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Low-dimensional Topology

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5 July – 10 July 2026

ABSTRACT. The workshop brought together experts from across all areas of low-dimensional topology, including knot theory, computational topology, 3-manifolds and 4-manifolds. There were sixteen talks of 50 minutes, and six shorter talks of 25 minutes each, highlighting research in all areas of low-dimensional topology and leading to many discussions of recent results and open problems. Furthermore we had lively evening sessions of five-minute lightning talks, with roughly twelve speakers per evening.

Mathematics Subject Classification (2020): 57K10, 57K20, 57K30, 57K40.

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Introduction by the Organizers

The workshop *Low-dimensional Topology* was organized by Marc Lackenby (Oxford), Jessica Purcell (Monash), Arunima Ray (Melbourne), and Saul Schleimer (Warwick). The workshop took place at Oberwolfach from 5 July to 10 July 2026 and was attended by nearly 50 researchers from countries including Australia, Canada, Germany, Ireland, Israel, Italy, Korea, Spain, Switzerland, the UK, and the USA.

The meeting brought together a broad cross-section of the low-dimensional topology community, including senior researchers, early-career mathematicians, postdocs, and graduate students. It provided an opportunity for researchers from different subfields to refocus on central problems, exchange techniques, discuss new directions, and begin fresh collaborations. The talks were very high quality, and

many mathematical discussions occurred over meals, in the library and meeting rooms, and during informal walks.

We had sixteen research talks of 50 minutes and six of 25 minutes. The talks covered a wide range of topics, including knot concordance, hyperbolic 3-manifolds, taut foliations and flows, computational topology, contact geometry, 4-manifolds and the surfaces within them, exotic smooth 4-manifolds, and classification problems for 3- and 4-manifolds.

Highlights included talks by Michael Willis explaining his recent work giving a combinatorial method to detect exotic 4-manifolds, and an announcement by Lily Li of a proof that 3-manifolds admit only finitely many pseudo-Anosov flows. There were also excellent talks on new progress in contact geometry by Joan Licata, and progress on computational problems discussed by Filippo Baroni and Rhuaidi Burke.

Although the talks ranged across several subfields, the speakers were very successful in establishing connections between various topics in low-dimensional topology. For example, knot concordance gives a bridge between knot theory and 4-manifolds. Speakers Paula Truöl and Steven Sivek spoke on different aspects of classical knots and their connections to 4-dimensions, such as the algebra of concordance groups, ribbon concordance, and slice obstructions, via new invariants and computation. These talks demonstrated the continuing vitality of the interaction between knot theory, 4-dimensional topology, Heegaard Floer homology, and other fields.

In addition to the scheduled research talks, we had evening sessions of five-minute lightning talks, with roughly twelve speakers per evening over three evenings. These sessions were especially lively and valuable. They gave many participants a chance to present recent results, advertise open problems, explain new examples, or introduce themselves and their work to the community. The format was highly effective: the short talks generated many follow-up conversations, and helped to ensure that the meeting was not only a sequence of formal lectures, but also a broad forum for mathematical exchange.

Overall, the workshop achieved its aims of strengthening connections between researchers working in different parts of low-dimensional topology, showcasing important recent developments, and creating an excellent environment for discussion and collaboration. The breadth of topics, the quality of the talks, the enthusiasm of the participants, and the success of the lightning-talk sessions all contributed to a very productive and memorable meeting.

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Workshop: Low-dimensional Topology

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Abstracts

Roots of Alexander polynomials of random positive 3-braids

NATHAN M. DUNFIELD

(joint work with Giulio Tiozzo)

Motivated by an observation of Dehornoy, we study the roots of Alexander polynomials of knots and links that are closures of positive 3-strand braids. We give experimental data on random such braids and find that the roots exhibit marked patterns, which we refine into precise conjectures. We then prove several results along those lines, for example that generically at least 69% of the roots are on the unit circle, which appears to be sharp. We also show there is a large root-free region near the origin. We further study the equidistribution properties of such roots by introducing a Lyapunov exponent of the Burau representation of random positive braids, and a corresponding bifurcation measure. In the spirit of Deroin and Dujardin, we conjecture that the bifurcation measure gives the limiting measure for such roots, and prove this on a region with positive limiting mass. We use tools including work of Gambaudo and Ghys on the signature function of links, for which we prove a central limit theorem.

In more detail, we found the following phenomena for random positive 3-strand braids based on a detailed analysis of the data. Because of the symmetry of the roots, we focus on the closed unit disk in $\mathbb{C}$.

- (1) The roots of the Alexander polynomial $\Delta_K(t)$ *nearly* all lie either on the unit circle or on a single continuous curve joining the roots of unity $\zeta_3 = e^{2\pi i/3}$ and $\bar{\zeta}_3$. Said curve meets the real axis in a single point in $(-1, -1/2)$.
- (2) Asymptotically, $2/3$ of the roots lie on the arc of the unit circle where $-1/2 < \Re(z)$ and those roots are evenly distributed there.
- (3) The roots on the curve from ζ_3 to $\bar{\zeta}_3$ are close to equidistributed with respect to arc length.
- (4) There are no roots in the region with $|z| < 1$ and $-1/2 < \Re(z)$. In fact, there are no roots with $|z| < 1$ to the right of the curve from ζ_3 to $\bar{\zeta}_3$.

Motivated by the notion of bifurcation current of Deroin and Dujardin, we combine these phenomena into an overarching conjecture as follows. Let Br_3 be the 3-strand braid group. Any $w \in \text{Br}_3$ has a braid closure $\hat{w}$ which is a link (a disjoint union of knots) of at most three components. Consider the product space $\Omega := \{\sigma_1, \sigma_2\}^{\mathbb{N}}$ with the measure $\mu^{\mathbb{N}}$ where $\mu(\sigma_1) = \mu(\sigma_2) = 1/2$. Choosing $(g_i)_{i \in \mathbb{N}} \in \Omega$ with respect to $\mu^{\mathbb{N}}$ gives a random walk $w_n := g_1 g_2 \cdots g_n$ and an associated sequence of links $\hat{w}_n$.

To state our conjecture about the roots of $\Delta_{\hat{w}_n}$, for any $w \in \text{Br}_3$ we define the set $Z_w := \{z \in \mathbb{C} \mid \Delta_{\hat{w}}(z) = 0\}$, whose elements are considered with multiplicity. Consider the uniform probability measure supported on Z_w , that is

$$\nu_w := \frac{1}{\#Z_w} \sum_{z \in Z_w} \delta_z.$$

We propose that the measures ν_{w_n} almost surely have a limit, which is moreover independent of the sample path:

Main Conjecture. There is a compactly supported measure ν_∞ on $\mathcal{C}$ such that for almost every sequence $(g_i)_{i \in \mathbb{N}} \in \Omega$ one has $\nu_{w_n} \rightarrow \nu_\infty$ weakly as $n \rightarrow \infty$. Moreover:

- (1) The support of ν_∞ is contained in the union of the unit circle, a continuous curve from ζ_3 to $\bar{\zeta}_3$, and the image of said curve under $z \mapsto 1/z$. The curve crosses the real axis at about -0.78 .
- (2) Divide the circle into left and right arcs at ζ_3 and $\bar{\zeta}_3$:

$$\mathcal{A}_L := \{t = e^{i\theta} \mid |\theta - \pi| < \pi/3\} \quad \mathcal{A}_R := \{t = e^{i\theta} \mid |\theta| < 2\pi/3\}.$$

Exactly $2/3$ of the mass of ν_∞ is on $\mathcal{A}_R$, and ν_∞ is a multiple of Lebesgue measure on that arc. Moreover, ν_∞ is absolutely continuous with respect to Lebesgue measure on $\mathcal{A}_L$, with total mass $(7 - 3\sqrt{5})/12 \approx 0.024316$. As a consequence, the total mass of the unit circle is $(5 - \sqrt{5})/4 \approx 0.690983$.

- (3) The measures along the curves joining ζ_3 to its complex conjugate are absolutely continuous with respect to Lebesgue measure on them.

For complete details, please see [1].

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Snails and Half-Apples

JOAN LICATA

(joint work with Matthias Scharitzer and Vera Vértési)

This talk presented joint work with Matthias Scharitzer and Vera Vértési which can be found in the Appendix of arXiv:2408.01079 [7]. The paper is currently under review.

I. Stratifications of topological spaces can be found throughout mathematics. Roughly speaking, a stratification is a decomposition of a topological space into subspaces E_i satisfying certain frontier properties. Rather than defining these precisely, we'll consider two examples likely to be familiar to topologists: triangulated manifolds and smooth functions. Each of these will illustrate a property also seen in Kotova-Stanzo's stratification on the space of smooth vector fields on planar surfaces [6].

In a triangulated 3-manifold, let X^j denote the j -skeleton and set E_i to be $X^{3-i} \setminus X^{2-i}$. Note that a generic k -dimensional submanifold intersects only strata of index less than or equal to k . For example, a path in the manifold will intersect only $E_0 \cup E_1$ (the open 3-cells and their faces) and will miss $E_2 \cup E_3$ (the edges and vertices).

Hatcher and Wagoner described a stratification on the space of smooth functions in which the 0-stratum consists of Morse functions [5]. A function that fails to be Morse by having a single birth/death singularity or by having a critical value of multiplicity 2 lies in the 1-stratum. Here we see that increasing indices measure deviation from the generic behaviour; a function can belong to the 2-stratum because it has a single more degenerate point (specifically, a swallowtail singularity) or two instances of the phenomena that appear in the 1-stratum: two births; a birth and a multiplicity-2 critical value; 2 multiplicity-2 critical values; or a multiplicity-3 critical value.

We now turn to smooth vector fields on planar surfaces. As a first simplification, we will study these via their integral curves, which are singular foliations. (This hides some technical details required for the precise argument.) Kotova-Stanzo enumerate features that may arise in such a singular foliation and assign these an integer value “cod”. Their main theorem states that for $i \leq 2$, a vector field V lies in the stratum E_i exactly when the cod values of its features sum to i . For example, nodes and hyperbolic saddles are examples of cod=0 features, while saddle-saddle connections are examples of cod=1 features. Thus a foliation with two saddle-saddle connections and otherwise generic features will lie in E_2 . This additivity mirrors the structure seen in the Hatcher-Wagoner stratification on smooth functions. As in the triangulated manifold example, a generic k -parameter family of vector fields will contain only foliations lying in strata of index less than or equal to k .

The classification for foliations in strata of index less than or equal to 2 is nearly combinatorial, so it is possible to enumerate the kinds of ensembles (finite concatenations of singularities) that appear in such foliations. Some of these have been given evocative names, such as snail, heart, lentil, and (half-)apple.

Before making the connection to contact geometry that motivates this work, we note that the talk suppressed any discussion of closed orbits; details on how these are handled may be found in the paper.

II. In three dimensions, contact geometry equips orientable manifolds with a nowhere-integrable, co-oriented 2-plane field ξ called a *contact structure*. This turns out to have surprising implications for the topological study of manifolds, as well as providing a fascinating mathematical playground. One immediate consequence of viewing 3-manifolds as contact manifolds is a more refined classification question: when is a topological product $\Sigma^2 \times I$ also a “contact product”?

The connection to Part I comes from the fact that when Σ is a surface embedded in a contact manifold (M, ξ) , the intersection $T\Sigma \cap \xi$ integrates to a singular foliation on Σ . Qualitatively, the non-integrable condition on ξ requires the planes to rotate along curves in M , and we use the co-orientation to assign signs to the singularities: when the coorientation of ξ agrees with a chosen normal to Σ at a point of tangency, the sign is positive. When these disagree, the sign is negative.

In many cases, this foliation not only recovers ξ on Σ , but also recovers ξ near Σ . However, contact structures are often studied up to contact isotopy, and the foliation may change dramatically under only small perturbations in ξ . In order

to capture the relevant level of detail, we consider the *dividing set* Γ of a signed foliation, which is a collection of simple closed curves that separates the positive and negative singularities of the foliation. An important result of Giroux states that the isotopy class of Γ on Σ determines ξ up to isotopy on a neighbourhood of Σ . We characterise $(\Sigma \times I, \xi)$ as a contact product when the isotopy class of Γ on $\Sigma \times \{t\}$ is fixed for all t . (Experts will note that this explanation elides both the contact vector field and the possibility that Γ may not exist.)

III. We may now connect Parts I and II: a contact structure ξ on $\Sigma \times I$ induces a movie of foliations on the surfaces $\Sigma \times t$, and this in turn produces a sequence of isotopy classes of dividing sets. Under some suppressed conditions, this decomposes $\Sigma \times I$ into products where Γ is constant, or more usefully, into elementary cobordisms called *bypass slices* where Γ changes in a single controlled way. Our main theorem is a statement about the relationship between two distinct decompositions of this type. One may easily see – although this was not presented for reasons of time – that moves known as Far Commutation and Trivial Insertion relate decompositions associated to isotopic contact structures. In fact, these suffice.

Theorem 1 ([1] 2023, [7] 2026). Two decompositions of $(\Sigma \times I, \xi)$ into bypass slices are related by a sequence of Far Commutations and Trivial Insertions.

We remark that the Breen-Honda-Huang theorem covers all dimensions, whereas our result holds only in dimension 3. In both cases, the theorem is used to prove the Giroux Correspondence relating contact structures to their supporting open books; we were motivated by the desire for a hands-on argument that avoids the subtleties of higher-dimensional convex surface theory.

We conclude with a brief remark about the proof. Just as a contact structure on $\Sigma \times I$ corresponds to a path of foliations on Σ , a family ξ_s on $\Sigma \times I$ corresponds to a 2-parameter family of singular foliations on Σ . Kotova-Stanzo lets us enumerate the configurations that may arise in a generic 2-parameter family of foliations, so the proof becomes a case chase to verify that these preserve the equivalence class of the decomposition into bypass slices. Many of the ideas used in the proof were first introduced by Giroux, and much of the work is devoted to extending these from the 1-parameter setting to the 2-parameter setting [2], [3], [4].

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Non-smoothable surfaces in the 4-sphere

DANIEL GALVIN

(joint work with Anthony Conway)

I presented the following theorem and explained two key corollaries.

Theorem 1. Let $b: \mathbb{Z}^h \times \mathbb{Z}^h \rightarrow \mathbb{Z}$ be a non-singular, symmetric, definite, bilinear form. If $b \cong b' \oplus \langle 1 \rangle$ where b' is even, i.e. $b'(x, x)$ is even for all x , then there exists a locally flat embedding of a non-orientable genus h surface $F \subset S^4$ such that

- (1) the integral intersection form of the branched double cover of F is isomorphic to b ; and
- (2) the knot group is order two, i.e. $\pi_1(S^4 \setminus F) \cong \mathbb{Z}/2$.

I now explain the corollaries. Some background is in order. It has been known since the work of Freedman [5] and Donaldson [3] that 4-manifolds exhibit radically different behaviors in the smooth and topological categories. To name but two examples, there are smooth 4-manifolds that are homeomorphic but not diffeomorphic, and there are topological manifolds that do not admit any smooth structure. The difference in categories is also witnessed by knotted surfaces. Examples of *exotic surfaces* (i.e. surfaces that are topologically isotopic but not smoothly isotopic) were first discovered by Finashin-Kreck-Viro [4] whereas examples of *non-smoothable* surfaces (i.e. locally flat surfaces that are not isotopic to any smoothly embedded surface) first appeared in [6, 7] though there have been many examples since.

The difficulty in finding differences between the smooth and topological categories typically increases as the amount of ambient algebraic topology decreases. The most challenging questions in the area therefore generally involve the 4-sphere S^4 . While the best known example is the smooth 4-dimensional Poincaré conjecture, there are also the questions of whether there exist exotic oriented surfaces in S^4 and whether there exist non-smoothable surfaces in S^4 . The first corollary addresses the latter, which is Question 4.32 on the K3-problem list [1].

Corollary A. For every $h \geq 9$, there is a non-smoothable surface of non-orientable genus h in S^4 .

We now describe the second corollary. First, some terminology is in order. A non-orientable surface in S^4 is *unknotted* if it is isotopic to a connected sum of unknotted projective planes. In turn, a projective plane in S^4 is *unknotted* if it is isotopic to the branch locus of one of the branched covers $\mathbb{C}P^2 \rightarrow S^4$ or $\overline{\mathbb{C}P}^2 \rightarrow S^4$ induced by complex conjugation.

The fundamental group of the complement, also called the *knot group*, of such non-orientable unknotted surfaces has order 2. Whether or not the converse holds, i.e. whether $\mathbb{Z}/2$ -surfaces are necessarily unknotted, has been asked in Question 4.29(a) on the K3-problem list as well as [2, Question 1.1] among other places.

Corollary B. For every $h \geq 9$, there exists a $\mathbb{Z}/2$ -surface with non-orientable genus h that is topologically knotted.

In fact, by virtue of the surfaces in Corollary A being non-smoothable (and unknotted surfaces being by definition smoothable), Corollary B is immediately deducible from Corollary A.

The proof of Corollary A proceeds by applying Theorem 1 to construct a non-orientable genus nine $\mathbb{Z}/2$ -surface $F \subset S^4$ such that the branched double cover has intersection form $E_8 \oplus \langle 1 \rangle$. A famous theorem of Donaldson [3] then implies that the branched double cover is non-smoothable, and we immediately deduce the same for the embedding F . Taking ambient connected-sums of F with copies of an unknotted projective plane yields the examples for higher non-orientable genera.

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Fixed points of pseudo-Anosov maps and 3-manifold invariants

DAVID FUTER

(joint work with Tarik Aougab and Samuel J. Taylor)

Let S be an orientable finite-type surface with $\chi(S) < 0$. Consider a pseudo-Anosov homeomorphism $f: S \rightarrow S$. This talk addresses the question of how combinatorial and geometric data associated to f determines its number of fixed points. It helps to introduce the (mild) simplifying assumption that f is *strongly irreducible*, which means that for every essential simple curve α on S , the curves α and $f(\alpha)$ intersect essentially. For a surface of sufficient complexity, this means the displacement of f on the curve graph of S is at least 2.

Under this hypothesis, the number of $\# \text{Fix}(f)$ of fixed points of f is well-predicted by the *dilatation* or *stretch factor* $\lambda = \lambda_f$. Our main theorem is:

Theorem 1. If $f: S \rightarrow S$ is a strongly irreducible pseudo-Anosov map, then

$$\log \# \text{Fix}(f) \asymp \log \lambda_f.$$

Here, the notation $a \asymp b$ means that the two quantities are equal up to additive and multiplicative error depending only on the topology of S . Similarly, $a \prec b$ means that a is upper-bounded by a linear function of b , where the implicit constants again depend only on S .

Theorem 1 fits into the following context. First, a theorem of Thurston [7] says that $\# \text{Fix}(f^n) \sim \lambda^n$. So the novelty of Theorem 1 is that it counts fixed points of f rather than fixed points of sufficiently large powers. Second, Theorem 1 asserts an upper and a lower bound on $\# \text{Fix}(f)$. The upper bound is not new; indeed, the stronger bound $\log \# \text{Fix}(f) \leq \log \lambda_f + O(\log |\chi(S)|)$ follows from the theory of Markov partitions of surfaces [2]. So the real content of Theorem 1 is the lower bound on $\# \text{Fix}(f)$, which is entirely new.

It is also classically known that a pseudo-Anosov homeomorphism f minimizes the number of fixed points among all maps homotopic to f . (The result is due to Thurston [7], with a nice account given by Birman and Kidwell [3].) Thus, if f is a strongly irreducible pseudo-Anosov, Theorem 1 produces a lower bound on $\# \text{Fix}(f')$ for any $f' \sim f$. We may thus write $\# \text{Fix}(\varphi) = \# \text{Fix}(f)$ for the mapping class $\varphi = [f]$ of a pseudo-Anosov f .

Finally, we note that the conclusion of Theorem 1 is false without the hypothesis that f is strongly irreducible. Indeed, if α is a simple closed curve such that $f(\alpha) \cap \alpha = \emptyset$, and T_α denotes a Dehn twist about α , then the tail of the sequence $[f \circ T_\alpha^n]$ consists of pseudo-Anosov mapping classes with no more fixed points than f , and with stretch factors limiting to ∞ .

If f is a pseudo-Anosov map that is weakly reducible (meaning, not strongly irreducible) then our work can still estimate $\log \# \text{Fix}(f)$ up to bounded additive and multiplicative error. However, the quantity appearing in the estimate is a fairly complex sum involving subsurface projections; see [1, Theorem 1.4] for details.

Every homeomorphism $f: S \rightarrow S$ defines a *mapping torus*, namely the 3-manifold

$$M_f = S \times [0, 1] / (x, 1) \sim (f(x), 0).$$

In this situation, the map f is called the *monodromy* of the fibration of M_f over S^1 . By a theorem of Thurston [7], a fibered 3-manifold M_f is hyperbolic if and only if f is (isotopic to) a pseudo-Anosov map. A somewhat surprising consequence of Theorem 1 is the following relationship between two different invariants of M_f : hyperbolic volume and Heegaard Floer homology.

Corollary 2. Let Y be a closed orientable 3-manifold. Let $K \subset Y$ be a knot whose complement $M = Y \setminus K$ is hyperbolic and fibered over S^1 , with fiber a surface S with $\partial S = K$, and strongly irreducible monodromy f . Then

$$\mathrm{Vol}(Y \setminus K) \prec \log \mathrm{rank} \widehat{\mathrm{HFK}}(Y, K),$$

where the constants in the coarse inequality $\prec$ depend only on the genus $g(S)$.

The reason why Corollary 2 follows from Theorem 1 is that the volume of $M_f = Y \setminus K$ is upper-bounded by a constant times λ_f , by the work of Kojima and McShane [5], while the number of fixed points of f provides a lower bound on the rank of $\widehat{\mathrm{HFK}}(Y, K)$, by the work of Ni [6] and Ghiggini and Spano [4].

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Ribbon concordance and fibered predecessors

STEVEN SIVEK

(joint work with John A. Baldwin, Jonathan Hanselman)

We say two knots $K_0, K_1 \subset S^3$ are *concordant* if there is a smooth, properly embedded annulus $A \subset S^3 \times [0, 1]$ with boundary components $K_0 \times \{0\}$ and $K_1 \times \{1\}$. If the $[0, 1]$ -coordinate is a Morse function on A with no local maxima, then we say that K_0 is *ribbon concordant* to K_1 , or $K_0 \leq K_1$.

Ribbon concordance was introduced by Gordon [6], who observed that some notions of knot complexity seem to behave monotonically with respect to it. For example, if $K_0 \leq K_1$ then Gilmer proved that the Alexander polynomial of K_0 divides that of K_1 ; Silver and Kochloukova proved that if K_1 is fibered then so is K_0 ; and Zemke [9] proved that the knot Floer homology $\widehat{\mathrm{HFK}}(K_0)$ injects into $\widehat{\mathrm{HFK}}(K_1)$, and so deduced an inequality $g(K_0) \leq g(K_1)$ between their Seifert genera. This culminated in work of Agol [1], who proved Gordon’s conjecture that ribbon concordance is in fact a partial order.

Gordon also conjectured that ribbon concordance satisfies a descending chain condition, meaning that every infinite sequence

$$\cdots \leq K_3 \leq K_2 \leq K_1 \leq K_0$$

must eventually stabilize. In light of Agol’s theorem, this would be implied by the following stronger conjecture, which was the subject of this talk.

Conjecture 1. For any fixed knot $K \subset S^3$, there are only finitely many knots $J \subset S^3$ such that J is ribbon concordant to K .

Conjecture 1 remains open, but we did manage to prove the following.

Theorem 2 ([2, 3]). For any fixed knot $K \subset S^3$, there are only finitely many *fibred* knots ribbon concordant to K .

Remark 3. Given the work of Silver and Kochloutkova mentioned above, Conjecture 1 follows immediately whenever the knot K is fibered.

To prove Theorem 2, we suppose that $J \leq K$ for some fibered knot J , and we use K to bound the complexity (in some appropriate sense) of the *monodromy* of J . We first outline the proof when J is hyperbolic, in which case its monodromy is freely isotopic to a pseudo-Anosov map whose dilatation will be the right notion of complexity.

To start, Zemke [9] provides us with an inequality

$$(1) \quad \text{rank } \widehat{HFK}(J) \leq \text{rank } \widehat{HFK}(K),$$

but this is not enough because there can be infinitely many J with the same knot Floer homology. Perhaps more usefully, we can apply a deep relationship between the knot Floer homology of a fibered knot and its monodromy, due to Ni [8] and Ghiggini–Spano [5]: the summand $\widehat{HFK}(J, g(J) - 1)$ is determined by the symplectic Floer homology of the monodromy of J . When J is hyperbolic and its monodromy is a pseudo-Anosov map $\phi : S \rightarrow S$, then this relationship is simply

$$(2) \quad \#\text{Fix}(\phi) = \text{rank } \widehat{HFK}(J, g(J) - 1) - 1,$$

and combining this with (1) gives us

$$\#\text{Fix}(\phi) < \text{rank } \widehat{HFK}(K).$$

It turns out that this is not the right bound to use either, though, because this might still leave us with infinitely many knots to consider.

On the other hand, if we can bound the fixed point counts of sufficiently many iterates ϕ^n , then we get a bound on the dilatation $\lambda(\phi)$, via the relation

$$\lambda(\phi) = \lim_{n \rightarrow \infty} (\#\text{Fix}(\phi^n))^{1/n}.$$

This is a more reasonable goal, because for any constant C and any fixed genus there are only finitely many conjugacy classes of monodromy with dilatation at most C ; there are also only finitely many possible genera, because we know that $g(J) \leq g(K)$, so this will show that there are only finitely many corresponding fibered knots J .

If $J \subset S^3$ has monodromy ϕ , then ϕ^n arises naturally as the monodromy of a knot J_n in the n -fold cyclic branched cover $\Sigma_n(J)$, where J_n is the lift of the branch locus J . In this case we restate (2) with ϕ^n in place of ϕ to get

$$(3) \quad \#\text{Fix}(\phi^n) < \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n).$$

We can also take n -fold branched covers of the entire ribbon concordance from J to K , and for suitable n — any large enough prime will do — we get what Daemi, Lidman, Vela-Vick, and Wong [4] call a *ribbon $\mathbb{Z}/2$ -homology cobordism* from $J_n \subset \Sigma_n(J)$ to $K_n \subset \Sigma_n(K)$. They prove that these satisfy the analogue of Zemke’s rank inequality (1):

$$(4) \quad \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(K), K_n).$$

Levine [7] constructed an explicit chain complex for $\widehat{HFK}(\Sigma_n(K), K_n)$ with at most $(\delta(K)!)^n$ generators, where $\delta(K)$ is the arc index of K , so we combine this with (3) and (4) to get

$$\#\text{Fix}(\phi^n) < (\delta(K)!)^n$$

for any large enough prime n . In the limit as $n \rightarrow \infty$ this means that $\lambda(\phi) \leq \delta(K)!$, and so there are finitely many monodromies up to conjugacy and hence finitely many hyperbolic J as explained above.

Having understood the case where J is hyperbolic, we can try to repeat this proof for more general J . In this case we put the monodromy $h : S \rightarrow S$ of J in Nielsen–Thurston form, and the argument only tells us about the outermost (i.e., boundary-adjacent) component S_0 of this decomposition. If the restriction $h|_{S_0}$ is pseudo-Anosov, then we learn that its dilatation is again at most $\delta(K)!$; if it is periodic, then this corresponds either to a cabling operation of bounded genus or a connected sum. Either way, there are only finitely many options up to conjugacy for the monodromy on S_0 . But what about the non-outermost components of this Nielsen–Thurston decomposition?

These other components tell us how J can be expressed as a satellite knot, because the reducing set for h is a collection of curves that suspend to satellite tori in the mapping torus $S^3 \setminus J$ of h . If we express J as such a satellite, with companion $C \subset S^3$ and pattern $P \subset S^1 \times D^2$, then the monodromy on S_0 tells us nothing about the companion C . But we get around this by proving a new inequality for knot Floer homology of satellite knots:

Theorem 4. Let J be a satellite knot with companion C and pattern P of nonzero winding number. Then

$$\dim_{\mathbb{Z}/2} \widehat{HFK}(C) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(J).$$

Theorem 4 holds not just for satellite knots in S^3 , where it is already new, but for a generalized notion of satellites in other 3-manifolds: in particular

$$\dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(C), C_n) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n)$$

for many values of n , including again all large enough primes. So we combine this with (4) as before to deduce that $\lambda(\psi) \leq \delta(K)!$, where ψ is any pseudo-Anosov component of the Nielsen–Thurston decomposition. Now there are only finitely many possible monodromies for each component, just as before, and only finitely many ways to glue these components together to produce the monodromy of our original knot J , establishing Theorem 2.

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Certifying hyperbolicity of fibred 3-manifolds and knots

FILIPPO BARONI

Consider the algorithmic problem of deciding whether a given triangulated orientable 3-manifold is hyperbolic (that is, whether its interior admits a complete hyperbolic metric of finite volume). This problem is known to be decidable as a consequence of the geometrisation theorem (see [7], or [6] for a self-contained exposition). In terms of computational complexity, if the input 3-manifold M is guaranteed to be irreducible and with non-empty boundary, then the decision problem is known to lie in the complexity class co-NP (see [5]) – meaning that, if M is *not* hyperbolic, then this can be certified in time polynomial in the number of tetrahedra of the given triangulation. When M *is* hyperbolic, however, it is not known whether this can be certified efficiently. In this talk, we present a partial result in this direction, restricted to the class of fibred 3-manifolds.

Theorem 1. Hyperbolicity of a triangulated orientable fibred 3-manifold can be certified in time polynomial in the number of tetrahedra and the absolute value of the Euler characteristic of a fibre.

Note that the absolute value of the Euler characteristic of a fibre is not bounded above by a polynomial in the number of tetrahedra, so this result does not imply that hyperbolicity of fibred 3-manifolds is in NP. Suppose, however, that we are given a diagram of a fibred knot K in S^3 with n crossings. The exterior of K can be triangulated with $O(n)$ tetrahedra, and the genus of the fibre is also bounded above by a linear function of n . Hence, we obtain the following corollary.

Corollary 2. Hyperbolicity of fibred knots in S^3 (given as diagrams) is in NP.

Our choice to restrict to fibred 3-manifolds is mostly pragmatic. While certifying hyperbolicity of a general 3-manifold seems elusive, a classical theorem of Thurston

[9] states that a fibred 3-manifold with fibre F is hyperbolic if and only if $\chi(F) < 0$ and the monodromy of the fibration is pseudo-Anosov. Therefore, in order to prove Theorem 1, we need to be able to efficiently certify three things:

- (1) that a certain properly embedded surface F is a fibre of a given 3-manifold M ;
- (2) that a certain homeomorphism $\varphi: F \rightarrow F$ is the monodromy of the fibration;
- (3) that φ is pseudo-Anosov.

The standard framework for algorithmically manipulating surfaces in triangulated 3-manifolds is provided by normal surface theory [4]. Schleimer [10, 11] proved that, if F is a normal fibre of a closed atoroidal triangulated 3-manifold M , then the fact that F is a fibre can be certified in time polynomial in the number of tetrahedra of M and the logarithm of the weight of F . Loosely speaking, this certificate consists of a sequence of alternately normal and almost normal fibres isotopic to F . In our work, we develop a different certificate for the fact that F is a fibre, employing specific properties of *least-weight* normal fibres. Our certificate is more complex than Schleimer's, but it has the advantage of carrying sufficient information to also certify the monodromy of the fibration. More precisely, we are able to prove the following.

Proposition 3. Let M be an orientable fibred 3-manifold triangulated with t tetrahedra, and let F be a least-weight normal fibre of M ; denote its monodromy by φ .

- (1) There exists a certificate that F is a fibre of M , which can be verified in time polynomial in t and the logarithm of the weight of F .
- (2) Let a_0 be an essential curve in F . Then $\varphi(a_0)$ is isotopic in F to a curve a_1 in F with $w(a_1) \leq 2^{O(t)} \cdot w(a_0)$, where w denotes the number of intersections with the 2-skeleton of M . Moreover, the fact that a_1 is isotopic to $\varphi(a_0)$ can be certified in time polynomial in t , $|\chi(F)|$, and $\log w(a_0)$.

In order to complete the certification of hyperbolicity of M , we use the fact that knowing the images of an essential curve $a_0 \subset F$ under $\varphi^1, \dots, \varphi^N$ for a sufficiently large N is enough to determine whether φ is pseudo-Anosov. To bootstrap this argument, however, one needs to start with a sufficiently “short” essential curve a_0 . We encounter substantial and unexpected difficulties in proving that such a curve exists. We do succeed, but the effort required seems disproportionate to the result we obtain – namely, that if F is a least-weight normal fibre of an orientable fibred hyperbolic 3-manifold M triangulated with t tetrahedra, then F contains an essential curve a_0 with

$$w(a_0) = O(t^7 \cdot \chi(F)^2).$$

It seems likely that this result is far from optimal.

Question 4. In the setting of the previous paragraph:

- (1) Can the dependence on $\chi(F)$ be removed?
- (2) Can the exponent of t be reduced (possibly to 1)?

- (3) Can the assumption that M is hyperbolic be removed?
- (4) Can the result be generalised to arbitrary normal surfaces in arbitrary triangulated 3-manifolds?

With the results stated so far, we are able to provide as a certificate of hyperbolicity of M a normal fibre F , a “short” essential curve $a_0 \subset F$, and the curves $\varphi^1(a_0), \dots, \varphi^N(a_0)$ for some $N \geq 1$, and everything is verifiable in time polynomial in t , $|\chi(F)|$, and N . We have previously stated that, for N sufficiently large, this is enough to determine whether φ is pseudo-Anosov; let us now make this more precise. It is a seminal result of Masur and Minsky [8] that a mapping class φ of a surface F is pseudo-Anosov if and only if the distance in the curve graph of F between an essential curve a_0 and its image $\varphi^n(a_0)$ grows linearly with n ; if φ is not pseudo-Anosov, then this distance is instead bounded. Later work of Gadre and Tsai [3] and Webb [12] makes this statement effective, providing explicit constants N and D_0 (depending on F , φ , and a_0) such that φ is pseudo-Anosov if and only if the distance in the curve graph of F between a_0 and $\varphi^N(a_0)$ is at least D_0 . In our setting, one can take $N = O(\chi(F)^6 \cdot t)$.

The final ingredient for verifying that φ is pseudo-Anosov is then an algorithm to efficiently compute distances in the curve graph of a surface. One such algorithm is provided by Bell and Webb in [2], who describe how to compute the distance in the curve graph between two normal curves in time polynomial in the logarithms of their weights. The running time of this algorithm, however, is at least exponential in the absolute value of the Euler characteristic of the surface, which is not sufficient for the requirements of Theorem 1. There is, however, a version of this algorithm that we developed in [1], which we now state.

Theorem 5 ([1, Corollary 3.12]). There is an algorithm that takes as input two normal curves a and b in a triangulated orientable surface F , and returns an estimate D of their distance $d(a, b)$ in the curve graph of F such that

$$D \leq d(a, b) \leq K \cdot D + L,$$

where K and L are constants depending polynomially on $|\chi(F)|$. The algorithm runs in time polynomial in $|\chi(F)|$ and the logarithms of the weights of a and b .

In the setting of our certificate, we can apply this algorithm to the curves a_0 and $\varphi^N(a_0)$, thus verifying in time polynomial in t and $|\chi(F)|$ that φ is pseudo-Anosov, and hence that M is hyperbolic.

The following generalisations of Theorem 1 remain open.

Question 6. Is hyperbolicity of fibred 3-manifolds in NP? Even more optimistically, is hyperbolicity of 3-manifolds in NP?

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Finiteness of Anosov and pseudo-Anosov flows on hyperbolic 3-manifolds

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(joint work with David Gabai)

There’s a longstanding program initiated by Smale as described in his 1966 ICM address: Fixing a closed manifold M , what are the Anosov flows it supports?

An active field of study lies at the intersection of 3-manifold topology and dynamical systems regarding Smale’s program for 3-manifolds while generalizing Anosov flows to include pseudo-Anosov flows. Coarsely, Anosov flows are locally modeled on suspension flows of Anosov maps on tori, and pseudo-Anosov flows are locally modeled on suspension flows of pseudo-Anosov maps on higher genus surfaces. The work discussed in this talk is motivated by the following finiteness conjecture:

Conjecture 1. Fixing a closed 3-manifold, there are only finitely many Anosov and pseudo-Anosov flows up to orbit equivalence.

There has been much activity and progress over the years on this conjecture from various viewpoints. More recently, work of Barthelmé–Bowden–Mann proved the finiteness conjecture for Reeb Anosov flows using the classification of transitive pseudo-Anosov flows up to orbit equivalence via free homotopy classes of closed orbits due to Barthelmé–Mann and Barthelmé–Frankel–Mann [2][1]. Zung [8] and Baldwin–Sivek–Zung [3] extended this to show finiteness for pseudo-Anosov flows admitting positive Birkhoff sections in rational homology spheres. Chaidez–Pan [5] proved the finiteness conjecture for pseudo-Anosov Reeb flows, which they define.

Barthelmé–Tsang–Zung [4] have recently used a theorem of Li [7] as the first step in proving the finiteness conjecture for pseudo-Anosov flows without perfect fits.

In this talk, we give an outline for a proof of the following result that resolves the finiteness conjecture for Anosov and pseudo-Anosov flows on hyperbolic 3-manifolds.

Theorem 2. Fixing a closed hyperbolic 3-manifold, there are only finitely many Anosov and pseudo-Anosov flows on it up to orbit equivalence.

First, produce the stable and unstable laminations of the Anosov/pA flows. They are transverse essential laminations. The first step is to prove a pairs version of Gabai’s Kneser normal form theorem [6].

Step 1. Let M be a closed orientable atoroidal 3-manifold. There exists a triangulation τ such that any pair Λ^s, Λ^u of transverse essential laminations coming from stable and unstable laminations of pseudo-Anosov flows can be simultaneously isotoped to be normal to τ while respecting their transversality.

Being able to normalize both laminations to a triangulation only dependent on M offers us the first hint of finiteness, because there is only a finite number of pairs of normal branched surfaces that carry pairs of normal laminations.

Eventually, we want to show finiteness of dynamic pairs, which are pairs of transverse branched surfaces not only carrying the stable and unstable laminations but also carrying dynamical information through a vector field on the branched surfaces. To split normal branched surfaces into dynamic pairs, we need to first ensure that they are transverse to each other. Heuristically, the failure of transversality looks like leaves of the stable and unstable laminations being closely parallel over large areas.

So we quantify what it means for two leaves of the stable and unstable laminations to be closely parallel. The *parallel locus*, which we define in the talk, measures how much of a stable leaf lies inside a neighborhood of an unstable leaf, where the neighborhood is determined by the fibered neighborhood of the unstable branched surface.

Step 2. If there’s an infinite sequence $\{\phi_i\}$ of Anosov or pA flows whose stable and unstable laminations Λ^s, Λ^u admit leaves with arbitrarily large parallel locus, there exists a subsequence $\{\phi_{i_k}\}$ with the following properties after isotopy:

- The stable laminations converge to a limit lamination Λ_{lim}^s ;
- The unstable laminations converge to a limit lamination Λ_{lim}^u ;
- The two limits $\Lambda_{\text{lim}}^s, \Lambda_{\text{lim}}^u$ share a nonempty sublamination Λ_{lim} .

Further examining this shared limit sublamination from stable and unstable laminations tells us that such a sublamination is dynamically impossible. So we conclude that arbitrarily large parallel loci is similarly impossible.

In fact, we show it’s possible to isotope the laminations so that every parallel locus can be covered by radius- R disks for some uniform constant R depending only on the manifold M . The last step, then, is to show that this condition gives the desired finiteness result, aggregated over local patches.

Step 3. Let $X \cong D^2 \times I$ be a subset of M^3 such that a uniform splitting of the stable branched surface Σ^s intersects X as $D^2 \times t$ sheets except in a small neighborhood of the branch locus, and let Σ^u be transverse to Σ^s on the cylindrical boundary $\partial D^2 \times I$ as a bounded number of parallel transversals. Then upon another uniform finite splitting, we obtain pairs of dynamic branched surfaces $\Sigma_{\text{split}}^u, \Sigma_{\text{split}}^s$ such that inside X , it's one of finitely many pairs. The number of possibilities depends only on the number of clumps of parallel transversals on the cylindrical boundary.

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Smooth stable isotopy of topologically isotopic surfaces

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(joint work with Daniel Galvin and Mark Powell)

A *stabilisation* of a smooth, compact, connected, orientable 4-manifold X is the effect of taking the connected sum of X with some number of copies of $S^2 \times S^2$. Topological results about 4-manifolds sometimes become true smoothly, after stabilisation. In the most famous instance, results of Wall [11, 12] in the simply-connected case and Gompf [4], generally, show that homeomorphic closed, orientable, smooth 4-manifolds are stably diffeomorphic.

In some cases, a similar phenomenon is known to occur for isotopy classes of self-maps of a 4-manifold. Given $g \geq 1$ and a diffeomorphism $f: X \rightarrow X$, one may assume after isotopy that f fixes a 4-ball, use that to form the connected sum $X \# g(S^2 \times S^2)$, then extend f by the identity, to get a diffeomorphism of this stabilisation of X . We call this a *stabilisation* of f . When X is closed and simply-connected, Kreck [6] and Quinn [8] (cf. [2]) showed that topologically isotopic diffeomorphisms of X are stably smoothly isotopic. Results of Krannich–Kupers [5] and Gabai [1] combine to extend this to the case of free fundamental group, and

Orson–Powell–Randal-Williams [7] further extend this to a larger class of fundamental groups. It remains open whether this result holds for every compact 4-manifold.

Here is a natural variant of the stabilisation question for embedded surfaces.

Question 1. If two smooth surfaces $\Sigma_1, \Sigma_2 \subseteq X$ are topologically isotopic, are they necessarily smoothly isotopic in some stabilisation of X ?

The first named author provided a positive answer in the case that $\pi_1(X)$ is trivial [3]. Our main result gives a positive answer for some cases in which X is not simply-connected. The question remains open, in general.

Theorem 2. Let X be a smooth, orientable, closed 4-manifold. Suppose $\Sigma_1, \Sigma_2 \subseteq X$ are connected closed surfaces. Suppose at least one of the following holds.

- (1) $[\Sigma_i] = 0 \in H_2(X; \mathbb{Z}/2)$, or
- (2) $H_2(\pi_1(X); \mathbb{Z}/2) = 0$.

Then if Σ_1 and Σ_2 are topologically isotopic, they are smoothly isotopic in some stabilisation of X .

We do not need to assume the surfaces are orientable. The theorem is stated in its simplest form, but also holds when X and/or the surfaces have boundary, in which case the surfaces must be proper. Many groups satisfy condition (2), including: free groups, finite groups of odd order, and classical knot groups.

The theorem provides new examples of the phenomenon of stabilisation dissolving the exoticness of surfaces knots, outside of the $\pi_1(X) = 1$ setting. For example, Torres [10] leveraged existing constructions for exotic, non-simply-connected 4-manifolds to show there exist infinite families of exotic nullhomologous tori in $S^2 \times T^2$ and $S^1 \times S^3$. Our theorem shows these are smoothly isotopic after stabilisation.

The proof of our theorem runs through the result of Gabai [1], which states that if diffeomorphisms $f, g \in \text{Diff}(X)$ are smoothly *pseudo-isotopic*, via a smooth pseudo-isotopy F with vanishing Hatcher-Wagoner obstruction, then they are smoothly stably isotopic. The strategy of the proof is thus to obtain, for some $n \geq 0$, a diffeomorphism of pairs $(X \# n(S^2 \times S^2), \Sigma_1) \cong_{C^\infty} (X \# n(S^2 \times S^2), \Sigma_2)$ that is smoothly pseudo-isotopic to the identity, via a smooth pseudo-isotopy F with vanishing Hatcher-Wagoner obstruction.

In more detail, the steps of the proof are:

- Our initial hypothesis is that there exists a homeomorphism

$$f: (X, \Sigma_1) \xrightarrow{\cong_{C^0}} (X, \Sigma_2)$$

topologically isotopic to the identity. We stably smooth this homeomorphism to a diffeomorphism

$$g: (X \# m(S^2 \times S^2), \Sigma_1) \xrightarrow{\cong_{C^\infty}} (X \# m(S^2 \times S^2), \Sigma_2)$$

that is now only stably topologically pseudo-isotopic to the identity. This smoothing is achieved by killing the Casson-Sullivan stable smoothing obstruction

$$\text{cs}(f|_{X \setminus \nu \Sigma_i}) \in H^3(X \setminus \nu \Sigma_i, \partial; \mathbb{Z}/2).$$

This step is the heart of the proof; we show the obstruction is dual to a meridian of the surface in order to kill it with enough control that we do not fully destroy the initial hypothesis of being topologically isotopic to the identity.

- Next we consider the problem of smoothing the stable topological pseudo-isotopy from g to the identity. Such problems were studied in detail in [7] and we make essential use of the results there. This is the stage where (1) or (2) from the theorem are used. This stage may involve changing g to some new diffeomorphism h .
- Finally, we have a smooth pseudo-isotopy F from a diffeomorphism

$$h: (X \# n(s^2 \times S^2), \Sigma_1) \xrightarrow{\cong_{C^\infty}} (X \# n(s^2 \times S^2), \Sigma_2)$$

to the identity. We can kill its Hatcher-Wagoner obstruction by composing with an inertial pseudo-isotopy with the same Hatcher-Wagoner obstruction, guaranteed to exist by the work of Singh [9].

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The Conway knot has infinite concordance order

PAULA TRUÖL

(joint work with Chiara Donatone, Marc Kegel, Lukas Lewark)

The main focus of this talk is the result stated in the title, which we proved in [5]. Additionally, we will prove a more general result that enables us to establish infinite concordance order of knots using satellite operations, null-homologous twists, and Rasmussen's s invariant [20]. Let us first provide some definitions and context before making this more precise.

Two knots K and J in the 3-sphere S^3 are called *concordant* if they cobound a smoothly and properly embedded annulus $S^1 \times [0, 1]$ in $S^3 \times [0, 1]$. Equivalently, K and J are concordant if and only if the knot $K \# -J$ is *slice*, i.e. if it bounds a smoothly and properly embedded disk in B^4 , the 4-ball bounded by S^3 . Concordance is an equivalence relation, and knots up to concordance form a group, called the *concordance group* $\mathcal{C}$, where the group operation is induced by connected sum. Slice knots represent the identity element, and the additive inverse of a concordance class $[K]$ is $[-K]$. Here, $-K$ is the knot obtained by taking the image of K under an orientation-reversing diffeomorphism of S^3 with the opposite orientation on K . See also [13] for an introduction to knot concordance.

The concordance group is countable and abelian. Levine defined a surjective homomorphism onto the *algebraic concordance group* [12], whose isomorphism type is known to be $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$ [11]. The kernel of this map—the set of *algebraically slice knots*—is also large. In particular, it contains subgroups isomorphic to $\mathbb{Z}^\infty$ and $(\mathbb{Z}/2\mathbb{Z})^\infty$, respectively, [9, 14], as well as a $\mathbb{Z}^\infty$ -summand [4]. Furthermore, the filtration of the topological concordance group due to Cochran–Orr–Teichner [2, 3] reveals additional structure within the kernel. However, the kernel has not yet been fully determined, and the isomorphism type of $\mathcal{C}$ remains unknown.

Understanding the structure of a group is intimately related to understanding its torsion subgroup. For the concordance group $\mathcal{C}$, the structure of its torsion subgroup is an important open problem in the field. For instance, it is unknown whether there exists a torsion element in $\mathcal{C}$ with an order other than two, and whether every concordance class of knots of order two is represented by a strongly negatively amphichiral knot; see [1, Problem 1.38]. We call the order of a knot K in $\mathcal{C}$ its *concordance order*, denoted by $\text{ord}(K)$. This is the least positive integer $n \geq 1$ for which the connected sum $K^{\#n}$ of n copies of K is slice, or infinity if no such n exists.

Early examples of the computation of $\text{ord}(K)$ for knots K included applying the Fox–Milnor condition [6] to show that the figure-eight knot has order two, and using the lower bound from the signature [18, Theorem 9.1] to show that the trefoil knot has infinite order. Jumping ahead in time, a significant recent development in the study of the concordance group $\mathcal{C}$ is the following celebrated result of Lisa Piccirillo.

Theorem 1 ([19]). The Conway knot C is not slice.

Piccirillo's proof completed the classification of slice knots with up to 12 crossings. More importantly, her ingenious method of proof opened up a new area of research concerning the use of so-called *RBG links*, while also drawing attention to the folklore trace embedding lemma.

Theorem 1 solved a long-standing, fifty-year-old open problem posed by John Horton Conway. He had observed that the sliceness status of the Conway knot was an interesting but difficult problem because this knot has a *mutant*: the Kinoshita–Terasaka knot, which is slice. As mutants, the two knots share many invariants, such as the Alexander, Jones, HOMFLY, and Kauffman polynomials [17]; the hyperbolic volume [22]; and the two-fold branched covers. Both knots have trivial Alexander polynomials, so they are algebraically and topologically slice [7, 8], and all classical obstructions to sliceness (those that can be derived from the Seifert form of a knot) vanish.

In summary, determining the non-sliceness of the Conway knot was difficult because all known sliceness invariants vanished for this knot. In particular, all known additive invariants vanish for C . Despite this, we prove the following.

Theorem 2 ([5]). The Conway knot has infinite concordance order.

This concludes the computation of concordance orders for knots with at most 11 crossings; see [15]. Furthermore, our proof provides a new method of establishing infinite order for knots, essentially based on the following statement. Here, $P(K)$ denotes the satellite knot with pattern P and companion K ; see, for example, [21, Section 4D].

Theorem 3 ([5]). Let K be a knot in S^3 that can be obtained from the unknot U via a finite sequence of concordances and right-handed null-homologous twists. Suppose that there is a pattern P such that $s(P(K)) \neq s(P(U))$. Then $\text{ord}(K) = \infty$.

In Theorem 3, the Rasmussen s invariant from [20] can be replaced with any other knot concordance invariant that is non-decreasing under right-handed null-homologous twists. Manolescu–Marengon–Sarkar–Willis showed that s is non-decreasing under such twists, since it satisfies an adjunction inequality in the punctured connected sum of $\overline{\mathbb{C}P}^2$ s; see [16, Corollary 1.9 and Theorem 1.11].

Piccirillo's proof of Theorem 1 uses a clever construction which involves finding a knot K' with the same 0-trace as C . According to the trace embedding lemma (see [19, Lemma 1.3] and [10]), this gives K' the same sliceness status as C . However, computing the s invariant for K' shows that K' is not slice.

To find a pattern P for which we can apply Theorem 3 with $K = C$ the Conway knot, we generalise Piccirillo's construction. Specifically, given a knot K that can be unknotted by changing one crossing in a diagram of K , we demonstrate the existence of a pattern P_K such that the knots $K \# J$ and $P_K(J)$ have the same sliceness status for all knots J . Indeed, we use an RBG link construction to show that $K \# J$ and $P_K(J)$ have the same 0-trace and employ the trace embedding lemma. For $K = C$, we obtain $P_C(U) = K'$. Moreover, by construction, $P_C(-C)$ and $C \# -C$ have the same sliceness status, so $P_C(-C)$ is slice. By the computation

of $s(K')$ from [19], this implies

$$s(P_C(-C)) = 0 \neq 2 = s(K') = s(P_C(U)).$$

Applying Theorem 3 to $K = -C$ now yields the desired result.

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A complete set of contact Kirby moves

MARC KEGEL

(joint work with Eric Stenhede and Vera Vértési)

A *contact structure* on a smooth, oriented 3-manifold M is a tangential hyperplane field $\xi \subset TM$, that can be written as the kernel of a 1-form α , such that $\alpha \wedge d\alpha$ is a volume form for M . A smooth link L in a contact 3-manifold (M, ξ) is called *Legendrian* if its tangent vectors lie in the contact planes, i.e. $TL \subset \xi$. Legendrian links in the standard contact 3-space $(\mathbb{R}^3, \ker xdy + dz) \subset (S^3, \xi_{st})$ are determined by their *front projections*, i.e. the projection to the yz -plane.

The *contact framing* of a Legendrian knot L is obtained by pushing L in a direction transverse to ξ . Contact (± 1) -surgery is topological Dehn surgery with coefficient ± 1 measured with respect to the contact framing, together with a preferred extension of the contact structure over the glued-in solid torus. A *contact surgery diagram* consists of a Legendrian link together with surgery coefficients ± 1 and uniquely describes the contactomorphism type of a contact 3-manifold.

A fundamental theorem of Ding and Geiges [2] states that every closed, connected contact 3-manifold is obtained from (S^3, ξ_{st}) by contact (± 1) -surgery along a Legendrian link.

The literature contains several examples of *contact Kirby moves*, i.e. explicit operations on contact surgery diagrams which replace one diagram by another without changing the contactomorphism type of the surgered contact manifold. Besides planar isotopies and Legendrian Reidemeister moves, one can insert or remove a *standard cancelling pair*, consisting of two vertical Legendrian push-offs carrying opposite surgery coefficients. Ding and Geiges also introduced contact handle slides [3]. These moves are extremely useful, but they do not form a complete calculus. For instance, the parity of the number of surgery components is preserved by Legendrian isotopy, cancelling pairs and contact handle slides, while there are odd-component contact surgery diagrams for (S^3, ξ_{st}) .

In joint work with Eric Stenhede and Vera Vértési [7], we introduce two further types of moves, called *lantern moves* and *chain moves*. As their names suggest, they arise from the lantern and chain relations in mapping class groups. Together with the previously known moves they give a complete contact Kirby calculus.

Theorem 1. Two contact surgery diagrams describe contactomorphic contact 3-manifolds if and only if their front projections are related by a sequence of

- planar isotopies,
- Legendrian Reidemeister moves,
- insertions or removals of standard cancelling pairs,
- the two standard contact handle slides,
- standard lantern moves, and
- standard chain moves.

These moves are shown below. The shaded region in the standard contact Kirby moves is assumed to be disjoint from all components of the contact surgery link except those explicitly involved in the move.



FIGURE 1.1. A standard cancelling pair.

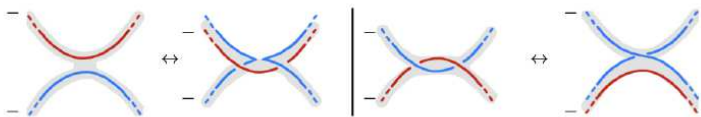


FIGURE 1.2. The two standard contact handle slides needed in Theorem 1.1.

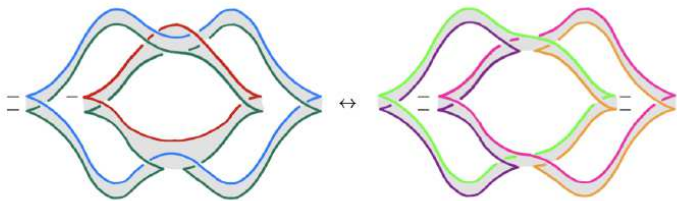


FIGURE 1.3. The standard lantern move.

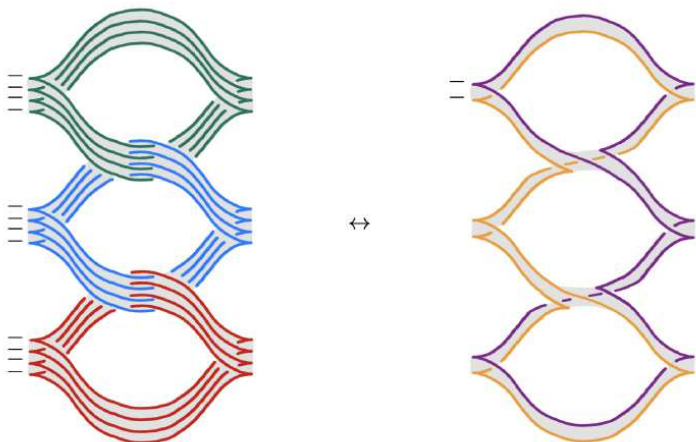


FIGURE 1.4. The standard chain move.

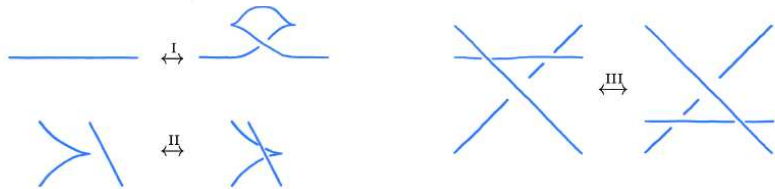


FIGURE 3.4. Legendrian Reidemeister moves for Legendrian links. Horizontal and vertical reflections of these moves are also allowed, with crossings adjusted appropriately. No other Legendrian strand of the diagram is allowed to appear in these pictures.

By analyzing how combinatorial properties of contact surgery diagrams change under the standard Kirby moves, we see that they are independent of each other.

Theorem 2. None of the standard cancelling pair, contact handle slides, lantern move, or chain move can be expressed as a sequence of the others together with planar isotopies and Legendrian Reidemeister moves.

One application is a general method for proving that quantities defined from contact surgery diagrams are genuine contact invariants by verifying invariance under the standard moves. As an example, we apply this to the surgery formula for Gompf's d_3 -invariant, giving a diagrammatic proof of its well-definedness [6, 7].

In general, it would be even more interesting to do the same for invariants of contact structures that do not depend only on the underlying tangential 2-plane field. Such invariants are usually defined using holomorphic curves.

We finish by sketching the proof of the contact Kirby theorem. The main input is Avdek's ribbon-move framework [1], which in turn is based on Giroux's correspondence between contact structures and open book decompositions [5]. A Legendrian surgery link can be placed on pages of an open book supporting (S^3, ξ_{st}) , and contact surgery then changes the monodromy by composing it with Dehn twists along the corresponding curves. Up to known contact Kirby moves, Avdek's results allow us to compare two surgery links for the same contact manifold on a common page. The problem is thereby converted into comparing two factorizations of the same element of the mapping class group of that page.

We then use Gervais' presentation of the mapping class group [4]. Its relevant relations are cancellation, commutativity, braid, lantern and chain relations between Dehn twists. These relations have direct counterparts in the surgery calculus: cancelling Dehn twists give cancelling pairs, braid relations give the standard contact handle slides, and the lantern and chain relations give precisely the new lantern and chain moves. Commutativity corresponds to rearranging disjoint curves. Positive stabilizations of the open book can be realized by a cancelling pair together with a handle slide. The main technical step is to show that each mapping class group relation has a unique standard local model at the level of front projections.

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Unknotting number of knots

JUNGHWAN PARK

(joint work with Hayato Imori, Duncan McCoy, Masaki Taniguchi)

The unknotting number $u(K)$ of a knot $K \subset S^3$ is the minimal number of crossing changes required to transform K into the unknot. Although the definition is elementary, the invariant is difficult to compute and can behave in unexpected ways. For example, Scharlemann proved that knots of unknotting number one are prime [8], while recent work of Brittenham and Hermiller shows that the unknotting number is not additive under connected sum [1]. The talk discussed two approaches to this invariant: one using equivariant singular instanton theory, and one using Donaldson theory and lattice combinatorics for special alternating knots.

The first part concerned same-sign unknotting operations. A natural strengthening of Scharlemann's theorem asks whether every band sum of two nontrivial knots must have unknotting number greater than one. This is closely related to the question of whether unknotting number is monotone under ribbon concordance. Indeed, Miyazaki proved that a band sum of K_1 and K_2 is ribbon concordant from the connected sum $K_1 \# K_2$ [6].

In joint work with Imori and Taniguchi, we use equivariant singular instanton Floer theory with the Chern–Simons filtration to obstruct unknotting sequences all of whose operations have the same sign [4]. The input is the instanton local class homomorphism

$$\Omega_{E,\omega}: \mathcal{C} \longrightarrow \Theta_{R_\omega}^{E,\omega},$$

constructed from enriched singular instanton $\mathcal{S}$ -complexes. Suppose that K is slice and that there is a ribbon concordance $K_1 \# K_2 \rightarrow K$. If one summand, say K_1 , has $\Omega_{E,\omega}([K_1]) \neq [0]$ for some $\omega \in (0, 1/2)$, then any unknotting sequence for K by null-homologous twists must contain both positive and negative twists. In particular, under the same hypothesis, if K is obtained as a band sum of two nontrivial knots, then K has unknotting number greater than one.

The proof proceeds by contradiction. A same-sign sequence of null-homologous twists gives, after blowing up, a normally immersed $\mathbb{Z}$ -concordance whose complement has cyclic fundamental group. Composing with the ribbon concordance gives a negative definite cobordism of pairs. The filtered singular instanton cobordism inequality, together with its equality case, forces the existence of an irreducible $SU(2)$ representation of this cyclic group with prescribed meridional holonomy. This is impossible, since representations of a cyclic group in $SU(2)$ are reducible. This yields, for instance, the following application: if K is a non-slice strongly quasipositive knot and $Wh(K)$ denotes its positive Whitehead double, then

$$J_K = Wh(K) \# -Wh(K)$$

has untwisting number $tu(J_K) = 2$, and every untwisting sequence of minimal length must use twists of both signs.

The same construction also has a 3-manifold analogue: it gives evidence for monotonicity of the Dehn surgery number under ribbon homology cobordism.

The second part of the talk concerned alternating knots and diagrams. For any oriented non-split k -component alternating link L , the classical signature gives the lower bound

$$u(L) \geq \frac{|\sigma(L)| + k - 1}{2}.$$

In joint work with McCoy, we prove that for special alternating links, sharpness of this lower bound is visible in every alternating diagram [5]. More precisely, if L is an oriented non-split special alternating link, then the following are equivalent: the unlinking number is equal to $(|\sigma(L)| + k - 1)/2$; the 4-ball crossing number is equal to the same number; and L can be unlinked by that many crossing changes in any alternating or minimal diagram. For a special alternating knot K , this gives

$$u(K) = |\sigma(K)|/2$$

if and only if K can be unknotted by $|\sigma(K)|/2$ crossing changes in any alternating or minimal diagram.

The proof again uses 4-dimensional topology. Starting from equality in the signature lower bound, the double branched cover bounds a smooth spin negative definite 4-manifold containing disjoint spherical classes of self-intersection -2 , as in the framework of Cochran–Lickorish and Nagel–Owens [2, 7]. Gluing this filling to the positive-definite Goeritz 4-manifold associated to an alternating diagram and applying Donaldson’s diagonalization theorem [3] produces a constrained lattice embedding. In the special alternating case, the Goeritz lattice and this embedding can be analyzed combinatorially, and the required crossing changes can be read off from the diagram. As applications, we obtain new unknotting-number computations for several special alternating knots with 11 and 12 crossings, and an additivity result for connected sums of special alternating knots satisfying the sharp signature bound.

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Continents for drawing Cannon-Thurston maps and drilling veering triangulations

HENRY SEGERMAN

(joint work with Anna Parlak, Saul Schleimer)

I described some of the algorithms and ideas involved in the veering code base (joint work with Anna Parlak and Saul Schleimer). Specifically, a *continent* is a ball in the universal cover of a veering triangulation, formed from a finite collection of tetrahedra, with no “overhangs”. That is, for each edge e on the boundary of the continent, the sum of the angles at e coming from tetrahedra inside the continent is either π or zero, and the zero angle edges form a single loop on the boundary of the continent. This loop is also called the *coast* of the continent.

[2, Proposition 5.3] states that the universal cover of a veering triangulation can be exhausted by nested continents, with consecutive continents differing by a single tetrahedron. We implemented the algorithm used to prove this result in our code base *Veering* [4]. This allows us to build arbitrarily large continents which serve as finite approximations of the universal cover.

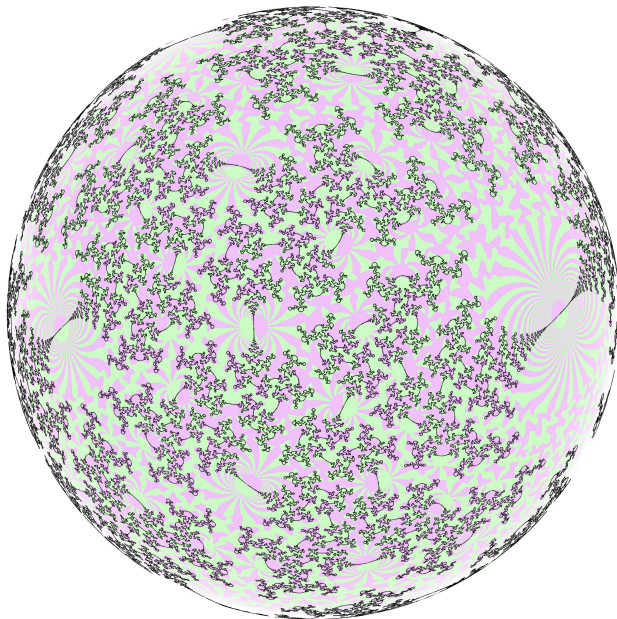


FIGURE 1. Two approximations to the Cannon-Thurston map associated to the figure-eight knot complement. Thurston’s technique is in black; ours is the common boundary of the green and purple regions.

The original use-case for this code was to be able to draw Cannon-Thurston maps. This is based on joint work in progress with Jason F. Manning and Saul Schleimer. Given a finite volume hyperbolic three-manifold M , a *Cannon-Thurston map* for M is a continuous equivariant map of a circle into the boundary of the universal cover of M . Cannon and Thurston’s original map has as its domain circle the boundary of an elevation of a fibre of M . We generalise to use the *veering circle*. This recovers the previous Cannon-Thurston map for veering triangulations corresponding to fibrations, but produces new examples for other veering triangulations.

Thurston generated pictures of the Cannon-Thurston map for the figure-eight knot complement by approximating an elevation of a fibre with a large disk in the universal cover. We replace this disk with a large continent. Here the veering circle is approximated by the coast of the continent. Figure 1 shows a comparison of the two techniques. The extra structure of a continent allows us to improve on Thurston’s image: we more closely approximate the true sphere-filling path while doing considerably less work.

Anna Parlak and I repurposed the continent code for the task of computationally drilling veering triangulations along a collection of *flow cycles*. Flow cycles, introduced by Landry, Minsky, and Taylor [3], combinatorially encode orbits of a pseudo-Anosov flow. Here we again generate an approximation to the full universal cover, guided this time by the need to know the relative ordering of leaves in the universal cover of the orbit space. This information allows us to construct the tetrahedra of the drilled veering triangulation by identifying the maximal rectangles in the drilled orbit space.

This algorithm allowed us to explore several questions computationally. First, with Matthias Goerner, we find an orbit of the suspension flow on the figure eight knot complement that is not ambiently isotopic to the geodesic in its homotopy class. We conjecture that every (drilled) pseudo-Anosov flow admits such orbits. Second, for each non-layered veering triangulation $\mathcal{V}$ in the veering census, we identify the minimal-length flow cycle of $\mathcal{V}$ that yields a layered veering parent. This data may be helpful for improving the existing theoretical results on Birkhoff sections for pseudo-Anosov flows. Finally, we show that all edge-orientable veering triangulations with four ladders in each boundary component and at most sixteen tetrahedra have a common veering ancestor. This supports a conjecture of Ghys stating that all Anosov flows with orientable invariant foliations are almost orbit equivalent.

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Branched covering representation of 4-manifolds

VALENTINA BAIS

(joint work with Riccardo Piergallini, Daniele Zuddas)

In this talk, we discuss branched coverings of piecewise linear 4-manifolds. Recall that, given two piecewise linear n -manifolds M and N , a piecewise linear homeomorphism

$$p: M \longrightarrow N$$

is a d -fold branched covering if and only if there exists a codimension two closed subcomplex $B \subset N$ such that the restriction of p over $N \setminus B$ is an ordinary d -fold covering. The *branch set* of p is the minimal codimension two closed subcomplex of N with this property and is usually denoted by B_p . One can show that the manifold M is determined up to piecewise homeomorphism by the pair (N, B_p) and the monodromy representation $\omega_p: \pi_1(N \setminus B_p) \rightarrow S_d$ associated to the restriction of p over $N \setminus B_p$. Here S_d denotes the permutation group of d letters.

By a classical result of Alexander [1], any closed connected oriented PL n -manifold M admits a branched covering

$$p: M \longrightarrow S^n$$

of some degree $d \geq 1$. This means that any such manifold M can be presented by means of a codimension two closed subpolyhedron $B_p \subset S^n$ and a group homomorphism $\omega_p: \pi_1(S^n \setminus B_p) \rightarrow S_d$. We remark that such a presentation is not optimal, since the degree d depends on the cardinality of a chosen triangulation of M and the branch set B_p is not a PL submanifold. However, the following improvements of Alexander's result in low dimensions are known. The easiest case is when $M = \Sigma_g$ is a closed oriented surface of genus g : one gets a simple d -fold branched covering $\Sigma_g \longrightarrow S^2$ for any degree $d \geq 2$ by stabilizing the quotient map induced by the hyperelliptic involution. Moving to the case $n = 3$, Hilden [3], Hirsch [4] and Montesinos [6] independently proved that any closed oriented 3-manifold can be represented by means of a 3-fold dihedral covering of S^3 branched over a knot. The 4-dimensional case has been considered by Piergallini and Iori [7, 8], who showed that any closed connected oriented PL 4-manifold M can be represented by a simple d -fold branched covering $p: M \rightarrow S^4$ for any $d \geq 4$, where the branch set $B_p \subset S^4$ is a locally flat PL embedded surface if $d \geq 5$ while it is a locally flat PL self-transversally immersed surface if $d = 4$. The case when $d = 3$ and M has no 1-handles has been studied in [2] using trisections.

In general, one would hope to give a complete answer to the following question.

Question 1. Given two PL n -manifolds M and N and a natural number $d \geq 2$, is there a d -fold branched covering

$$M \longrightarrow N?$$

Criteria for the existence of a simple d -fold branched covering $p: M \rightarrow N$ with $N = \#_{k_1} \mathbb{CP}^2 \#_{k_2} \overline{\mathbb{CP}}^2$ or $N = \#_k (S^2 \times S^2)$ are given by Piergallini and Zuddas [9] in terms of the Betti numbers $b_2^\pm(M)$.

In a joint work with Riccardo Piergallini and Daniele Zuddas, we generalize the results in [9] to all the closed 4-manifolds N which admit a handle decomposition without 1- and 3-handles, thus providing a partial answer to the above Question for simply connected PL 4-manifolds. We recall that it is a long standing open problem in low dimensional topology that any closed PL simply connected 4-manifold N admits a handle decomposition with no 1- and 3-handles, see [5, Problem 4.18] in Kirby's list.

Theorem 2. Let M and N be two closed connected oriented PL 4-manifolds such that N admits a handle decomposition with no 1- and 3-handles. For any $d \geq 4$, there exists a d -fold branched covering

$$p: M \rightarrow N$$

if and only if there exists an isometric embedding of lattices

$$d \cdot I_N \hookrightarrow I_M,$$

where I_N and I_M respectively denote the intersection lattices of N and M . Moreover, if this is the case, one can build p in such a way that it is simple and its branch set $B_p \subset N$ is a locally flat PL embedded surface if $d \geq 5$ while it is a locally flat PL self-transversally immersed surface if $d = 4$.

We also show that there exists an embedding $d \cdot I_N \hookrightarrow I_M$ for some $d \geq 1$ if and only if $b_2^\pm(N) \leq b_2^\pm(M)$. Moreover, we provide a list of achievable values of d , depending on the parity and the signature of the intersection forms of M and N .

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PL Recognition of $\#_2(S^2 \times S^2)$ is PSPACE-Hard

RHUAIDI ANTONIO BURKE

For a fixed closed PL manifold X , the recognition problem asks whether an input triangulation of a closed PL manifold is PL-homeomorphic to X . Markov's work shows that such problems become undecidable in dimensions at least four for some fixed targets [5]. More recent constructions give unrecognisable targets of the form $\#_k(S^2 \times S^2)$, with k as low as nine [7]. On the other hand, it is unknown whether the standard PL 4-sphere is recognisable. The following result gives a complexity obstruction for a target much closer to S^4 .

Theorem 1. Fixed-target PL recognition of $\#_2(S^2 \times S^2)$ among closed triangulated PL 4-manifolds is PSPACE-hard under polynomial-time many-to-one reductions. More precisely, from a straight-line program A over the standard generators of Thompson's group F , one can construct in polynomial time a triangulation of a closed PL 4-manifold X_A such that

$$\text{val}(A) = 1 \text{ in } F \iff X_A \cong_{\text{PL}} \#_2(S^2 \times S^2).$$

In particular, the recognition problem is also NP-hard. We outline the three parts of the construction below.

The compressed algebraic input. The compressed word problem in F is complete for PSPACE [1]. A straight-line program is an acyclic circuit whose gates record letters and products; its output word can be exponentially longer than the circuit. For every gate of A we introduce a generator y_i and a defining relator. Together with the two generators and two relators in the standard presentation of F , these form a balanced presentation Q_A of F . Its distinguished output generator y_{out} represents $\text{val}(A)$. A presentation with k more relators than generators is k -Andrews–Curtis trivial if its relator tuple can be transformed into the list of generators followed by k empty relators. The allowed moves invert a relator, multiply it by another relator or its inverse, conjugate it, or permute the relators. Adjoining the two standard generators of F as relators makes Q_A 2-Andrews–Curtis trivial.

A surplus-controlled modification of the Miller–Tancer construction [6, 7] applied to (Q_A, y_{out}) produces a presentation P_A having two more relators than generators. It has the dichotomy

$$\begin{aligned} \text{val}(A) = 1 &\implies P_A \text{ is 2-Andrews–Curtis trivial,} \\ \text{val}(A) \neq 1 &\implies F \hookrightarrow G(P_A). \end{aligned}$$

The no-case follows from a normal-form argument for an amalgamated free product: suitably conjugated gadget words freely generate the two amalgamated subgroups. The yes-case is controlled at the level of relator tuples, so the two surplus relators remain visible throughout the Andrews–Curtis reduction. Every gate contributes only bounded-length presentation data, and the exponentially long word $\text{val}(A)$ is never expanded.

From presentations to closed 4-manifolds. For a finite presentation P , construct a 4-dimensional 2-handlebody $W(P)$: generators give dotted components in a Kirby diagram and relators give 0-framed attaching circles. Thus $\pi_1(W(P)) \cong G(P)$. The diagram has polynomially many crossings, and an algorithm due to Casali–Cristofori [2] converts it into a polynomial-size triangulation of $W(P)$ with its boundary coned to a distinguished vertex. Truncating this vertex recovers $W(P)$: the vertex is ideal unless its link is S^3 , in which case truncation removes the attached 4-ball. Doubling along the resulting boundary gives the closed PL 4-manifold

$$X(P) = D(W(P)) \cong_{\text{PL}} \partial(W(P) \times I).$$

The inclusion $\partial W(P) \rightarrow W(P)$ is surjective on fundamental groups, so van Kampen’s theorem gives $\pi_1(X(P)) \cong G(P)$.

If P is k -Andrews–Curtis trivial, its presentation complex 3-deforms to $\bigvee_k S^2$. The 5-manifold $W(P) \times I$ collapses onto the presentation complex and has $w_2 = 0$, since all 2-handles have even framing. Transport of spines under 3-deformation [3] and the uniqueness of the zero- w_2 orientable 5-dimensional thickening [4, 8] therefore give

$$W(P) \times I \cong_{\text{PL}} \natural_k(S^2 \times D^3), \quad X(P) \cong_{\text{PL}} \#_k(S^2 \times S^2).$$

Finally set $X_A = X(P_A)$. In the yes-case, 2-Andrews–Curtis triviality identifies X_A with $\#_2(S^2 \times S^2)$. In the no-case, $F \hookrightarrow G(P_A) \cong \pi_1(X_A)$, so X_A is not simply connected and cannot be PL-homeomorphic to the target.

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Towards a structure theory for rank two Kleinian groups

ALEX ELZENAAR

A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{Möb}^+(\mathbb{S}^3)$. If Γ is a Kleinian group, then Γ acts discontinuously on $\mathbb{H}^3$ and $\mathbb{H}^3/\Gamma$ is a hyperbolic orbifold. Further, Γ acts discontinuously on an open subset $\Omega(\Gamma) \subset \mathbb{S}^2 = \partial\mathbb{H}^3$, and $\Omega(\Gamma)/\Gamma$ is a (possibly empty, possibly disconnected, possibly punctured) Riemann surface.

Suppose that $\eta : \mathbb{C}^n \rightarrow \mathrm{PSL}(2, \mathbb{C})^m$ is a holomorphic parameterisation of m -sets of matrices with n complex parameters, such that $\langle \eta(\mathbf{0}) \rangle$ is Kleinian. (We have the holomorphism condition since when $n = 1$ this allows for the use of the rich theory of holomorphic motions [2, Chapter 12]; higher-dimensional analogues of this are not well-understood.) The set

$$\{\mathbf{z} \in \mathbb{C}^n : \langle \eta(\mathbf{z}) \rangle \text{ is discrete and } \mathbb{H}^3/\langle \eta(\mathbf{z}) \rangle \simeq_{\text{homeo.}} \mathbb{H}^3/\langle \eta(\mathbf{0}) \rangle\}$$

is very wild in general; for instance, not locally connected in the subspace topology from $\mathbb{C}^n$ (see e.g. [12]).

We now restrict to rank two groups. First suppose both generators are elliptic or parabolic. If the orders are $a, b \in \mathbb{N} \cup \{\infty\}$ then up to conjugacy there is a one-parameter family of such generating sets:

$$\Gamma_z = \left\langle \begin{bmatrix} e^{i\pi/a} & 1 \\ 0 & e^{-i\pi/a} \end{bmatrix}, \begin{bmatrix} e^{i\pi/b} & 0 \\ z & e^{-i\pi/b} \end{bmatrix} \right\rangle.$$

The set of all z such that Γ_z is discrete has been completely classified. The parabolic-parabolic discrete groups are classified by Keen and Series [9], Maskit and Swarup [11], and Ohshika and Miyachi [13] (free groups) and by Akiyoshi, Ohshika, Parker, Sakuma, Yoshida [1] (non-free groups). The groups with possibly elliptic generators have been classified by Elzenaar, Martin, and Schillewaert [4, 8] and Chesebro, Martin and Schillewaert [3]. Essentially, there is one large open set $\mathcal{R}$ of flexible discrete groups which is a quotient of a Teichmüller space of four-times marked spheres; the closure of this set parameterises all Γ_z which are discrete and isomorphic to $\mathbb{Z}/a\mathbb{Z} * \mathbb{Z}/b\mathbb{Z}$. In addition there are isolated discrete groups, either lying on the real axis or accumulating onto $\partial\mathcal{R}$. The groups on the real line are classified by standard results on Fuchsian groups [14, 10]. All the others are virtually (up to index 2) holonomy groups of orbifold structures on 2-bridge link complements with possibly singular unknotting tunnels. In particular, if a group Γ_z is discrete, torsion-free, non-elementary, and non-Fuchsian, then it is either a 2-bridge link group or lies in $\overline{\mathcal{R}}$. One application of this work is to computing the radius of convergence of q -rational numbers [7].

One can ask how much of this picture generalises to the general rank two case. It is easy to make some naïve conjectures, of which here are two:

Conjecture 1. There exists a Keen–Series pleating ray theory for genus two Schottky space.

The case of rational pleating rays is discussed in my recent preprint [5]. In an upcoming new version of this preprint, a description of irrational pleating rays

will also be given (arising from recent conversations with Bruno Dular) which will complete the picture of the coordinate system, modulo understanding the asymptotic behaviour of the pleating rays which relies (in the one-dimensional case) on explicit estimates which are not available in general.

Conjecture 2. If α, β, γ are disjoint simple closed curves on the boundary of a genus two handlebody and $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is a discrete, non-elementary, non-Fuchsian representation of the fundamental group of the genus two handlebody such that

$$(*) \quad \mathrm{tr}^2 \rho(\alpha) = \mathrm{tr}^2 \rho(\beta) = \mathrm{tr}^2 \rho(\gamma) = 4,$$

then either:

- (1) If all three of $\rho(\alpha), \rho(\beta), \rho(\gamma)$ are parabolic, then $\Gamma = \rho(F_2)$ is a maximal cusp group on the boundary of genus two Schottky space and $\Omega(\Gamma) \neq \emptyset$.
- (2) If exactly two are parabolic, then Γ is a tunnel number 1 link group in some ambient closed 3-manifold.
- (3) If 0 or 1 are parabolic, then Γ is a Dehn filling of a group appearing in 2.

In recent work [6] I show that 1. of Conjecture 2 is false. There exists a set of three curves α, β, γ and a discrete representation ρ such that $\rho(\alpha), \rho(\beta), \rho(\gamma)$ are all parabolic but $\rho(F_2)$ acts ergodically on the 2-sphere, i.e. $\Omega(\Gamma) = \emptyset$. Thus, unlike the case of elliptic or parabolic generators, the assumption $\Omega \neq \emptyset$ is necessary to deduce that a group realising three disjoint curves on the boundary of the genus 2 handlebody by parabolics is a maximal cusp group.

In upcoming work, I will give a structure theory for the groups which appear in 2. and 3. of Conjecture 2, i.e. tunnel number 1 link groups and their Dehn fillings. This is a first step to understanding the general solution set of (*).

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Three-manifolds that do not admit veering triangulations

LAYNE HALL

Veering triangulations are rigid combinatorial structures on cusped hyperbolic three-manifolds. They have been studied in recent years due to their deep correspondence with Pseudo-Anosov flows [3, 1]. A natural problem is to determine which manifolds admit such triangulations. Large-scale surveys suggest that cusped hyperbolic three-manifolds which have veering triangulations are rare [4, 6]. We discuss forthcoming work addressing this gap.

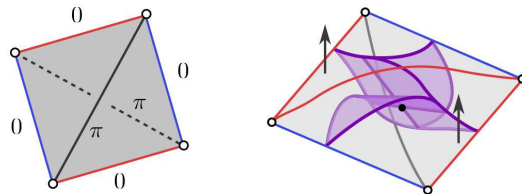


FIGURE 1. A veering tetrahedron (left) and one of its associated branchings (right).

Schmalian gave the first few examples of three-manifolds that do not admit veering triangulations [8]. Fix M , an oriented cusped hyperbolic three-manifold. To obstruct a veering triangulation on M , they pass to every double cover, then Dehn fill along various slopes. The fillings are then compared to Dunfield’s data on closed manifolds that do not admit cooriented taut foliations [2]. Thus, this test requires large amounts of data on closed manifolds. To produce bigger examples, we need a local criterion.

Definition 1. Let $\mathcal{T}$ be an ideal triangulation of M . We say that $\mathcal{T}$ is *branchable* if the dual spine of $\mathcal{T}$ can be smoothed to a branched surface with generic vertices.

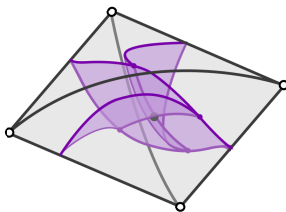


FIGURE 2. A branching on an ideal tetrahedron.

A branching has the local model shown in Figure 2.

An ideal triangulation is *essential* if no edge of the triangulation is homotopic into the boundary.

Theorem 2. Suppose that M admits a veering triangulation. Let $\mathcal{T}$ be an essential ideal triangulation of M . Then $\mathcal{T}$ is branchable.

One can decide whether a given triangulation is branchable with a straightforward search. Applying this to Shana Li’s recently compiled census [6], we deduce:

Theorem 3. At least 165,065 manifolds in the census of orientable cusped hyperbolic three-manifolds up to eleven tetrahedra do not admit veering triangulations.

These examples constitute roughly 23% of all the manifolds in the census.

The local nature of being nonbranchable means that it is amenable to ‘cut-and-paste’ operations. We use this to build the first explicit infinite family of hyperbolic three-manifolds that do not admit veering triangulations.

Theorem 4. There are infinitely many cusped hyperbolic three-manifolds that do not admit veering triangulations.

A veering triangulation has two natural branchings called the *veering branched surfaces*. When these branched surfaces are cooriented, they each impart a $\pi_1(M)$ -invariant partial order on the cusps of the universal cover $\widetilde{M}$. The key technical step for proving Theorem 2 is to extend the partial order to a total order. We build this total order by studying the *loom space* of a veering triangulation [7], and adapting the core idea of [9].

Now let $\mathcal{V}$ be a veering triangulation of M , and let $\mathcal{T}$ be an essential triangulation of M . By [5], we can connect $\mathcal{V}$ to $\mathcal{T}$ through a sequence of essential triangulations using Pachner 2 – 3 and 0 – 2 moves. However, without a total order, there are choices in how to transport a branching. More critically, a branching need not survive through tetrahedron-reducing moves; see Figure 3. A total order both resolves the ambiguity and ensures that we do not encounter these branch-breaking moves.

We conclude with a natural converse question.

Question 5. Given an arbitrary branching, how can we decide whether it arises from a veering triangulation?

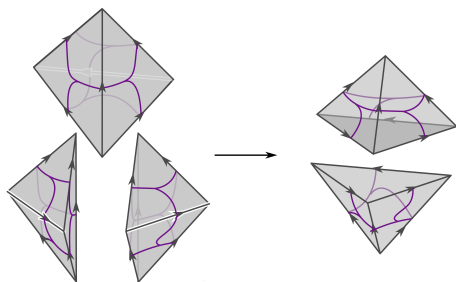


FIGURE 3. A branching that does not survive a $3 \rightarrow 2$ Pachner move. The oriented edges form a cycle in the local order on the cusps induced by the branching.

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Unclassifiability in manifold topology

ROBERT GOMPF

(joint work with Aristotelis Panagiotopoulos)

In descriptive set theory (DST), a hierarchy has been developed for measuring the difficulty of classifying sets with the cardinality of the continuum, analogously to computational complexity for finite problems and recursion theory for countably infinite problems. This theory can now be applied in many areas of mathematics. We discuss its application to various classification problems for open manifolds. Several basic problems reach the first nontrivial level:

Theorem 1. Diffeomorphism types of

- a) contractible, open 3-manifolds and
- b) exotic smoothings of $\mathbb{R}^4$

are not classifiable by any countable collection of real invariants.

As naively interpreted, this is obviously false: Both sets have the cardinality of the continuum (in ZFC set theory), so any bijection to $\mathbb{R}$ is a complete invariant. Of course, we can't even define this invariant explicitly, let alone compute it. Thus, we are asking for invariants we can define explicitly – more precisely, without the axiom of choice.

We prove (a) by considering the set $\mathcal{W}$ of diffeomorphism types of *Whitehead manifolds*. Such a manifold is constructed from an unknotted solid torus in S^3 by repeated Whitehead doubling to create an infinite nest of solid tori, then removing the infinite intersection from S^3 . The resulting 3-manifold is contractible but not diffeomorphic to $\mathbb{R}^3$. At each stage, we have a binary choice of which handedness the clasp should have, so $\mathcal{W}$ is naturally indexed by all binary sequences. That is, we have a surjection $\Pi_{\mathbb{N}}\mathbb{Z}_2 \rightarrow \mathcal{W}$. This map is not injective: Since the solid torus at each finite stage is unknotted in S^3 , throwing away the first n terms of a sequence does not change the resulting diffeomorphism type. In particular, we can change any finite collection of terms, so the indexing descends to a map $\Pi_{\mathbb{N}}\mathbb{Z}_2 / \oplus_{\mathbb{N}}\mathbb{Z}_2 \rightarrow \mathcal{W}$. This is still not injective since we can also shift the indices of a sequence without changing its image, but the point preimages are countable. Now (a) follows from a variant of a well-known result in DST:

Theorem 2. Suppose an equivalence relation “ $\sim$ ” on $\Pi_{\mathbb{N}}\mathbb{Z}_2$ descends to $\Pi_{\mathbb{N}}\mathbb{Z}_2 / \oplus_{\mathbb{N}}\mathbb{Z}_2$ and has countable equivalence classes. Then $\Pi_{\mathbb{N}}\mathbb{Z}_2 / \sim$ is not classifiable by any countable collection of real invariants.

More precisely, every Baire map $f: \Pi_{\mathbb{N}}\mathbb{Z}_2 \rightarrow \mathbb{R}^\omega$ satisfying the given hypotheses has some comeager point preimage (in the product topology, which is separable and completely metrizable). Since this preimage is then uncountable and the equivalence classes are countable, there are uncountably many equivalence classes that are not distinguished by f . It then follows from a deep theorem in logic [6], [5] that a countable, complete set of real invariants cannot be constructed without using the axiom of choice.

The proof of (b) uses the same idea. A family of smoothings of $\mathbb{R}^4$ is carefully constructed, indexed by $\Pi_{\mathbb{N}}\mathbb{Z}_2 / \oplus_{\mathbb{N}}\mathbb{Z}_2$. Unlike the previous case, a complete understanding of which members are diffeomorphic seems beyond the range of current technology. But at least the diffeomorphism classes are countable (via [2]), so the above theorem still applies.

While $\mathcal{W}$ cannot be classified by real invariants, we *have* classified it by $\Pi_{\mathbb{N}}\mathbb{Z}_2$ modulo the relation that two sequences are equivalent precisely when they can be made equal by suitable truncation. (This can be proved using JSJ decompositions of the ends.) It is natural to consider what else can be classified by more complicated objects. Descriptive set theorists have developed a hierarchy of classification problems (see [3]), a partially ordered set that they are mapping with benchmark

problems. We have already considered the first nontrivial level. Now we consider a standard benchmark that is higher by uncountably many levels, but still within the range of human understanding:

Definition 3. A classification problem is *complete for countable structures* if it is equivalent to classifying countable trees (allowing infinite valence) up to graph isomorphism.

Recently, Bergfalk and Smythe [1] have shown that the homeomorphism classification of open surfaces is complete for countable structures, and Iannella and Weinstein have claimed the same for 3-manifolds. Our work in progress shows the same for higher-dimensional manifolds, and exhibits finer detail:

Theorem 4. a) For each $n \geq 3$, there is a homotopy class of smooth n -manifolds whose proper homotopy, homeomorphism and diffeomorphism classifications are complete for countable structures.

b) For $n = 3$ or $n \geq 5$, there is a proper homotopy class of topological n -manifolds whose homeomorphism classification is complete for countable structures.

c) For $n = 4$, or for most $n \geq 8$ (those for which an exotic S^{n-1} exists) there is a homeomorphism class of smooth n -manifolds whose diffeomorphism classification is complete for countable structures.

The skips in dimension are likely an artifact of the particular tricks used so far in constructing the examples and ruling out unexpected equivalences. In fact, the following seems likely:

Metaconjecture 5. Any sufficiently complicated classification problem in manifold topology is complete for countable structures.

The flexibility of manifolds is crucial here. Complex manifolds can reach a higher level through their rigidity [4], but the examples in that paper are all diffeomorphic.

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Dehn surgery functions are never injective

LAURA WAKELIN

(joint work with Kyle Hayden and Lisa Piccirillo)

Given an n -tuple of slopes $(p_1/q_1, \dots, p_n/q_n)$ together with an n -component link $L = L_1 \cup \dots \cup L_n$ in $\mathbb{S}^3$, *Dehn surgery* describes how to build a closed oriented 3-manifold

$$\mathbb{S}_{L_1, \dots, L_n}^3(p_1/q_1, \dots, p_n/q_n).$$

Every closed oriented 3-manifold arises in this way [10, 17], but the Dehn surgery description is highly non-unique [4].

In 1978, Gordon conjectured that, for each slope p/q , the Dehn surgery function

$$K \mapsto \mathbb{S}_K^3(p/q)$$

from the set of knots to the set of closed oriented 3-manifolds with $H_1 \cong \mathbb{Z}/p\mathbb{Z}$ is neither surjective nor injective [5]. The non-surjectivity part was resolved in 1989 [6], but the non-injectivity part had only been established for slopes satisfying $p \equiv 1 \pmod{q}$ [1, 2, 3, 9, 11, 13, 15]. We complete the proof for all slopes [7].

Theorem 1. Dehn surgery functions are never injective.

The proof has two steps. First, we construct a rational surgery homeomorphism

$$\varphi : \mathbb{S}_K^3(p/q) \rightarrow \mathbb{S}_{K'}^3(p/q)$$

for some tactically constructed knots K and K' which both depend on p/q . Second, we distinguish the knots K and K' using a polynomial invariant.

RATIONAL RBG LINKS

Definition 2. A *rational p -RBG link* is a rationally framed 3-component link

$$(B, b) \cup (R, r) \cup (G, g)$$

together with orientation-preserving homeomorphisms $\psi_G : \mathbb{S}_{R,B}^3(r, b) \rightarrow \mathbb{S}^3$ and $\psi_B : \mathbb{S}_{R,G}^3(r, g) \rightarrow \mathbb{S}^3$ such that $H_1(\mathbb{S}_{B,R,G}^3(b, r, g); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.

A rational p -RBG link L gives rise to a rational surgery homeomorphism

$$\varphi_L : \mathbb{S}_{K_B}^3(p/q_B) \rightarrow \mathbb{S}_{K_G}^3(p/q_G)$$

[12, 14, 7]. Conversely, any rational surgery homeomorphism

$$\varphi : \mathbb{S}_K^3(p/q) \rightarrow \mathbb{S}_{K'}^3(p/q')$$

is presented by a rational p -RBG link L_φ [12, 14, 16].

One way to produce a rational p -RBG link is by taking duals and double duals:

$$(K, p/q) \cup (K, p/q)^* \cup (K, p/q)^{**}.$$

A straightforward 2×2 matrix computation allows us to express this as

$$(K, p/q) \cup (C_{r,s}(K), -s(1 - qr)/q) \cup (C_{qr^2,p}(C_{r,s}(K)), p(1 + q^2r^2)/q)$$

for r, s coprime with $ps - qr = 1$. However, this gives $(K_B, p/q_B) = (K_G, p/q_G)$. We rectify this by introducing an asymmetric interaction between (B, b) and (G, g) .

ZEROTH COEFFICIENT POLYNOMIAL

Recall that the HOMFLYPT polynomial $P_L(v, z)$ can be defined by a skein relation. Moreover, it decomposes as a sum of single-variable polynomials:

$$P_L(v, z) = \left(\frac{v}{z}\right)^{|L|-1} \sum_{i \geq 0} p_L^i(v) z^{2i}.$$

Definition 3. The *zeroth coefficient polynomial* of L is $\Gamma_L(\alpha) = p_L^0(v)|_{\alpha=-v^2}$.

Even after setting $K = U$ above, it is too difficult to compute Γ_{K_B} and Γ_{K_G} for infinitely many pairs K_B and K_G . However, since K_B and K_G are so similar, we can use a skein relation and linking formula [8] to show that $\Gamma_{K_B} = \Gamma_{K_G}$ if and only if Γ_J is a unit in $\mathbb{Z}[\alpha^{\pm 1}]$ for $J = C_{-s(1-qr),q}(T_{r,s})$. We can assume that $p > 1$ and choose r, s so that J is a non-trivial positive braid knot. We prove that the zeroth coefficient polynomial of such a knot can never be a unit. Hence $K_B \neq K_G$.

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Inscription Problems

ANDREW LOBB

(joint work with Josh Greene)

We discussed recent work on inscription problems. Such problems ask which finite configurations of points (up to similarity) occur as subsets of Jordan curves (with possibly some imposed regularity conditions on the curves). The motivational and unsolved problem is the Toeplitz conjecture which posits that every Jordan curve contains the vertices of a square.

With Josh Greene, we established that every smooth Jordan curve inscribes every cyclic quadrilateral. This is an optimal result in the sense both that it becomes false if one replaces smooth by, say, piecewise smooth; and that the circle is itself a smooth Jordan curve (and only inscribes cyclic quadrilaterals by definition). We also showed that a general Jordan curve inscribes a positive measure of similarity classes of rectangles.

The proofs run by interpreting inscription questions as questions about intersections of Lagrangians inside a symplectic manifold. Techniques of symplectic geometry can then be brought to bear.

Khovanov skein lasagna modules and exotica

MICHAEL WILLIS

(joint work with Qiuyu Ren)

The main goal of this talk was to discuss the following theorem, which was joint work with Qiuyu Ren.

Theorem 1. (Theorem 1.1 of [5]) The Khovanov skein lasagna module invariants $\mathcal{S}_0^{Kh}$ can detect exotic smooth structures on orientable, compact 4-manifolds with boundary.

That is to say, there exist such 4-manifolds X_1, X_2 which are homeomorphic, but for which the Khovanov skein lasagna modules $\mathcal{S}_0^{Kh}(X_1), \mathcal{S}_0^{Kh}(X_2)$ are not isomorphic, implying that X_1 and X_2 are not diffeomorphic. Theorem 1 is notable as providing the first detection result for such exotica using essentially combinatorial methods; all earlier such detection results required more analytic methods involving gauge theory and partial differential equations.

Remark 2. All discussions during this talk (and this abstract) take place in the relevant smooth category.

We began by indicating how a functorial link homology theory Z for links in S^3 (over some commutative ring R) could formally be upgraded into a *skein lasagna module* theory $\mathcal{S}_0^Z$ [3], which takes as input a 4-manifold X and (possibly empty) link $L \subset \partial X$ and produces an R -module $\mathcal{S}_0^Z(X; L)$. We emphasized how, in the case that Z was either the Khovanov or Lee homology theory, the only analysis

involved in the construction came from standard general position arguments allowing links and link cobordisms to be replaced with the combinatorics of link diagrams and sequences of critical points along a cobordism.

We next specialized to Lee homology, where results of [1, 4] on the structure and quantum filtrations levels of the homology theory over $\mathbb{Q}$ could be generalized as follows.

Theorem 3. (Theorems 1.6 and 1.8 of [5]) The Lee skein lasagna module over $\mathbb{Q}$ of a 4-manifold with no link on its boundary satisfies

$$\mathcal{S}_0^{Lee}(X) \cong \bigoplus_{(\alpha, \beta) \in H_2(X)^2} \mathbb{Q}\langle x_{\alpha, \beta} \rangle.$$

Moreover, for fixed $\alpha \in H_2(X)$, the *lasagna s-invariant* $s_\alpha(X)$ defined to be the quantum filtration level of $x_{\alpha, 0}$ satisfies

$$g_\alpha(X) := \min_{[\Sigma] = \alpha} g(\Sigma) \geq \frac{s_\alpha(X) + \alpha^2}{2}.$$

Here $g(\cdot)$ is the genus function, and α^2 is computed using the intersection form of X .

We then briefly discussed the main computational tool [2] which states that, for 4-dimensional 2-handlebodies X_K consisting of a single 2-handle attached to a 4-ball along a framed knot K , one can compute $\mathcal{S}_0^{Kh}(X_K)$ in terms of the Khovanov homologies of various cablings of K . Although the required formula is still complicated, involving a limiting procedure as the number of cables goes to infinity, it is enough to prove the following two theorems.

Theorem 4. (Theorem 1.4 of [5]) Let $f(K)$ denote the framing of K . If $f(K) \geq -\text{TB}(-K)$, where $-K$ is the mirror of K , and $\text{TB}(\cdot)$ denotes the maximal Thurston–Bennequin number, then $\mathcal{S}_0^{Kh}(X_K) \cong 0$ and $s_\alpha(X_K) = -\infty$ for all α (and thus give no topological information).

Theorem 5. (Theorem 1.13 of [5]) If $f(K) < -2N_-(K)$, where $N_-(\cdot)$ denotes the minimal number of negative crossings in any diagram for K , then $s_{\alpha=0}(X_K) = 0$ and $\mathcal{S}_0^{Kh}(X_K)$ is partially computable (and in particular is non-zero).

Moreover, if K is positive and $f(K) \leq 0$, then $s_{\alpha=1}(X_K) = s(K) - f(K)$ where $s(K)$ is the original Rasmussen s -invariant of K , and even more of $\mathcal{S}_0^{Kh}(X_K)$ can be computed.

In particular, the computations available from Theorem 5., although somewhat limited in scope, were enough to allow us to detect some exotic pairs of 2-handlebodies.

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Foliations forcing closed orbits

ELLIS BUCKMINSTER

(joint work with Audrey Rosevear)

Given a codimension 1 foliation $\mathcal{F}$ on a closed 3-manifold M , we associate to $\mathcal{F}$ the set

$$\text{Flows}(\mathcal{F}) = \{\varphi \text{ a flow on } M \mid \varphi \pitchfork \mathcal{F}\},$$

where we say $\varphi \pitchfork \mathcal{F}$ when every flow line of φ is transverse to every leaf of $\mathcal{F}$ it meets. Our work is motivated by the following broad question.

Question 1. How are dynamical properties of $\text{Flows}(\mathcal{F})$ informed by topological and geometric properties of $\mathcal{F}$?

As motivation for this question, we consider the example of a pseudo-Anosov mapping torus. Let Σ be a closed surface of genus ≥ 2 , and let $f: \Sigma \rightarrow \Sigma$ be a pseudo-Anosov map. We form the **mapping torus** M_f of f by taking the product of Σ with an interval $I = [0, 1]$, and gluing the ‘top’ ($\Sigma \times \{1\}$) to the ‘bottom’ ($\Sigma \times \{0\}$) via f , producing a closed 3-manifold. This construction comes with a natural **foliation by fibers** $\mathcal{F}_f$, i.e. the foliation by surfaces of the form $\Sigma \times \{t\}$, and a natural **suspension flow** φ , which is the quotient of the unit speed vertical flow on $\Sigma \times I$. The flow φ is transverse to $\mathcal{F}_f$ and is an example of a pseudo-Anosov flow.

Now consider any flow $\psi \pitchfork \mathcal{F}_f$. As ψ is transverse to the foliation by fibers, it induces a first return map on the surface Σ which is homotopic to f . Using Handel’s global shadowing theorem [2], some closed subset of (M_f, ψ) semiconjugates onto (M_f, φ) , implying that ψ inherits many dynamical properties of φ . This example motivates the following refinement of Question 1.

Question 2. Which foliations $\mathcal{F}$ force all flows in $\text{Flows}(\mathcal{F})$ to have ‘pseudo-Anosov-like dynamics’?

In search of a question specific enough to answer, we pick out the existence of closed orbits as a dynamical property of pseudo-Anosov flows and ask the following. Of course, there is much room for further work in the space between Questions 2 and 3.

Question 3. Which foliations $\mathcal{F}$ force all flows in $\text{Flows}(\mathcal{F})$ to have closed orbits?

Our work addresses this question in the case that $\mathcal{F}$ is a finite depth foliation. Such foliations were introduced by Gabai [1] and generalize foliations by fibers by allowing for a finite hierarchy of spiraling behavior. Thus, transverse flows have

much of the flavor of surface dynamics while still admitting more complicated behavior.

We give in Theorem 4 a complete answer to this question in the case of taut, C^2 , depth one foliations on closed, hyperbolic 3-manifolds. Perhaps the most surprising element of our work is a construction of aperiodic flows transverse to certain depth one foliations.

Theorem 4. Let $\mathcal{F}$ be a taut, C^2 , depth one foliation on a closed, hyperbolic 3-manifold. Then $\mathcal{F}$ forces closed orbits if and only if

- (1) $\mathcal{F}$ has a depth one region which is not a topological product, or
- (2) $\mathcal{F}$ is approximable by fibrations.

Furthermore, we give a characterization of case (2) above in terms of the juncture classes of the foliation. A version of Theorem 4 for finite depth foliations is in progress; we state this as Pre-Theorem 5 below.

Pre-Theorem 5. Let $\mathcal{F}$ be a taut, C^2 , finite depth foliation on a closed, hyperbolic 3-manifold. Then $\mathcal{F}$ forces closed orbits if and only if

- (1) $\mathcal{F}$ has a locally top depth region which is not a topological product, or
- (2) $\mathcal{F}$ is approximable by foliations of the above type.

To produce closed orbits for the positive direction, we leverage the spun pseudo-Anosov machinery of Landry–Minsky–Taylor [5] in case (1), and use the existence of closed orbits in pseudo-Anosov suspension flows in case (2).

The more interesting direction is the negative direction, which we show by constructing aperiodic flows transverse to the foliations in question. This is done via plug insertions, a technique for breaking closed orbits of a given flow without introducing new closed orbits. These were used by Wilson [7], Schweitzer [6], Harrison [3], and Kuperberg [4] to disprove the Seifert Conjecture in increasing levels of regularity. The key technical lemma of our work is understanding when a modified version of the Schweitzer plug can be inserted transverse to a given foliation.

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Stably exotic fillings of 3-manifolds

MARK POWELL

Let Y be a closed 3-manifold and let X_0 and X_1 be smooth, compact 4-manifolds with $\partial X_i \cong Y$ for $i = 0, 1$. We say that X_0 and X_1 are *stably exotic fillings* of Y if they are homeomorphic rel. Y but not stably diffeomorphic. That is, there is no k such that $X_0 \# k(S^2 \times S^2)$ is diffeomorphic to $X_1 \# k(S^2 \times S^2)$.

I discussed the following two theorems about this notion. The first says that in many cases, stably exotic fillings exist.

Theorem 1. (Kasprowski-Orson-Powell-Ray). Suppose Y :

- (1) is orientable; or
- (2) is a circle bundle over a surface F with $F \neq \mathbb{RP}^2$; or
- (3) admits a null-bordant Pin^+ structure.

Then Y admits a pair of stably exotic fillings.

In a fixed homeomorphism class, there are at most two stable diffeomorphism classes of exotic fillings. Stably exotic closed 4-manifolds were discovered by Kreck [4], who showed that they can be found realising any pair (π, w) of a finitely presented group π and a nontrivial orientation character $w: \pi \rightarrow C_2$. See also [3]. The case of nonempty boundary is more subtle. To illustrate this, our second theorem shows that stably exotic fillings cannot always be found.

Theorem 2. (Kasprowski-Orson-Powell-Ray). Suppose that Y contains a two-sided embedded $\mathbb{RP}^2$. Then Y does not admit stably exotic fillings.

Although we have provided two partial answers, the following remains open.

Question 3. Which 3-manifolds admit stably exotic fillings?

All the fillings used to prove Theorem 1 are nonorientable. In fact, Gompf [2] showed that this is necessarily the case, as homeomorphic orientable 4-manifolds are stably diffeomorphic. It is also open whether every orientable 3-manifold admits orientable exotic fillings; cf. [1].

In the talk, I discussed the proof of Theorem 1, focussing on case (1), that Y is orientable. The proof uses Kreck's modified surgery theory [5]. We construct two fillings that are homeomorphic, and whose union M along Y is nontrivial in the bordism group $\Omega_4(\xi)$ of the normal 1-type $\xi: B \rightarrow BO$. Here, the normal 1-type is a 2-coconnected fibration such that the stable normal bundle $\nu_M: M \rightarrow BO$ admits a 2-connected lift $\overline{\nu}_M: M \rightarrow B$ along ξ . Kreck's theorem tells us that two 4-manifolds are bordant (rel. boundary) over ξ , for some choices of normal 1-smoothing, if and only if they are stably diffeomorphic.

Here is the construction of the examples that prove Theorem 1 (1). Let V be a simply-connected spin filling of Y . Then

$$X_0 := V \# S^1 \tilde{\times} S^3 \# 11(S^2 \times S^2)$$

and

$$X_1 := V \# S^1 \tilde{\times} S^3 \# K_3$$

are homeomorphic rel. Y . A spectral sequence computation shows that the K_3 surface generates the relevant bordism group. We show that $X_0 \cup_Y X_1$, for any normal 1-smoothings, is bordant to K_3 . This implies by Kreck's theorem that X_0 and X_1 are not stably diffeomorphic.

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