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Minimizing energy

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What is the most efficient way to fence land when you've only got so many metres of fence? Or, to put it differently, what is the largest area bounded by a simple closed planar curve of fixed length?

We consider the answer to this question and others like it, making note of recent results in the same spirit.

1 A model problem

The question posed in the abstract is an *optimization problem.* We are given a fixed length of curve and are trying to maximize enclosed area. Equivalently, one could fix the enclosed area, and try to minimize the length of a closed curve bounding a region with the specified area. The planar *isoperimetric inequality* answers both forms of this question:

We consider a *simple* closed curve C in the plane.[1] Let L_C denote the length of the curve C, and let A_C denote the area of the region enclosed by the curve C. The *isoperimetric inequality* states that the area enclosed by the curve times 4π cannot exceed the square of the length of the curve, that is,

$$4\pi A_C \leq L_C^2.$$

In addition, the area enclosed by a curve of fixed length is the largest possible (that is, $4\pi A_C = L_C^2$) if and only if the curve is a circle. Notice that stating the

[1] By a *simple curve* we mean a curve that does not cross itself.

inequality is the same thing as saying that the circle maximizes enclosed area for a fixed perimeter and minimizes perimeter for a given enclosed area.

We sketch a few of the simple ideas behind the proof of such an inequality. First notice that given a circle with radius r, its circumference L is given as $L = 2\pi r$, and its area is $A = \pi r^2$. So

$$4\pi A = 4\pi^2 r^2 = (2\pi r)^2 = L^2.$$

Thus, $4\pi A_C = L_C^2$ if C is a circle with circumference L.

We now consider a simple, closed planar curve C of length L_C. One way to demonstrate that the inequality holds is to show that if the curve C **is not** a circle, then we can find a way to increase the enclosed area. In other words, the enclosed area for a curve of length L is always less than or equal to the enclosed area for a circle of circumference L.

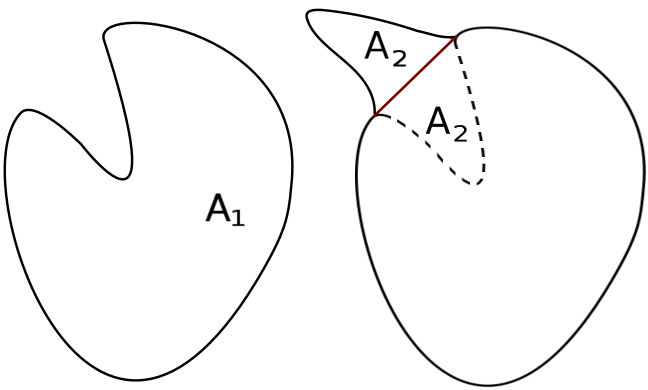

Figure 1: Both solid curves have length L, but the curve on the right bounds a region of larger area.

We first show that if the curve C does not bound a convex region, then the area can be increased without changing the curve's length L_C. Consider Figure 1. The region on the left has area A_1 but is not a convex region. After reflection across the red line segment, the region on the right that is bounded by the solid curve has area $A_1 + 2A_2$. Moreover, the length of the solid curve remains L. As long as the region on the right is not convex, one may repeat the process, reflecting part of the curve over a new line segment, to get a new curve of length L bounding a region with greater area than previously. One can continue this process until the region is convex.[2]

We next show that if a convex curve C does not possess a certain type of symmetry, then the area can also be increased without changing the length L_C. Let P, Q be points on C such that P and Q divide C into two curves of length $L/2$.[3] If C is not symmetric about the line segment connecting P and Q, we can increase the area by reflecting the region with the larger area over the line segment PQ, see Figure 2. If we consider the red (lighter, if you read this in black and white) curve and the reflection of this curve over PQ, the area enclosed is larger than the original area bounded by the blue (darker) curve. One can repeat this reflection process for any points P, Q that divide the curve as specified. This procedure can only stop when one reaches a circle.[4]

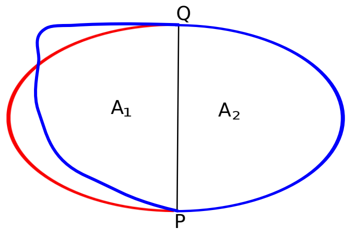

Figure 2: The region bounded by the segment PQ and the red curve has larger area (A_2) than the region bounded by PQ and the portion of the blue curve to the left of PQ (A_1).

To sum up, given a simple closed curve of length L, we have sketched the following string of inequalities:

$$\begin{aligned}&\text{Area of a non-convex region bounded by curve of length } L\\ &\qquad\leq \text{Area of a convex region bounded by a curve of length } L\\ &\qquad\leq \text{Area of a circle with circumference } L.\end{aligned}$$

Notice that we neglected a great deal of detail, especially in arriving at the

[2] This procedure involves taking limits of infinite sequences of curves, which is a delicate issue not to be addressed here.

[3] Notice that if C were not convex, the curve could possibly cross the segment PQ.

[4] Note that again we haven't explained how we pass to the limit of the possibly infinitely many intermediate curves constructed in this way.

final inequality; we have neither shown that an area-maximizing curve indeed exists nor that it is unique. A similar problem the reader might consider is how to prove that the largest area enclosed by a convex curve of length $L/2$ with endpoints on a line is a semi-circle.

2 A surface minimizing area

In three dimensions, the analogous question to the model problem can be considered from a physical point of view. Indeed, one might ask, "What is the shape of a soap bubble?"[5] The soap bubble, like every mechanical system, tries to reach a state of minimal potential (or tension) energy. Its tension energy is proportional to its surface area, so area minimizing soap bubbles are the same thing as tension energy minimizing soap bubbles. We only consider bubbles with a given enclosed volume (the air cannot escape the bubble nor be compressed). The answer to our question is the one we should expect – a soap bubble has the shape of a round sphere. The rigorous proof for this can be found in an area of mathematics called *calculus of variations*, and the word *calculus* should not come as a surprise. While the problem is much more complicated than those seen in a calculus class, at the core we are trying to minimize one quantity (surface area/tension energy) with another quantity (enclosed volume) acting as a constraint. Whereas in calculus one usually varies one variable (or possibly two or more variables) and asks questions like "For which value of this variable does the function under consideration attain a minimum (if at all)?", in calculus of variations we vary shapes of surfaces (and other more abstract "variables"). One can view calculus of variations as a kind of infinite-dimensional calculus.

Variational questions are frequently in one of the following forms:

- In a particular class of surfaces (determined by certain restrictions), does a minimizer to a specified energy[6] exist?
- If a minimizer exists, is it the only one? In other words, is there a *unique* minimizer?

Notice that the model problem we considered in Section 1 can be framed in exactly this way: "For a fixed number A, what is the smallest possible length of a simple, closed planar curve enclosing an area of A square units? If a curve exists that minimizes the length, what is the shape of such a curve?" The restrictions are the fixed enclosed area and the condition that the curve is simple

[5] For more on this and similar questions see the vivid presentations in [3] (available only in German).

[6] In analogy to tension energy and surface area, other quantities that appear in variational problems are frequently called *energies*. In our comparison to calculus, the energy is the function we want to minimize.

and planar, the length is the energy, and we already know the solution by the previous proof.

When we consider the problem in three dimensions, a surface takes the place of the curve, the surface's area takes the place of the curve's length, and the volume of the enclosed area takes the place of the area of the enclosed region. It is helpful to add a *topological condition* to any optimization question – that is, we restrict our attentions to objects of a certain rubber shape (distorting the object by stretching and squeezing is allowed, but no cutting or gluing). An important surface is the sphere. When we attach a handle to a sphere (and allow for some stretching and squeezing), we obtain a *torus*, a donut-shaped surface. Attaching more handles leads to different surfaces, see Figure 3. We might ask, given a fixed volume $V > 0$, what surfaces with zero, one, or two handles have the least surface area enclosing a volume V? In 1955, Hopf showed that among

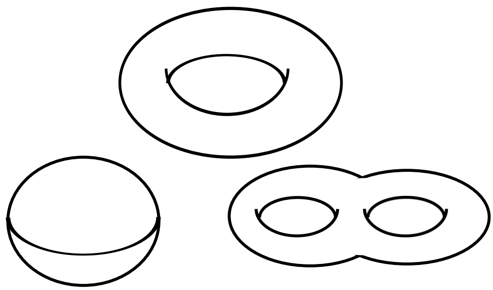

Figure 3: A zero-handled, one-handled, and two-handled surface: also known as a sphere, a torus, and a two-torus.

all zero-handled surfaces, the round sphere[7] has least area for a fixed enclosed volume [4]. In fact, Hopf proved an even stronger result. As any good calculus student knows, not every critical point[8] of a function is a minimizer (possibly not even a local minimizer). In the calculus of variations, mathematicians are not only interested in minimizers of an energy but also in critical points for the energy. Notice that the class of surfaces that are critical for an energy also contains all minimizers for that energy and is thus a potentially larger class. Hopf's result shows that the only zero-handled surface that is critical for surface area (with fixed enclosed volume) is the round sphere.

[7] By round, we mean the zero handled surface with constant radius. Note that the surface of the earth is zero handled, but not round because of the mountains and valleys.

[8] A critical point of a function is a point where the first derivative is zero.

In 1958, Alexandrov answered a slightly different question about the same energy [1]. He demonstrated that for any number of handles allowed, as long as the surface doesn't pass through itself (or *self-intersect*), the only surface that is critical for this energy is again the round sphere. In particular, no surface with handles has area less than or equal to the area of the round sphere (presuming fixed enclosed volume). The proofs of these two results were quite different and introduced new and very powerful techniques that provided inspiration to mathematicians in the years to come.

3 Recent breakthroughs

In recent years, a number of long-standing open questions have been answered. We mention a few of these results, which are similar in spirit to the ideas outlined above.

In 1970, Lawson conjectured that there is a unique one-handled surface without self-intersection in the three-dimensional sphere $\mathbb{S}^3$ that minimizes surface area [6].[9] In 2012, Brendle used a sophisticated analytical principle argument, involving *partial differential equations* and the so-called *maximum principle*, to verify the conjecture [2].

The round sphere is not just a minimizer for the energy described in the previous section. Among all surfaces, the round sphere also minimizes another energy, which relates to the curvature of a surface, the so-called *Willmore energy*. This energy is also known as the *bending energy* of a surface – it describes how much energy you would need to shape a thin membrane into the form of the surface by bending the membrane. It has applications in cell biology and image processing.

In 1965, Willmore conjectured that the Willmore energy for one-handled surfaces (tori) is always at least $2\pi^2$ [7]. Many mathematicians proved this conjecture under extra hypotheses. In 2012, Marques and Neves verified the conjecture in full generality [5]. One major step of their argument was to determine that there exists a unique one-handled surface without self-intersection in $\mathbb{S}^3$ that minimizes the Willmore energy. In fact, the one-handled surface that minimizes Willmore energy in $\mathbb{R}^3$ is directly related to the one-handled surface that minimizes surface area in $\mathbb{S}^3$!

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[9] Instead of looking at curves in the plane, we could have considered curves on a two-dimensional sphere $\mathbb{S}^2$ in the first example and asked similar questions. Similarly, instead of considering surfaces in the three-dimensional space $\mathbb{R}^3$ one can also consider surfaces in the *three-dimensional sphere* $\mathbb{S}^3$.

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