mathbb{Z}_3$ "anyon" in 3+1D, defined in (d).

with 2-form gauge fields. More precisely, in this example the action is

$$S = \frac{2\pi i}{3} \int_{M_4} B_2 \cup B_2 \tag{1}$$

where B_2 is a dynamical 2-form gauge field (more precisely, a 2-cocycle) and M_4 is a 4-manifold. To turn this into a wavefunction, we choose M_4 to be the cone of some 3-manifold M_3. Since B is locally flat, we may replace $B \to \delta b$. Applying Stoke's theorem, we obtain the following expression for the wavefunction purely on M_3

$$|\psi\rangle = \sum_{B=\delta b} e^{\frac{2\pi i}{3} \int_{M_3} b \cup \delta b} |B\rangle \tag{2}$$

The fact that the phase factor in the exponent cannot be decomposed locally in terms of B signals that we should not be able to prepare this wavefunction using a constant depth circuit, the property of a non-trivial QCA.

One way to visualize this wavefunction on the lattice is to specify it as a simultaneous $+1$ eigenstate of certain operators, which are products of generalized Pauli operators X and Z satisfying $ZX = e^{2\pi i/3} XZ$. The form of these commuting Pauli operators are shown in Fig. 1(b). An equivalent set of stabilizers is shown in 1(c), where a Pauli-Z is attached with a closed loop of an operator U. The action of the QCA α is defined as attaching such a closed loop to each Pauli-Z. Similarly, the action on Pauli-X turns it into U decorated by an appropriate product of $\alpha(Z)$ such that they commute.

In [1], we construct a QCA corresponding to the $SU(p)_1$ Walker-Wang model for every odd prime p. We moreover show that up to constant depth circuits and translations, the QCA has order 4 for $p = 3 \mod 4$ and order 2 for $p = 1 \mod 4$.

The order of these QCA suggests that there is a close relation between QCA and the Witt group of non-degenerate quadratic forms. Indeed, this correspondence has been proven rigorously in [5] in the special case of QCAs that map Pauli operators to Pauli operators, called *Clifford* QCA. Moreover, Clifford QCA have been classified in [6, 7] in arbitrary dimensions, where the odd-prime factors of the Witt group appears in every $4n-1$ spatial dimensions. We show that the invertible TQFT which is prepared by the QCA is given by the action

$$S = \frac{2\pi i}{p} \int_{M_{4n}} B_{2n} \cup B_{2n} \tag{3}$$

where B_{2n} is now a $2n$-cocycle, and use this to provide an explicit construction of these QCAs in [1] following an analogous procedure as above.

A remaining open question is to show that these QCAs cannot be realized as constant depth circuits. More generally, it is not known whether they are robust to arbitrary evolution of a local Hamiltonian. It would also be interesting to explicitly construct QCAs for all elements of the Witt group in higher dimensions. One example for $p = 2$ corresponds to $SU(2)_1$, which is an order 8 element in the Witt group. This QCA cannot be realized as a Clifford QCA. It has been constructed in 3 dimensions in [8], but its order has not been explicitly shown, and the corresponding QCAs in dimensions $4n-1$ remain to be constructed. More generally, it remains open whether there are QCAs corresponding to the Witt group of non-degenerated braided fusion categories [9], to which what was covered here is only the pointed part.

Returning to the topic of generalized symmetries. Just as lattice translation was an example in one dimension of a symmetry that can't be gauged but nevertheless admits a trivial product state, non-trivial QCA in higher dimensions also cannot be gauged (i.e. they do not admit decomposition into a local action). In the continuum, the obstruction to gauging of a finite group G in d spacetime dimensions is typically associated to the group cohomology class $H^d(G, U(1))$. However, these QCAs seem to be beyond these examples [10]. It would be interesting to see whether these QCAs admit a symmetric gapped ground states as symmetries, and whether they can intertwine with non-invertible symmetries in interesting ways on the lattice.

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Sheaves of modular functors from CFT and quantum Teichmüller theory

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(joint work with Andrea Ferrari and Simon Lentner)

Important structures shared by conformal field theories (CFTs) and topological quantum field theories (TQFTs) can be encoded in the notion of a modular functor. We shall here use a variant of this notion, assigning pairs $(\pi^S(\Sigma), \pi^M(\Sigma))$ to two-dimensional topological surfaces Σ, where $\pi^S(\Sigma)$ is a representation of the skein algebra $\mathsf{Skein}_q(\Sigma)$, and $\pi^M(\Sigma)$ is a projective representation of the mapping class group $\mathrm{MCG}(\Sigma)$ of Σ, both behaving naturally under cutting of Σ into pairs of pants. The structures characterising modular functors can be encoded in projective representations of the modular groupoid, assigning representations $\pi_\sigma = (\pi^S_\sigma, \pi^M_\sigma)$ of skein algebras *and* mapping class groups to pants decompositions σ decorated with trivalent graphs, and intertwining maps $E_{\sigma_2\sigma_1} : \pi_{\sigma_1} \to \pi_{\sigma_2}$ to pairs (σ_2, σ_1). Such an assignment defines a projective representation of the modular groupoid if for all triples $(\sigma_3, \sigma_2, \sigma_1)$ there exist complex numbers $\chi_{\sigma_3\sigma_2\sigma_1}$ with $|\chi_{\sigma_3,\sigma_2,\sigma_1}| = 1$ such that $E_{\sigma_3\sigma_2}E_{\sigma_2\sigma_1} = \chi_{\sigma_3\sigma_2\sigma_1}E_{\sigma_3\sigma_1}$. A description of the modular groupoid in terms of generators (certain pairs (σ_2, σ_1)) and relations (composites of generators) is known thanks to the work of Moore-Seiberg, Bakalov-Kirillov, and others.

In certain cases one can infer isomorphisms of modular functors from CFT and TQFT from results of Kazhdan-Lusztig type, establishing equivalences of braided tensor categories from vertex operator algebra (VOA), and quantum group representation theory, respectively. Recent progress has established the first results of Kazhdan-Lusztig type for logarithmic VOAs, having *non-semisimple* braided tensor categories of representations [CLR, GN]. Paradigmatic examples are the representation categories of the triplet VOAs $\mathcal{W}_k$, now known to be equivalent to the categories of representations of the small quantum $u_q(\mathfrak{sl}_2)$, defined for parameters q which are roots of unity $q^{2k} = 1$. These results may offer a basis for a future comparison of modular functors from logarithmic VOAs, and TQFTs.

In the non-semisimple cases one may expect two types of new features. One may, on the one hand, expect to find derived (differential graded) enhancements of the modular functors characterising CFTs and TQFTs. Our main interest here

will be another, ultimately related, feature of logarithmic VOAs like the triplet VOAs, which can originate from the existence of a continuous group G of outer automorphisms: The pairs $(\pi^S(\Sigma), \pi^M(\Sigma))$ can be twisted by local systems on Σ, representations $\rho : \pi_1(\Sigma) \to G$, which may be expected to lead to the definition of sheaves of modular functors over the character variety $\mathrm{Ch}_G(\Sigma)$.

Twisted conformal blocks of logarithmic VOAs

VOAs are certain infinite-dimensional algebras $\mathcal{V}$ having generators $V_{i,n}$ organised into generating series of the form $V_i(z) = \sum_{n\in\mathbb{Z}} V_{i,n} z^{-n-1}$. Conformal blocks are functionals $F(V_{i_1}(z_1) \dots V_{i_n}(z_n))$, depending on a Riemann surface C, a puncture $P \in C$, and a choice of coordinate on a neighbourhood D_P around P, which define multi-differentials on C that are analytic away from the diagonals $z_r = z_s$, having Laurent series expansions around the diagonals determined by the relations in $\mathcal{V}$. For certain classes of CFTs it is known that the spaces of conformal blocks are modules both for the skein algebras, and for the mapping class groups, defining modular functors.

For VOAs admitting continuous automorphisms $V_i \mapsto \sum_j g_{ij} V_j$, $g \in G$, and given local systems $\rho : \pi_1(\Sigma) \to G$, one may define twisted conformal blocks by weakening the analyticity condition to existence of a multi-valued analytic continuation away from the diagonals, with monodromies represented by ρ.

The main results of the joint work with A. Ferrari and S. Lentner presented at this workshop can be briefly summarised as follows. In the case of the triplet VOA one may consider the following construction. Let

$$F^\sigma_{\lambda,\kappa} := \sum_{n\in\mathbb{Z}^{3g-3+n}} e^{2\pi i\langle n,\kappa\rangle} f^\sigma_{\lambda-n}, \qquad \lambda,\kappa \in \mathbb{C}^{3g-3+n}, \tag{1}$$

where f^σ_λ are conformal blocks of the Virasoro VOA with central charge $c = 13 - 6(k - k^{-1})$ associated to a pants decomposition σ of Σ, with λ being labels for the representations assigned to the cutting curves of σ. We can show that the conformal blocks $F_{\lambda,\kappa}$ are conformal blocks of the triplet VOAs, twisted by a family $\rho_{\lambda,\kappa} : \pi_1(\Sigma) \to \mathrm{PSL}(2)$ of local systems, parameterised by the variables λ, κ appearing in (1). The spaces of conformal blocks constructed in this way carry a representation $\pi^S_\sigma(\rho_{\lambda,\kappa})$ of $\mathsf{Skein}_q(\Sigma)$, generated by the so-called Verlinde line operators. This representation is k^{3g-3+n}-dimensional for the triplet VOA $\mathcal{W}_k$ if $\Sigma = \Sigma_{g,n}$ has genus g, and n punctures. In ongoing work we hope to prove the conjecture that the conformal blocks $F^\sigma_{\lambda,\kappa}$ form a basis for the space of all conformal blocks of the triplet VOA $\mathcal{W}_k$ associated to $\Sigma_{g,n}$, and twisted by $\rho_{\lambda,\kappa}$.

Comparison with quantum Teichmüller theory and TQFT

There are at least two other sources of skein algebra representations, provided by TQFT and the quantisation of Teichmüller spaces $\mathcal{T}_{g,n}$, respectively. Quantisations of $\mathcal{T}_{g,n}$ have been constructed by Checkhov, Fock, and by Kashaev using convenient coordinates associated to triangulations τ of Σ. The quantum trace map of Bonahan and Wong then allows one to assign representations π^S_τ of $\mathsf{Skein}_q(\Sigma)$ to

any triangulation τ. There exist isomorphisms $F_{\tau_2\tau_1} : \pi^S_{\tau_1} \to \pi^S_{\tau_2}$ for any τ_1, τ_2, allowing us to identify representations π^S_τ associated to different triangulation τ.

It is known that the skein algebra $\mathsf{Skein}_q(\Sigma)$ has a large center for $q^{2k} = 1$, isomorphic to the coordinate ring of character variety $\mathrm{Ch}_G(\Sigma)$ parameterising conjugacy classes $[\rho]$ of representations $\rho : \pi_1(\Sigma) \to G$. For conjugacy classes $[\rho]$ representing points in an open dense subset $\mathcal{AL}$ of $\mathrm{Ch}_G(\Sigma)$, there exist essentially unique k^{3g-3+n}-dimensional irreps of $\mathsf{Skein}_q(\Sigma)$ by the unicity conjecture [BW19] proven in [FKL, GJS, KK]. The subset $\mathcal{AL} \subset \mathrm{Ch}_G(\Sigma)$ is called Azumaya locus.

There exist realisations of the algebraic structures characterising quantum Teichmüller theory on finite-dimensional vectors spaces, yielding k^{3g-3+n}-dimensional representations of the skein algebra by means of the quantum trace map. The unicity theorem implies the isomorphism of the representations of $\mathsf{Skein}_q(\Sigma)$ associated to $[\rho] \in \mathcal{AL}$ from TQFT, quantum Teichmüller theory, and CFT.

The known examples for relations between CFTs and TQFTs suggest that these relations can be strengthened to isomorphisms of modular functors. For the infinite-dimensional version of the quantum Teichmüller theory, it has been argued in [T07, TV] that this theory defines an infinite-dimensional analog of a modular functor. For each decorated pants decomposition σ one may construct triangulations τ_σ, representations π^M_σ of the mapping class group, and isomorphisms $T_\sigma : \pi^M_{\tau_\sigma} \to \pi^M_\sigma$, defining a projective representation of the modular groupoid by means of $E_{\sigma_2\sigma_1} := T_{\sigma_2} F_{\tau_{\sigma_2}\tau_{\sigma_1}} (T_{\sigma_1})^{-1}$.

In order to adapt these results to the variants of quantum Teichmüller theory assigning finite-dimensional skein algebra representations to surfaces Σ, one may start from the framework provided by [Is], defining representations $\pi_\tau(\rho) = (\pi^S_\tau(\rho), \pi^M_\tau(\rho))$ for any triangulation τ of Σ, and any $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2)$ corresponding to points from an open subset of $\mathrm{Ch}_G(\Sigma)$. Existence of representations $\pi_\sigma(\rho)$ defining operators $E_{\sigma_2\sigma_1}(\rho)$ generating a projective representation of the modular groupoid will now follow along the lines of [T07, TV] from the spectral theorem of linear algebra. The representations $\pi_\sigma(\rho)$ will be defined on open subsets U_σ of $\mathrm{Ch}_G(\Sigma)$. The complex numbers $\chi_{\sigma_3\sigma_2\sigma_1}$ appearing in $E_{\sigma_3\sigma_2} E_{\sigma_2\sigma_1} = \chi_{\sigma_3\sigma_2\sigma_1} E_{\sigma_3\sigma_1}$, define a class $[\chi] \in H^2(\mathcal{AL}, U(1))$. Following the paradigm provided by [TV], one can calculate $\chi_{\sigma_3\sigma_2\sigma_1}$ by using the known description of the modular groupoid in terms of generators and relations. As the compactified moduli spaces $\overline{\mathcal{M}}(\Sigma)$ of complex structures on Σ embed into $\mathrm{Ch}_{\mathrm{PSL}(2)}(\Sigma)$, one should be able to compare $[\chi]$ to the class of the projective line bundle $(\lambda_{\mathrm{H}})^{\frac{c}{2}}$ within $H^2\big(\overline{\mathcal{M}}(\Sigma), U(1)\big)$, with λ_{H} being the Hodge line bundle, and find a match.

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Transversal Clifford-Hierarchy Gates via Non-Abelian Surface Codes

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(joint work with Sakura Schafer-Nameki)

Introduction. Fault-tolerant realization of non-Clifford gates is a key bottleneck for scalable, universal quantum computation. Before presenting our work [1], we recall the definition of the Clifford hierarchy of unitary operators [2, 3, 4]. The first level, $\mathcal{C}^{(1)} = \mathcal{P}$, is defined as the Pauli group generated by

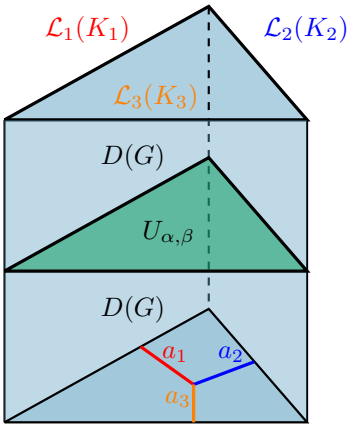
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1}$$

on each qubit. The second level, $\mathcal{C}^{(2)}$, consists of the Clifford gates, defined as those unitaries U that conjugate Paulis to Paulis, i.e. $UPU^\dagger \in \mathcal{P}$ for all $P \in \mathcal{P}$, such as $S = \text{diag}(1, i)$. Circuits composed only of Clifford gates can be efficiently simulated classically [2]. Higher levels of the Clifford hierarchy, $\mathcal{C}^{(k)}$ for $k \geq 3$ contain the non-Clifford gates, defined recursively by $\mathcal{C}^{(k)} = \{U : UPU^\dagger \in \mathcal{C}^{(k-1)} \ \forall P \in \mathcal{P}\}$, an example (here $N = 2^{k-3}$) is $T^{1/N} = \text{diag}(1, e^{i\pi/(4N)})$.

In two spatial dimensions (2D), Pauli stabilizer codes with geometrically local checks are constrained via the Bravyi-Konig (BK) theorem: any locality-preserving logical unitary on a D-spatial dimensional topological Pauli stabilizer code is contained in the D-th level of the Clifford hierarchy [5]. In particular, for $D = 2$ such logical gates are therefore at most Clifford, precluding transversal or constant-depth implementations of non-Clifford gates such as the $T = \text{diag}(1, e^{i\pi/4})$ gate.

Main Results. We will now state our main results [1]. First, we show how group cocycles can give rise to transversal diagonal gates in group surface codes (Theorem 4). We then apply it to the dihedral groups D_{4N} and construct transversally in 2D the $T^{1/N}$ gates (Corollary 1). Finally, we demonstrate that for $8N = 2^n$ these can be realized purely in terms of qubits (Theorem 5).

Theorem 4. Tranversal Gates from SPT-Stacking. Consider the quantum double $D(G)$ of a finite group G on a spatial triangle [6, 7]. Let the three gapped boundaries be labelled by subgroups $K_1, K_2, K_3 \subseteq G$, chosen to encode a single



logical qubit. Let $\alpha \in H^2(G, U(1))$ be a group 2-cocycle whose restriction to each boundary is trivial in group cohomology, i.e. there exist $\beta^{(i)} : K_i \to U(1)$ such that

$$\alpha|_{K_i} = \delta\beta^{(i)} \qquad \text{for } i \in \{1,2,3\}\,. \tag{2}$$

Define the tranversal unitary $U_{\alpha,\beta}$ by stacking the 2D purely spatial SPT circuit for α on a single time-slice, with the boundary counterterm $\beta^{(i)}$ on each 1D boundary.

Then the induced logical gate $U_{\alpha,\beta}$ is diagonal in the $\overline{Z}$ basis $\{\overline{|0\rangle}, \overline{|1\rangle}\}$, with

$$\begin{aligned} U_{\alpha,\beta}\overline{|0\rangle} &= \overline{|0\rangle}\,, \\ U_{\alpha,\beta}\overline{|1\rangle} &= \frac{\alpha(g_1,g_2)\beta^{(3)}(g_1g_2)}{\beta^{(1)}(g_1)\,\beta^{(2)}(g_2)}\,\overline{|1\rangle}\,, \end{aligned} \tag{3}$$

for any state labeled by $g_1 \in K_1$, $g_2 \in K_2$ representing $\overline{|1\rangle}$.

$U_{\alpha,\beta}$ preserves the logical codespace (since it is constructed from an automorphism of $D(G)$). Furthermore, its lattice realization involves only operators acting on $O(1)$ lattice sites: a local error can thus enlarge its support only by $O(1)$ sites.

Applied to the dihedral groups this theorem allows us to realize non-Clifford gates at any level of the Clifford hierarchy and beyond:

Corollary 1. ***Transversal Non-Clifford Gates from $D(D_{4N})$.*** *For any integer $N \geq 1$, consider the order-$8N$ dihedral group (of symmetries of a $4N$-gon)*

$$G = D_{4N} = \langle r, s \mid r^{4N} = s^2 = id,\ srs = r^{-1} \rangle\,. \tag{4}$$

We consider the setup of Theorem 4 for this group G and choose the three gapped boundaries to be determined by the subgroups

$$K_1 = \langle rs \rangle \cong \mathbb{Z}_2\,, \quad K_2 = \langle s \rangle \cong \mathbb{Z}_2\,, \quad K_3 = \langle r \rangle \cong \mathbb{Z}_{4N}\,. \tag{5}$$

Consider the non-trivial cocycle

$$\alpha_N \in H^2(D_{4N}, U(1)) = \mathbb{Z}_2\,, \tag{6}$$

whose derivation and expression can be found in App. A of the paper.

- *For $K = \langle s \rangle$ or $K = \langle rs \rangle$, $\alpha|_K = 1$ so β on these subgroups can also be taken to be identically 1. Note also that $\alpha_N(id, id) = \alpha_N(rs, s) = 1$.*
- *For $K = \langle r \rangle$, a solution to the full set of equations $\alpha_N|_{\langle r \rangle} = \delta\beta_N$ has $\beta_N(r) = e^{i\pi/(4N)}$.*

Then, on the state labeled by $g_1 = rs$, $g_2 = s$, $g_1 g_2 = r$, Eq. (3) evaluates to the unitary

$$U_{\alpha,\beta} = T^{1/N} = P\left(\frac{\pi}{4N}\right) = \mathrm{diag}\left(1, e^{i\pi/(4N)}\right) . \tag{7}$$

Theorem 5. Qubit-only realization for $8N = 2^n$. When $8N = 2^n$, the quantum double $D(D_{4N}) = D(D_{2^{n-1}})$ can be implemented on a qubit-only physical Hilbert space in 2D, with n qubits on each lattice edge. The unitary $U_{\alpha,\beta}$ in (7) becomes a geometrically local, topologically protected qubit circuit implementing the logical $T^{2^{3-n}} = P(\pi/2^{n-1})$ gate, in the n-th level of the Clifford hierarchy. In particular, for $n \geq 3$ it is non-Clifford.

Concretely, we use the supersolvable property of dihedral groups to map the local Hilbert space

$$\mathcal{H} = \{|g\rangle \ , \ g \in D_{2^{n-1}}\} \tag{8}$$

to one spanned by n qubits on a Hilbert space

$$\mathcal{H} = (\mathbb{C}^2)^n , \tag{9}$$

in which each factor is associated to a supersolvable generator. The first $n-1$ qubits are those for the normal $\mathbb{Z}_{2^{n-1}}$ subgroup and correspond to $r^{2^{n-2}}, r^{2^{n-3}}, \cdots, r$ while the last qubit is for the non-normal $\mathbb{Z}_2$ generated by s. The map is therefore:

$$\left|r^a s^j\right\rangle \mapsto |\mathrm{bin}(a)\rangle\,|j\rangle \ , \tag{10}$$

where $\mathrm{bin}(a)$ denotes the binary representation of $a \in \{0, 1, 2, .., 2^{n-1} - 1\}$ and $j \in \{0, 1\}$. The D_{4N} operators can similarly be mapped to qubit operators, forming a non-abelian Clifford-hierarchy stabilizer group [1]. Interfaces to $\mathbb{Z}_2$ codes can be realized using condensable algebras [8, 9, 10, 11].

Conclusions and future directions. Our work provides an alternative to the Bravyi-Konig theorem, by considering, instead of higher dimensions, purely 2D topological codes, but increasing complexity by allowing non-Abelian groups and Clifford-hierarchy gates on the physical qubits. The realization in terms of qubits and purely in 2D should nevertheless give this approach an edge in terms of implementation. The next most important aspect to explore is the Quantum-Error-Correction model, investigating the feasibility of the proposed just-in-time decoder geared towards non-Abelian surface codes in [12]. On a theoretical level it would be interesting to apply this systematic approach to explore gates that arise from automorphisms of $D^\omega(G)$ for any finite (non-Abelian) group G with non-trivial 3-cocycle $\omega \in H^3(G, U(1))$ [13] or codes based on Hopf algebras.

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Spaces of conformal blocks beyond semisimplicity, rigidity and finiteness

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(joint work with Lukas Müller)

The monodromy data of a two-dimensional conformal field theory can be described by means of the *spaces of conformal blocks*, certain (projectively) flat vector bundles over a suitable moduli space of Riemann surfaces that can equivalently be seen as (projective) representations of mapping class groups of surfaces, see e.g. [2, 6, 10] for an overview and [4] for a description in the language of vertex operator algebras. In good cases, the spaces of conformal blocks (when correctly defined) can be organized into a *modular functor*, a notion developed in [19, 13, 2] for this purpose. Roughly, a modular functor is a consistent system of mapping class group representations that in particular satisfies gluing properties.

In joint work in progress with L. Müller [17], we give, building on preparations in [15, 16], the minimal requirements for modules over a vertex operator algebra that ensure the coherent definition of spaces of conformal blocks on disks with marked intervals. The structure that needs to be specified is for disks with two and three marked intervals and is subject to a short list of conditions. Under

these assumptions, we provide an extension to a modular functor, for a notion of modular functor that includes singularities. The construction can be applied beyond semisimplicity, rigidity and finiteness. In the infinite, but rigid case, an investigation of the open sector has been carried out in [22].

In the finite rigid case, a framework described in [7, 8], the construction recovers the open part of the open-closed modular functor built from a possibly non-semisimple modular category $\mathcal{A}$ that associates $\mathcal{A}$ to the interval and its Drinfeld center $Z(\mathcal{A})$ to the circle [12, 14]. Given a symmetric Frobenius algebra $F \in \mathcal{A}$, one obtains correlators defined over disks with marked intervals that can be extended to the entire open sector via the *modular microcosm principle* [20]. If F is special, one obtains this way a consistent system of open-closed correlators for the Lyubashenko modular functor [12] associated to the modular category $Z(\mathcal{A})$ [21]. This generalizes the construction of Fuchs-Runkel-Schweigert, partly with Fjelstad [5, 3], to the non-semisimple setting.

The construction [5, 3] in the semisimple case uses the Reshetikhin-Turaev topological field theory [18] to construct the correlators associated to a special symmetric Frobenius algebra; in other words, the correlators have a three-dimensional origin, which has been interpreted as a holographic principle [11]. In [21] such a correspondence is also present, but it is set up by means of factorization homology [1] instead. This holographic principle in terms of factorization homology can be used to give a definition of the torus partition function. From this partition function one may extract, by evaluation on projective objects, certain coefficients generalizing the coefficients of the partition functions from [5] (even though we have to insist that in the non-semisimple setting these coefficients are not matrix elements with respect to a basis!). Remarkably, also in the non-semisimple case [21], these coefficients remain non-negative integers. For the Cardy case, one recovers the Cartan matrix that was already proposed as torus partition function in [9].

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Discussion on the physical applications of higher dagger categories

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This extended abstract reports on an informal discussion session on the recently developed theory of higher dagger categories. The discussion involved short talks by Thomas Bartsch and Lukas Müller and was moderated by Luuk Stehouwer.

Higher dagger categories have applications to

(1) the categorical structure on a unitary topological field theory (TFT);
(2) the categorical structure on the collection of topological defects of a quantum field theory.

These two applications are a priori distinct, but they do interact. More specifically, if the TFT is the symTFT of a unitary quantum field theory, both structures are present and result in extra symmetries (such as time reversal symmetry) depending on the tangential structure. These two applications were briefly discussed by our speakers in that order.

A basic principle of quantum mechanics is that symmetries act (anti)unitarily on the Hilbert space of states, thereby preserving quantum amplitudes and probabilities [3]. In Euclidean quantum field theory, this property can be understood as a consequence of reflection positivity: reflecting the operator content of correlation functions across a spacetime hyperplane corresponds to complex conjugation, so that reflection-symmetric correlators are positive definite [4]. For ordinary (invertible) symmetries described by a group G, this leads to the familiar conclusion that local observables / states transform in unitary representations of G.

Recent years have seen an expansion of the traditional notion of symmetry in quantum theory to so-called generalised symmetries, which correspond to topological defects whose fusion is governed not by a group law but by more general algebraic structures such as (higher) fusion categories [6]. This naturally raises the question: what additional structures do such defects need to be equipped with in order to be compatible with reflection positivity? When topological defects form a fusion category $\mathcal{C}$ (e.g. in spacetime dimension two), one is led to equip $\mathcal{C}$ with a suitable dagger-structure that encodes the reflection of defects and their junctions across spacetime hyperplanes [7]. As a consequence, algebras canonically associated to $\mathcal{C}$, such as the tube algebra, acquire the structure of C*-algebras that act on observables / states via *-representations, generalising the notion of unitary representations of groups. In particular, this ensures that the action of topological defects preserves quantum amplitudes and probabilities in a manner consistent with their interpretation as symmetry transformations [8].

In spacetime dimensions greater than two, topological defects are expected to organise into higher-categorical structures (such as fusion 2-categories in dimension three). Compatibility with reflection positivity again suggests that the latter should be equipped with appropriate dagger-structures encoding the reflection of extended defects and their higher junctions. However, recent work [10] shows that, in contrast to the 1-categorical case, there is a variety of inequivalent dagger-structures available in the higher-categorical setting, which raises the natural question: which of these dagger-structures are physically relevant and what are their implications for physical observables?

Intuitively, the different choices of dagger structures are related to the ability of reflecting extended operators in different directions. For example, we can reflect lines along a hyperplane transversal to them, which reverses the direction of composition for topological point operators on the line. Alternatively, we can reflect along a hyperplane parallel to the line, instead reversing the direction of the fusion of topological lines. All possible reflections generate the orthogonal group $O(n)$, suggesting that it plays a fundamental role in the study of higher dagger categories. The full mathematical picture and its physical implication has not yet been fully fleshed out, but some parts of the theory have recently been developed.

A central conjecture is that the group $O(n)$ acts on the collection (or more specifically the $(\infty,1)$-category) AdjCat_n of n-categories which admit all adjoints for higher morphisms. The different $\mathbb{Z}_2$-subgroups act by sending a category $\mathcal{C}$ to the category $\mathcal{C}^{i\text{-op}}$ where the direction of i-morphism is reversed. Intuitively,

the adjoints allow us to rotate morphisms. This has also been discussed in David Ayala's talk at this conference.

A higher $O(n)$-dagger category is roughly speaking an n-category with adjoints $\mathcal{C}$ equipped with fixed point data for the $O(n)$-action mentioned above, together with data how induced actions on certain subcategories are trivial.

For $n = 1$ the fixed point data unpacks to a 1-category $\mathcal{C}$ together with functors $d\colon \mathcal{C} \to \mathcal{C}^{1\text{-op}}$, and a natural isomorphism $d^2 \cong \mathrm{id}_{\mathcal{C}}$ satisfying a further condition. It doesn't make sense to require the condition $d(f) = f$, because the left and right side have different sources and targets. However, we can ask $d(c) = c$ and $d^2 = \mathrm{id}_{\mathcal{C}}$ (instead of isomorphic). Imposing these two extra conditions leads to the definition of a dagger category. Equivalently—and more naturally from the perspective of category theory—we can specify the data implementing these conditions leading to the notion of a coherent dagger category [12].

The talk also briefly commented on the application of the theory of higher dagger categories to the higher categorical structure of topological defects. This application is a priori not related to the application to higher unitarity. The theory developed in [11, 9] reproduces known higher categorical structures on the collection of defects, which depend on tangential structures necessary to describe the corresponding topological field theory.

Conclusion of the discussion: Interactions during the discussion session exposed connections to work of several attendants of the workshop. The role of the cobordism hypothesis [14] (and the tangle hypothesis) was emphasized. This connects to open problems in determining the $SO(3)$-action on fully dualizable objects of a 3-category [16, 5], and this difficulty explains the lack of concrete computations in higher dimensions. In this light, the recent preprint [2] proves a relationship between $SO(3)$-dagger categories and spherical fusion categories. The reason why pivotal bicategories [15] (and their higher analogues such as Gray categories with duals [13] and disk-like n-categories [17]) have played an important role in the mathematical literature on topological defects, is that they are expected to correspond to $SO(n)$-dagger categories. The observation is also relevant in work of Ayala–Francis [1], and is connected to the appearance of ribbon structures in Reshetikhin–Turaev theory. The theory of higher dagger categories is still very much in its infancy. For physics applications to unitary quantum field theory, it will be crucial to construct the higher dagger category of higher fusion categories.

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Applications of Topology in Algebra and Physics

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Topology has proven to be an indispensable tool in the study of physics, in particular in the area of quantum field theory. Strongly coupled quantum systems exhibiting long range entanglement can give rise to excitations which are topological in nature, while being totally devoid of local propagating degrees of freedom. Such theories are known as topological quantum field theories (TQFTs) and are reasonable candidates for the long distance limit of many interesting interacting quantum field theories. The Cobordism Hypothesis constructs TQFTs from fully dualizable objects in a suitable symmetric monoidal higher category.

In this discussion session we further saw how ideas of topology can be applied to physical systems that are completely algebraic in nature, and may not have continuum realization; these are called stabilizer codes and this discussion was specifically geared towards Pauli stabilizer codes. These physical systems are realized as lattice models, and while they are not continuous theories they can also exhibit topological excitations which are entirely described by homological algebra rather than category theory. The classification of such systems is an important question, and a proposed classification uses tools from algebraic L-theory to classify the topological excitations along with their statistics. Algebraic L-theory is a generalization of the ideas of surgery theory, coming from topology, to homotopy theory. Moreover, there is a very precise analogy between Pauli stabilizer codes, TQFTs, and manifolds.

While we have only scratched the surface in the study of general stabilizer codes, it is not unreasonable to believe that other frameworks developed in homotopy theory such as algebraic K-theory can be useful in classifying stabilizer codes. We ended the discussion with a few open questions and interesting avenues for future work.

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