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AISSA AIBECHE
CHÉRIF AMROUCHE
PETER LEWINTAN

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
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Germany
<https://www.mfo.de>

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REGULARITY FOR VECTOR FIELDS, HODGE DECOMPOSITION AND GREEN FORMULA IN LIPSCHITZ DOMAINS

AISSA AIBECHE, CHÉRIF AMROUCHE, AND PETER LEWINTAN

ABSTRACT. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, neither necessarily simply connected nor with connected boundary Γ , we prove a new Hodge-type decomposition: every $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ splits uniquely into a gradient and a field in $\mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega)$ with tangential $\mathbf{L}^2$ -trace. As applications we remove the additional topological hypotheses from Costabel's $\mathbf{H}^{\frac{1}{2}}$ -regularity theorem, and extend the classical Green formula to two $\mathbf{H}(\mathbf{curl})$ -fields. We define tangential differential operators on the non-smooth boundary Γ and revisit the space $\mathbf{H}^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$.

1. INTRODUCTION

Throughout this paper, we denote by $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain (not necessarily simply connected) with boundary Γ (not necessarily connected) and exterior normal vector field $\boldsymbol{\nu}$. For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$ the classical Green formula reads

$$\int_{\Omega} (\mathbf{u} \cdot \mathbf{curl} \mathbf{v} - \mathbf{v} \cdot \mathbf{curl} \mathbf{u}) = \langle \mathbf{u} \times \boldsymbol{\nu}, \mathbf{v} \rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{\frac{1}{2}}(\Gamma)}. \quad (1.1)$$

Its left-hand side is well defined as soon as **both** vector fields lie in $\mathbf{H}(\mathbf{curl}, \Omega)$. Its right-hand is not: an $\mathbf{H}(\mathbf{curl})$ -field has a tangential trace $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, but has, in general, no trace at all, so the duality on the right asks of $\mathbf{v}$ exactly the $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ -regularity that $\mathbf{H}(\mathbf{curl}, \Omega)$ does not provide. Giving (1.1) a meaning for two $\mathbf{H}(\mathbf{curl})$ -fields is therefore a question about the tangential trace space – and on a Lipschitz boundary it is a delicate one.

For smooth domains the picture is classical: the tangential trace maps $\mathbf{H}(\mathbf{curl}, \Omega)$ onto $\mathbf{H}^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$, the surjectivity going back to Paquet [15]. On Lipschitz domains, Tartar [16, Thm. 2] characterized the range of the tangential trace and proved surjectivity, in a manuscript that was never published; his description is a single condition coupling tangentiality and the surface divergence. Buffa et al. [7] developed a systematic functional-analytical framework, introducing new spaces V'_γ, V'_π of tangential fields of order $-\frac{1}{2}$, using Hodge decompositions for them, and deriving a Green formula for two $\mathbf{H}(\mathbf{curl})$ -fields. Their boundary term is expressed through duality pairings in these non-standard spaces and the authors suppose Γ connected and Ω simply connected:

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$$\begin{aligned}
\gamma \langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v}) \rangle_\pi &= \int_\Gamma \gamma_\tau(\Phi) \cdot \pi_\tau(\Psi) + v'_\gamma \langle \nabla_\Gamma q, \gamma_\tau(\Psi) \rangle_{V_\gamma} + v'_\pi \langle \mathbf{curl}_\Gamma p, \pi_\tau(\Psi) \rangle_{V_\pi} \\
&= \int_\Gamma \gamma_\tau(\Phi) \cdot \pi_\tau(\Psi) - \langle \operatorname{div}_\Gamma \gamma_\tau(\Phi), q \rangle_\Gamma + \langle \mathbf{curl}_\Gamma \pi_\tau(\Psi), p \rangle_\Gamma,
\end{aligned}$$

where

$$\mathbf{u} = \Phi + \nabla p, \quad \mathbf{v} = \Psi + \nabla q \quad \Phi, \Psi \in \mathbf{H}^1(\Omega), p, q \in H^1(\Omega).$$

As the decompositions above are not unique and the duality pairings lie in non-standard spaces, this Green's formula, as it stands, is difficult to put into practice. On the regularity side, Costabel [9, Thm. 2] proved that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega)$ with either $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$ or $\mathbf{u} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$ belongs to $\mathbf{H}^{\frac{1}{2}}(\Omega)$ – again for Ω simply connected with connected boundary. Simply connectedness is not an artefact there: it is what permits a curl-free field to be written as a gradient, and thereby reduces the vectorial problem to *scalar* elliptic regularity on a Lipschitz domain, where the results of Nečas and of Dahlberg-Jerison-Kenig apply, see also [6] for an overview. A first, indirect answer to the validity of Green's formula (1.1) when both functions lie in $\mathbf{H}(\mathbf{curl}, \Omega)$ follows from [1, Lem. 3.5], which implies that for any $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ there exists an $\mathbf{w} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{curl} \mathbf{v} = \mathbf{curl} \mathbf{w}$, see also [12, Cor. 5.5] for another proof. However, this does not give a formula whose RHS is anti-symmetric and is expressed in the **standard** duality $\mathbf{H}^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{\frac{1}{2}}(\Gamma)$. That is what we present in this work. We develop a new approach for extending this result to the case of bounded Lipschitz domains (not necessarily simply connected and whose boundary need not be connected). This is achieved showing a new Hodge-type decomposition. More precisely, we prove Costabel's theorem [9, Thm. 2] with all additional topological hypotheses removed, see our regularity Lemma 4.3. Our new Hodge-type decomposition (Lemma 4.6) reads: Every $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ splits *uniquely* as follows:

$$\mathbf{u} = \boldsymbol{\varphi} + \nabla \chi \quad \text{with} \quad \chi \in H^1(\Omega) \quad \text{and} \quad \boldsymbol{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega)$$

whose trace is a genuine tangential $\mathbf{L}^2$ -field, $\boldsymbol{\varphi}|_\Gamma \in \mathbf{L}^2_\tau(\Gamma)$. We then show the following Green formula (Theorem 5.1) for $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$:

$$\int_\Omega (\mathbf{u} \cdot \mathbf{curl} \mathbf{v} - \mathbf{v} \cdot \mathbf{curl} \mathbf{u}) = \int_\Gamma (\boldsymbol{\varphi} \times \boldsymbol{\nu}) \cdot \boldsymbol{\psi} + \langle \operatorname{div}_\Gamma(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi \rangle_\Gamma - \langle \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{\nu}), \lambda \rangle_\Gamma$$

where $\mathbf{u} = \boldsymbol{\varphi} + \nabla \chi$ and $\mathbf{v} = \boldsymbol{\psi} + \nabla \lambda$ are the decompositions from Lemma 4.6. Observe that $\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{\nu})$ and $\operatorname{div}_\Gamma(\boldsymbol{\psi} \times \boldsymbol{\nu})$ are in $H^{-\frac{1}{2}}(\Gamma)$. Explicit characterizations of the tangential trace operator for $\mathbf{H}(\mathbf{curl})$ -functions in Lipschitz domains are essential for applications in boundary value problems for elliptic systems and inverse problems for PDEs. These applications include:

- Maxwell's equations,
- Navier-Stokes equations,
- PDEs in optics and electromagnetism,
- PDEs in continuum mechanics and elasticity.

The paper is organized as follows. Section 2 fixes notation. Section 3 introduces the tangential divergence on $\mathbf{L}_\tau^2(\Gamma)$ and the density results we need. Section 4 contains the regularity result and the new decomposition, Section 5 the Green formula, and Section 6 the characterization of the tangential trace space.

2. NOTATIONS

To keep our presentation succinct, we stay in the three-dimensional Euclidean space and focus only on Hilbert spaces. We use boldface to denote vector fields and spaces of vector valued functions and recall the standard definitions

$$\begin{aligned} H^1(\Omega) &= \{\varphi \in L^2(\Omega) : \nabla \varphi \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}(\mathbf{curl}, \Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}(\mathbf{div}, \Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathbf{div} \mathbf{u} \in L^2(\Omega)\}. \end{aligned}$$

It is known that any $\varphi \in H^1(\Omega)$ admits a trace $\varphi|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$, $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ admits a tangential trace $\gamma_\tau(\mathbf{u}) := \mathbf{u} \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ (the dual space of $\mathbf{H}^{\frac{1}{2}}(\Gamma)$), and any $\mathbf{u} \in \mathbf{H}(\mathbf{div}, \Omega)$ admits a normal trace $\mathbf{u} \cdot \boldsymbol{\nu} \in H^{-\frac{1}{2}}(\Gamma)$. We denote by $\langle \cdot, \cdot \rangle_\Gamma$ the natural duality pairing in $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ or $\mathbf{H}^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{\frac{1}{2}}(\Gamma)$ extending the $L^2(\Gamma)$ inner product. We repeat the Green's formula for $\mathbf{u} \in \mathbf{H}(\mathbf{div}, \Omega)$:

$$\int_\Omega \mathbf{u} \cdot \nabla \chi + \int_\Omega \chi \mathbf{div} \mathbf{u} = \langle \mathbf{u} \cdot \boldsymbol{\nu}, \chi \rangle_\Gamma \quad \forall \chi \in H^1(\Omega). \quad (2.1)$$

Furthermore, the space of tangential $L^2(\Gamma)$ -vector fields is denoted by

$$\mathbf{L}_\tau^2(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma) : \mathbf{u} \cdot \boldsymbol{\nu} = 0\}.$$

3. TANGENTIAL DIVERGENCE

For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ we have $\mathbf{div} \mathbf{curl} \mathbf{u} = 0$ in the distributional sense, so $\mathbf{curl} \mathbf{u} \in \mathbf{H}(\mathbf{div}, \Omega)$ and admits a normal trace $\mathbf{curl} \mathbf{u} \cdot \boldsymbol{\nu} \in H^{-\frac{1}{2}}(\Gamma)$. Applying the Green's formula (2.1), we obtain on the one hand

$$\int_\Omega \nabla \chi \cdot \mathbf{curl} \mathbf{u} = \langle \mathbf{curl} \mathbf{u} \cdot \boldsymbol{\nu}, \chi \rangle_\Gamma \quad \forall \chi \in H^1(\Omega), \quad (3.1)$$

and on the other hand by (1.1)

$$-\int_\Omega \nabla \chi \cdot \mathbf{curl} \mathbf{u} = \langle \mathbf{u} \times \boldsymbol{\nu}, \nabla \chi \rangle_\Gamma \quad \forall \chi \in H^2(\Omega). \quad (3.2)$$

Observe that, since Ω is only Lipschitz, the trace of a function $\chi \in H^2(\Omega)$ belongs only to $H^1(\Gamma)$, so the tangential gradient $\nabla_\Gamma \chi \in \mathbf{L}^2(\Gamma)$ and $\partial_\nu \chi \in L^2(\Gamma)$. However, as $\nabla \chi \in \mathbf{H}^1(\Omega)$ we have $\nabla \chi|_\Gamma \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$. So, the RHS in (3.2) is well defined. On Γ , we have $\nabla \chi = \nabla_\Gamma \chi + \boldsymbol{\nu} \partial_\nu \chi \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, where each term on the right-hand side belongs to $L^2(\Gamma)$ but not necessarily to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Assuming further that $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$, we get

$$\langle \mathbf{u} \times \boldsymbol{\nu}, \nabla \chi \rangle_\Gamma = \int_\Gamma (\mathbf{u} \times \boldsymbol{\nu}) \cdot \nabla_\Gamma \chi = -\langle \mathbf{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)}. \quad (3.3)$$

We used above the following definition:

Definition 3.1. For $\boldsymbol{\lambda} \in \mathbf{L}_\tau^2(\Gamma)$ we set

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \mu \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} := - \int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \mu \quad \forall \mu \in H^1(\Gamma).$$

By this definition, we clearly have the following property: the operator

$$\operatorname{div}_\Gamma : \mathbf{L}_\tau^2(\Gamma) \rightarrow H^{-1}(\Gamma)$$

is linear and continuous.

Remark 3.2. In the context of Definition 3.1, if $\mu = \chi|_\Gamma$ for a $\chi \in H^2(\Omega)$, then we have

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \mu \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = - \int_\Gamma \boldsymbol{\lambda} \cdot \nabla \chi.$$

If, moreover, $\chi \in H^2(\Omega) \cap H_0^1(\Omega)$, then we obtain

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \mu \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = - \int_\Gamma \boldsymbol{\lambda} \cdot \boldsymbol{\nu} \partial_\nu \chi = 0.$$

Returning to (3.3), for $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ with $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$ and $\chi \in H^2(\Omega)$, it holds

$$\langle \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu}, \chi \rangle_\Gamma = \langle \operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)}. \quad (3.4)$$

We now aim to extend this identity to functions $\chi \in H^1(\Omega)$. The left-hand side is well defined, since $\operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu} \in H^{-\frac{1}{2}}(\Gamma)$.

The following density result goes back to Nečas:

Lemma 3.3. The space $\{\chi|_\Gamma \in H^1(\Gamma) : \chi \in H^2(\Omega)\}$ is dense in $H^1(\Gamma)$.

Proof. Recall the following result (see [14], Lemma 4.4, Chapter 5, page 274): the space

$$\{(v, \partial_\nu v) \in H^1(\Gamma) \times L^2(\Gamma); v \in H^2(\Omega)\}$$

is dense in $H^1(\Gamma) \times L^2(\Gamma)$. So let $g \in H^1(\Gamma)$ and apply the above last density result for the data $(g, 0)$. There exists $v_k \in H^2(\Omega)$ such that

$$v_k|_\Gamma \rightarrow g \quad \text{in } H^1(\Gamma) \quad \text{and} \quad \partial_\nu v_k \rightarrow 0 \quad \text{in } L^2(\Gamma). \quad \square$$

We include also the next density lemma from [5]:

Lemma 3.4 ([5]). The space

$$\mathcal{D}(\overline{\Omega}; \mathbb{R}^3) \text{ is dense in } \{\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega) : \mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)\}.$$

In particular, $H^2(\Omega)$ is dense in $\{\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega) : \mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)\}$.

The additional assumption $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$ enables a correct understanding of (3.4) in case $\chi \in H^1(\Omega)$:

Lemma 3.5. Let $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ such that $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. Then, $\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) \in H^{-\frac{1}{2}}(\Gamma)$ and $\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) = \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu}$. Moreover, there exists a constant $C(\Omega) > 0$ such that

$$\|\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu})\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}.$$

Proof. Observe that $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$ implies that $\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) \in H^{-1}(\Gamma)$.

Let $\mu \in H^1(\Gamma)$ be arbitrary. By Lemma 3.3 we find a sequence of functions $\chi_k \in H^2(\Omega)$ such that $\chi_k|_\Gamma \xrightarrow{k \rightarrow \infty} \mu$ in $H^1(\Gamma)$. Then

$$\begin{aligned} \langle \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu}, \mu \rangle_\Gamma &= \lim_{k \rightarrow \infty} \langle \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu}, \chi_k \rangle_\Gamma \\ &\stackrel{(3.4)}{=} \lim_{k \rightarrow \infty} \langle \operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}), \chi_k \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} \\ &= \langle \operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}), \mu \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)}. \end{aligned}$$

Hence, $\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) = \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu}$ in $H^{-1}(\Gamma)$. But, since $\operatorname{curl} \mathbf{u} \cdot \boldsymbol{\nu} \in H^{-\frac{1}{2}}(\Gamma)$ we conclude $\operatorname{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) \in H^{-\frac{1}{2}}(\Gamma)$ and (3.4) holds for any $\chi \in H^1(\Omega)$. $\square$

4. HODGE-TYPE DECOMPOSITIONS

One of the essential tools for obtaining Hodge decompositions is the resolution of a Dirichlet or Neumann problem for the Laplacian. Recall the following result:

Theorem 4.1. *Assume that*

$$v \in H^1(\Omega), \quad \text{with} \quad \Delta v \in L^2(\Omega). \quad (4.1)$$

i) *Then we have the following equivalence (see Nečas [14], Section 5)*

$$v|_\Gamma \in H^1(\Gamma) \quad \Longleftrightarrow \quad \partial_\nu v \in L^2(\Gamma). \quad (4.2)$$

ii) *Moreover, if*

$$v|_\Gamma \in H^1(\Gamma) \quad \text{or} \quad \partial_\nu v \in L^2(\Gamma), \quad (4.3)$$

then $v \in H^{\frac{3}{2}}(\Omega)$ (see Dahlberg-Jerison-Kenig, [13]).

We recall the following regular Hodge decomposition for functions in $\mathbf{H}(\operatorname{curl}, \Omega)$:

Lemma 4.2 (regular Hodge decomposition for $\mathbf{H}(\operatorname{curl})$). *Let $\mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega)$. Then, there exist $\boldsymbol{\Psi} \in \mathbf{H}^1(\Omega)$ and $\theta \in H^1(\Omega)$ such that*

$$\mathbf{v} = \boldsymbol{\Psi} + \nabla \theta \quad \text{in } \Omega. \quad (4.4)$$

For a proof we refer the reader to [4, Thm. 3.6.7] or [11, Lem. 2.4].

To generalize the Green's formula (1.1), we employ a different decomposition. To this end, we extend Costabel's auxiliary result [9, Thm. 2] to the case where neither the domain itself is assumed to be simply connected, nor its boundary is necessarily connected. Before that, recall that for bounded $\mathcal{C}^{1,1}$ -domains, if $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega)$, either of the two boundary conditions $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ or $\mathbf{u} \cdot \boldsymbol{\nu} \in H^{\frac{1}{2}}(\Gamma)$ implies $\mathbf{u} \in \mathbf{H}^1(\Omega)$, see e.g. [10]. However this regularity does not hold when the domain Ω is only Lipschitz.

The following lemma is an extension of the main Theorem from [9] with the topological hypotheses removed.

Lemma 4.3. *Let $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega)$.*

(a) If $\mathbf{u} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$, then $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. Moreover, there exists a constant $C(\Omega) > 0$ such that

$$\|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} + \|\mathbf{u} \times \boldsymbol{\nu}\|_{\mathbf{L}^2(\Gamma)} \leq C(\Omega) \left(\|\mathbf{u}\|_{\mathbf{H}(\text{curl}, \Omega)} + \|\mathbf{u}\|_{\mathbf{H}(\text{div}, \Omega)} + \|\mathbf{u} \cdot \boldsymbol{\nu}\|_{L^2(\Gamma)} \right).$$

(b) If $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$, then, $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\mathbf{u} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$. Moreover, there exists a constant $C(\Omega) > 0$ such that

$$\|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} + \|\mathbf{u} \cdot \boldsymbol{\nu}\|_{L^2(\Gamma)} \leq C(\Omega) \left(\|\mathbf{u}\|_{\mathbf{H}(\text{curl}, \Omega)} + \|\mathbf{u}\|_{\mathbf{H}(\text{div}, \Omega)} + \|\mathbf{u} \times \boldsymbol{\nu}\|_{\mathbf{L}^2(\Gamma)} \right).$$

Proof. We cannot adapt the proof of [9, Thm. 2] directly since our domain is not necessarily simply connected, which is crucial for the argumentation in [9]. Instead, we make use of the explicit extension operator for $\mathbf{H}(\text{curl}, \Omega)$ -functions from Hiptmair et al. [12, Cor. 4.1], see also [8, Lem. 2.2]. More precisely, there exists a universal extension operator $\mathcal{E}_1 : \mathbf{H}(\text{curl}, \Omega) \rightarrow \mathbf{H}(\text{curl}, \mathbb{R}^3)$ such that

$$\mathcal{E}_1 \mathbf{u} = \mathbf{u} \text{ a.e. in } \Omega \text{ and } \|\mathcal{E}_1 \mathbf{u}\|_{\mathbf{H}(\text{curl}, \mathbb{R}^3)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\text{curl}, \Omega)}. \quad (4.5)$$

Now let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^3; [0, 1])$ be such that $\eta \equiv 1$ in a ball $B \supset \overline{\Omega}$. Consider the function $\mathbf{f} := \text{curl}(\eta \mathcal{E}_1 \mathbf{u}) \in \mathbf{L}^2(\mathbb{R}^3)$. Then, $\mathbf{f}$ is compactly supported and in the distributional sense, $\text{div } \mathbf{f} = 0$ in $\mathbb{R}^3$. There exists a vector potential $\mathbf{w} \in \mathbf{H}^1(\mathbb{R}^3)$ such that

$$\text{curl } \mathbf{w} = \mathbf{f} \quad \text{and} \quad \text{div } \mathbf{w} = 0 \quad \text{in } \mathbb{R}^3. \quad (4.6)$$

Define $\mathbf{z} := (\eta \mathcal{E}_1 \mathbf{u} - \mathbf{w})|_B$. Then $\mathbf{z} \in \mathbf{L}^2(B)$ and, in the distributional sense, $\text{curl } \mathbf{z} = 0$ in B . Since the ball B is simply connected, we find $\theta \in H^1(B)$ such that $\mathbf{z} = \nabla \theta$ in B . Hence

$$\Delta \theta = \text{div } \mathbf{u} \quad \text{in } \Omega.$$

Now, assuming $\mathbf{u} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$, we have $\partial_\nu \theta = \mathbf{u} \cdot \boldsymbol{\nu} - \mathbf{w} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$ by the regularity of $\mathbf{w}$. It follows from Theorem 4.1 that $\theta \in H^{\frac{3}{2}}(\Omega)$ since $\text{div } \mathbf{u} \in L^2(\Omega)$. Hence, $\mathbf{u} = \mathbf{z} + \mathbf{w} = \nabla \theta + \mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$.

Furthermore, since $\Delta \theta \in L^2(\Omega)$ and $\partial_\nu \theta \in L^2(\Gamma)$ it follows from Nečas [14] that $\theta|_\Gamma \in H^1(\Gamma)$ and, so, $\nabla \theta \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. As $\mathbf{w} \in \mathbf{H}^1(\Omega)$, we also have $\mathbf{w} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. Therefore, $\mathbf{u} \times \boldsymbol{\nu} = \nabla \theta \times \boldsymbol{\nu} + \mathbf{w} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$, which completes the proof of part (a).

Conversely, assuming $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$, we have $\nabla \theta \times \boldsymbol{\nu} = \mathbf{u} \times \boldsymbol{\nu} - \mathbf{w} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. Using again Theorem 4.1, this is equivalent to $\theta|_\Gamma \in H^1(\Gamma)$ and implies $\theta \in H^{\frac{3}{2}}(\Omega)$. So, we have $\mathbf{u} = \mathbf{z} + \mathbf{w} = \nabla \theta + \mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$. Finally, $\mathbf{u} \cdot \boldsymbol{\nu} = \nabla \theta \cdot \boldsymbol{\nu} + \mathbf{w} \cdot \boldsymbol{\nu} \in L^2(\Gamma)$ and then part (b) is complete. $\square$

Remark 4.4. Instead of the $\mathbf{H}(\text{curl}, \mathbb{R}^3)$ -extension in (4.5) we can also argue in the following way: For $\mathbf{u} \in \mathbf{H}(\text{curl}, \Omega)$ we have $\text{div}_\Gamma(\mathbf{u} \times \boldsymbol{\nu}) \in H^{-\frac{1}{2}}(\Gamma)$. Let us denote again by B a ball containing $\overline{\Omega}$. Then we find $\mathbf{v} \in \mathbf{H}(\text{curl}; B \setminus \overline{\Omega})$ such that

$$\begin{cases} \mathbf{v} \times \boldsymbol{\nu} = \mathbf{0} & \text{on } \partial B, \\ \mathbf{v} \times \boldsymbol{\nu} = \mathbf{u} \times \boldsymbol{\nu} & \text{on } \Gamma. \end{cases}$$

Then,

$$\mathbf{U} := \begin{cases} \mathbf{u} & \text{in } \Omega, \\ \mathbf{v} & \text{in } B \setminus \overline{\Omega}, \end{cases}$$

fulfills $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, B)$ with $\mathbf{U} \times \boldsymbol{\nu} = \mathbf{0}$ on ∂B . And we can proceed as in the proof above considering $\mathbf{U}$ instead of $\eta \mathcal{E}_1 \mathbf{u}$.

Remark 4.5. Costabel does not need such an $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ –extension in his proof of [9, Thm 2] since his domain is simply connected by hypothesis.

We are now able to present the following Hodge-type decomposition:

Lemma 4.6. *Let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then, there exist $\chi \in H^1(\Omega)$ unique up to a constant and a unique $\boldsymbol{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega)$ with $\boldsymbol{\varphi}|_{\Gamma} \in \mathbf{L}_{\tau}^2(\Gamma)$ such that*

$$\mathbf{u} = \boldsymbol{\varphi} + \nabla \chi \quad \text{in } \Omega. \quad (4.7)$$

Moreover, there exists a constant $C(\Omega) > 0$ such that

$$\|\nabla \chi\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Gamma)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl}, \Omega)}$$

Proof. Let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$. We find $\chi \in H^1(\Omega)$ satisfying

$$\begin{cases} \Delta \chi = \operatorname{div} \mathbf{u} & \text{in } \Omega, \\ (\nabla \chi - \mathbf{u}) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma, \end{cases} \quad (4.8)$$

with $\|\nabla \chi\|_{\mathbf{L}^2(\Omega)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$.

We consider $\boldsymbol{\varphi} := \mathbf{u} - \nabla \chi$. Then, $\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{\varphi} = 0$ in the distributional sense, $\mathbf{curl} \boldsymbol{\varphi} = \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0$ on Γ . By Lemma 4.3 we obtain $\boldsymbol{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\boldsymbol{\varphi} \times \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma)$. Hence, also $\boldsymbol{\nu} \times (\boldsymbol{\varphi} \times \boldsymbol{\nu}) \in \mathbf{L}^2(\Gamma)$. Since for the vector triple product, we have $\boldsymbol{\nu} \times (\boldsymbol{\varphi} \times \boldsymbol{\nu}) = \boldsymbol{\varphi} - (\boldsymbol{\varphi} \cdot \boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\varphi}$, we conclude $\boldsymbol{\varphi}|_{\Gamma} \in \mathbf{L}_{\tau}^2(\Gamma)$.

Finally, to show uniqueness of the decomposition (4.7) fix $\mathbf{u} = \mathbf{0}$. Then, $\nabla \chi = \mathbf{0}$, so $\chi = \text{const}$ and $\boldsymbol{\varphi} = \mathbf{0}$. $\square$

Remark 4.7. In contrast to the regular decomposition (Lem. 4.2) into H^1 -fields we have here a $H^{\frac{1}{2}}$ -part with tangential L^2 -trace and the decomposition (4.7) is *unique*.

5. GREEN'S FORMULA

To derive a suitable Green's formula for $\mathbf{H}(\mathbf{curl})$ -functions, Buffa et al. [7] employ a regular Helmholtz decomposition involving H^1 -functions:

$$\mathbf{u} = \boldsymbol{\Phi} + \nabla p, \quad \mathbf{v} = \boldsymbol{\Psi} + \nabla q \quad \boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathbf{H}^1(\Omega), p, q \in H^1(\Omega)$$

and proved the following Green's formula for $\mathbf{H}(\mathbf{curl})$ -functions:

$$\int_{\Omega} (\mathbf{u} \cdot \mathbf{curl} \mathbf{v} - \mathbf{v} \cdot \mathbf{curl} \mathbf{u}) = {}_{\gamma} \langle \gamma_{\tau}(\mathbf{u}), \pi_{\tau}(\mathbf{v}) \rangle_{\pi}, \quad (5.1)$$

where the boundary terms are then handled using duality pairings in non-standard function spaces, cf. [7, Eq. (36)]:

$${}_{\gamma} \langle \gamma_{\tau}(\mathbf{u}), \pi_{\tau}(\mathbf{v}) \rangle_{\pi} = \int_{\Gamma} \gamma_{\tau}(\boldsymbol{\Phi}) \cdot \pi_{\tau}(\boldsymbol{\Psi}) + {}_{V_{\gamma}'} \langle \nabla_{\Gamma} q, \gamma_{\tau}(\boldsymbol{\Psi}) \rangle_{V_{\gamma}} + {}_{V_{\pi}'} \langle \mathbf{curl}_{\Gamma} p, \pi_{\tau}(\boldsymbol{\Psi}) \rangle_{V_{\pi}}$$

$$= \int_{\Gamma} \gamma_{\tau}(\Phi) \cdot \pi_{\tau}(\Psi) - \langle \operatorname{div}_{\Gamma} \gamma_{\tau}(\Phi), q \rangle_{\Gamma} + \langle \operatorname{curl}_{\Gamma} \pi_{\tau}(\Psi), p \rangle_{\Gamma},$$

with $\pi_{\tau}(\cdot) = \nu \times (\cdot \times \nu)$, $\gamma_{\tau}(\cdot) = \cdot \times \nu$, $V_{\gamma} = \gamma_{\tau}(\mathbf{H}^{\frac{1}{2}}(\Gamma))$ and $V_{\pi} = \pi_{\tau}(\mathbf{H}^{\frac{1}{2}}(\Gamma))$. Their integration by parts formula and its duality interpretation are derived under Γ connected and Ω simply connected; we assume neither. In [7, Sec. 5] Buffa et al. mention that “the generalization to non-simply connected geometries would require some additional work.” Indeed, a generalization towards keeping the simply connectedness of Ω and dropping the connectedness assumption of Γ would follow similar lines as in [9] but involving [1, Lem. 3.5]. In contrast, dropping also the simply connectedness of Ω requires some sophisticated analysis. However, our approach is based on the new Hodge-type decomposition (4.7). In what follows, we present two alternative versions of Green’s formula for $\mathbf{H}(\operatorname{curl})$ -functions. These variants are based on the decomposition (4.7), which involves $\mathbf{H}^1$ -functions and $\mathbf{H}^{\frac{1}{2}}$ -functions, and utilize the natural duality pairing $\mathbf{H}^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{\frac{1}{2}}(\Gamma)$, cf. Theorems 5.1 and 5.3 below.

Theorem 5.1. *Let $u, v \in \mathbf{H}(\operatorname{curl}, \Omega)$ and $\chi, \lambda \in \mathbf{H}^1(\Omega)$ as well as $\varphi, \psi \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\operatorname{curl}, \Omega)$ with $\varphi|_{\Gamma}, \psi|_{\Gamma} \in \mathbf{L}_{\tau}^2(\Gamma)$ come from the decomposition (4.7). Then*

$$\int_{\Omega} (u \cdot \operatorname{curl} v - v \cdot \operatorname{curl} u) \quad (5.2)$$

$$= \int_{\Gamma} (\varphi \times \nu) \cdot \psi + \langle \operatorname{div}_{\Gamma}(\psi \times \nu), \chi \rangle_{\Gamma} - \langle \operatorname{div}_{\Gamma}(\varphi \times \nu), \lambda \rangle_{\Gamma}. \quad (5.3)$$

Proof. By Lemma 4.6 we find $\chi, \lambda \in \mathbf{H}^1(\Omega)$ and $\varphi, \psi \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\operatorname{curl}, \Omega)$ with $\varphi \times \nu, \psi \times \nu \in \mathbf{L}^2(\Gamma)$ and $\varphi \cdot \nu = \psi \cdot \nu = 0$ on Γ such that

$$u = \varphi + \nabla \chi \quad \text{and} \quad v = \psi + \nabla \lambda.$$

Then

$$\begin{aligned} \int_{\Omega} (u \cdot \operatorname{curl} v - v \cdot \operatorname{curl} u) &= \int_{\Omega} (\varphi \cdot \operatorname{curl} \psi - \psi \cdot \operatorname{curl} \varphi) + \int_{\Omega} \nabla \chi \cdot \operatorname{curl} \psi \\ &\quad - \int_{\Omega} \nabla \lambda \cdot \operatorname{curl} \varphi =: I_1 + I_2 + I_3. \end{aligned}$$

We will show how each of these integrals can be transformed properly.

Term I_1 : By Lemma 3.4 we find $\psi_k \in \mathcal{D}(\overline{\Omega}; \mathbb{R}^3)$ such that

$$\psi_k \xrightarrow{k \rightarrow \infty} \psi \text{ in } \mathbf{H}(\operatorname{curl}, \Omega) \quad \text{and} \quad \psi_k \times \nu \xrightarrow{k \rightarrow \infty} \psi \times \nu \text{ in } \mathbf{L}^2(\Gamma).$$

Then, we also have $\nu \times (\psi_k \times \nu) \xrightarrow{k \rightarrow \infty} \nu \times (\psi \times \nu) = \psi - (\psi \cdot \nu)\nu = \psi$ in $\mathbf{L}^2(\Gamma)$ and by classical Green’s formula (1.1) we obtain

$$\begin{aligned} \int_{\Omega} (\varphi \cdot \operatorname{curl} \psi - \psi \cdot \operatorname{curl} \varphi) &= \lim_{k \rightarrow \infty} \int_{\Omega} (\varphi \cdot \operatorname{curl} \psi_k - \psi_k \cdot \operatorname{curl} \varphi) \\ &\stackrel{(1.1)}{=} \lim_{k \rightarrow \infty} \int_{\Gamma} (\varphi \times \nu) \cdot \psi_k \\ &= \lim_{k \rightarrow \infty} \int_{\Gamma} (\varphi \times \nu) \cdot (\nu \times (\psi_k \times \nu)) \end{aligned}$$

$$= \int_{\Gamma} (\boldsymbol{\varphi} \times \boldsymbol{\nu}) \cdot \boldsymbol{\psi}.$$

Term I_2 : We use the density of $H^2(\Omega)$ in $H^1(\Omega)$. So, let $\chi_k \in H^2(\Omega)$ be such that $\chi_k \xrightarrow{k \rightarrow \infty} \chi$ in $H^1(\Omega)$. Then

$$\begin{aligned} \int_{\Omega} \nabla \chi \cdot \operatorname{curl} \boldsymbol{\psi} &= \lim_{k \rightarrow \infty} \int_{\Omega} \nabla \chi_k \cdot \operatorname{curl} \boldsymbol{\psi} \stackrel{(3.2)}{=} - \lim_{k \rightarrow \infty} \langle \boldsymbol{\psi} \times \boldsymbol{\nu}, \nabla \chi_k \rangle_{\Gamma} \\ &\stackrel{(3.3)}{=} \lim_{k \rightarrow \infty} \langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi_k \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)}. \end{aligned}$$

But, from Lemma 3.5 we know that

$$\langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi_k \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = \langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi_k \rangle_{\Gamma},$$

so that we conclude

$$\lim_{k \rightarrow \infty} \langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi_k \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = \langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi \rangle_{\Gamma}.$$

Term I_3 : To be treated like I_2 .

In total, the proof is complete. $\square$

Proposition 5.2. *The right-hand side of our Green formula (5.2)*

$$b_{\Gamma}(\mathbf{u}, \mathbf{v}) := \int_{\Gamma} (\boldsymbol{\varphi} \times \boldsymbol{\nu}) \cdot \boldsymbol{\psi} + \langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi \rangle_{\Gamma} - \langle \operatorname{div}_{\Gamma}(\boldsymbol{\varphi} \times \boldsymbol{\nu}), \lambda \rangle_{\Gamma}$$

gives rise to a bounded, anti-symmetric bilinear form $b_{\Gamma}(\cdot, \cdot)$ on $\mathbf{H}(\operatorname{curl}, \Omega) \times \mathbf{H}(\operatorname{curl}, \Omega)$.

Proof. The linearity in both components follows from the uniqueness from Lemma 4.6 and the observation that the constant in χ is harmless because

$$\langle \operatorname{div}_{\Gamma} \boldsymbol{\varphi}, \alpha \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} \stackrel{\text{Def. 3.1}}{=} - \int_{\Gamma} \boldsymbol{\varphi} \cdot \nabla_{\Gamma} \alpha = 0 \quad \forall \boldsymbol{\varphi} \in \mathbf{L}_{\tau}^2(\Gamma), \alpha \in \mathbb{R}.$$

The boundedness follows from several previous estimates. By Lemma 4.6 we have $\|\boldsymbol{\varphi} \times \boldsymbol{\nu}\|_{\mathbf{L}^2(\Gamma)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}$ and since $\boldsymbol{\psi}$ is tangential we have $\boldsymbol{\psi} = \boldsymbol{\nu} \times (\boldsymbol{\psi} \times \boldsymbol{\nu})$ and estimate $\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Gamma)} = \|\boldsymbol{\psi} \times \boldsymbol{\nu}\|_{\mathbf{L}^2(\Gamma)} \leq C(\Omega) \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}$ by Lemma 4.6. Furthermore,

$$\|\operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu})\|_{H^{-\frac{1}{2}}(\Gamma)} \stackrel{\text{Lem. 3.5}}{\leq} C(\Omega) \|\boldsymbol{\psi}\|_{\mathbf{H}(\operatorname{curl}, \Omega)} \stackrel{\text{Lem. 4.6}}{\leq} C(\Omega) \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}.$$

Setting $\chi_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} \chi$ we deduce by Poincaré's inequality:

$$\|\chi - \chi_{\Omega}\|_{H^{\frac{1}{2}}(\Gamma)} \leq C(\Omega) \|\chi - \chi_{\Omega}\|_{H^1(\Omega)} \leq C(\Omega) \|\nabla \chi\|_{\mathbf{L}^2(\Omega)} \stackrel{\text{Lem. 4.6}}{\leq} C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}.$$

Hence,

$$\begin{aligned} |\langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi \rangle_{\Gamma}| &= |\langle \operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu}), \chi - \chi_{\Omega} \rangle_{\Gamma}| \\ &\leq \|\operatorname{div}_{\Gamma}(\boldsymbol{\psi} \times \boldsymbol{\nu})\|_{H^{-\frac{1}{2}}(\Gamma)} \|\chi - \chi_{\Omega}\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\leq C(\Omega) \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}, \Omega)} \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl}, \Omega)} \end{aligned}$$

and similar for the remaining term. In total we deduce

$$|b_{\Gamma}(\mathbf{u}, \mathbf{v})| \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl}, \Omega)} \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}, \Omega)}.$$

The anti-symmetry follows from the property of the scalar triple product:

$$(\psi \times \nu) \cdot \varphi = -(\varphi \times \nu) \cdot \psi. \quad \square$$

We present a variant of the Green's formula from Theorem 5.1 involving in the decompositions only three functions:

Theorem 5.3. *Let $u, v \in \mathbf{H}(\text{curl}, \Omega)$. Let $\Psi \in \mathbf{H}^1(\Omega)$, $\theta \in H^1(\Omega)$ be given by (4.4) and $\varphi \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\text{curl}, \Omega)$ be given by (4.7). Then*

$$\int_{\Omega} (u \cdot \text{curl } v - v \cdot \text{curl } u) = \langle u \times \nu, \Psi \rangle_{\Gamma} - \langle \text{div}_{\Gamma}(\varphi \times \nu), \theta \rangle_{\Gamma}.$$

Proof. By Lemma 4.2 we find $\Psi \in \mathbf{H}^1(\Omega)$ and $\theta \in H^1(\Omega)$ such that $v = \Psi + \nabla \theta$. Then

$$\begin{aligned} \int_{\Omega} (u \cdot \text{curl } v - v \cdot \text{curl } u) &= \int_{\Omega} (u \cdot \text{curl } \Psi - \Psi \cdot \text{curl } u) - \int_{\Omega} \nabla \theta \cdot \text{curl } u \\ &\stackrel{(1.1)}{=} \langle u \times \nu, \Psi \rangle_{\Gamma} - \int_{\Omega} \nabla \theta \cdot \text{curl } \varphi \\ &= \langle u \times \nu, \Psi \rangle_{\Gamma} - \langle \text{div}_{\Gamma}(\varphi \times \nu), \theta \rangle_{\Gamma} \end{aligned}$$

where we have used a decomposition $u = \varphi + \nabla \chi$ with $\chi \in H^1(\Omega)$ and $\varphi \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\text{curl}, \Omega)$ with $\varphi \in \mathbf{L}_{\tau}^2(\Gamma)$ and argued for the remaining integral like in the transformation of the integral I_2 in the proof of the previous Theorem 5.1. $\square$

6. SURJECTIVITY ON $\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$

In this final section, we investigate the expression $\text{div}_{\Gamma}(u \times \nu)$ for $u \in \mathbf{H}(\text{curl}, \Omega)$. Since the normal vector field ν belongs only to $\mathbf{L}^{\infty}(\Gamma)$ for Lipschitz domains, the scalar product $\nu \cdot \lambda$ is not well defined for $\lambda \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$. To characterize tangential vector fields in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$, we introduce the following definition inspired by [16, Rem. 3].

Definition 6.1. *A vector field $\lambda \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is called tangential if and only if*

$$\langle \lambda, \nabla \psi \rangle_{\Gamma} = 0 \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega). \quad (6.1)$$

In this case we write $\lambda \in \mathbf{H}_{\tau}^{-\frac{1}{2}}(\Gamma)$.

Example: Let $u \in \mathbf{H}(\text{curl}, \Omega)$. Then $u \times \nu \in \mathbf{H}_{\tau}^{-\frac{1}{2}}(\Gamma)$. Indeed, $u \times \nu \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and from (3.1) and (3.2) we get (6.1) with $\lambda = u \times \nu$.

Remark 6.2.

- i) Recall that for any $\mu \in \mathbf{H}_{\nu}^{\frac{1}{2}}(\Gamma) =: \{\mu \in \mathbf{H}^{\frac{1}{2}}(\Gamma) : \mu_{\tau} = 0\}$, where $\mu_{\tau} := \mu - (\mu \cdot \nu)\nu$, there exists a $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\nabla \psi = \mu$ on Γ , cf. [3, Sec. 4.2]. So, $\lambda \in \mathbf{H}_{\tau}^{-\frac{1}{2}}(\Gamma)$ means, that $\lambda \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and

$$\langle \lambda, \mu \rangle_{\Gamma} = 0 \quad \forall \mu \in \mathbf{H}_{\nu}^{\frac{1}{2}}(\Gamma).$$

ii) Let $\boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$ satisfying $\boldsymbol{\lambda} \cdot \boldsymbol{\nu} = 0$ on Γ . Then, $\boldsymbol{\lambda} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and

$$\langle \boldsymbol{\lambda}, \nabla \psi \rangle_\Gamma = \int_\Gamma \boldsymbol{\lambda} \cdot \boldsymbol{\nu} \partial_\nu \psi = 0 \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega).$$

So, the definition (6.1) is a natural extension of tangentiality to $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ -fields and we have $\mathbf{L}_\tau^2(\Gamma) \hookrightarrow \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma)$.

We recall the following definition, which goes back to Tartar [16, Thm 2], see also [2, Def. 2.1]:

Definition 6.3. Let $\boldsymbol{\lambda} \in \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma)$. Its surface divergence $\operatorname{div}_\Gamma \boldsymbol{\lambda}$ is given via

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \psi \rangle := -\langle \boldsymbol{\lambda}, \nabla \psi \rangle_\Gamma \quad \forall \psi \in H^2(\Omega). \quad (6.2)$$

By this definition, we clearly have the following property: the operator

$$\operatorname{div}_\Gamma : \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-1}(\Gamma)$$

is linear and continuous.

Remark 6.4. This definition coincides with the Definition 3.1 for $\mathbf{L}_\tau^2(\Gamma)$ -fields. In the following let $\boldsymbol{\lambda} \in \mathbf{L}_\tau^2(\Gamma)$. We first show, how to obtain (6.2) from Definition 3.1. Note that $\boldsymbol{\lambda} \in \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma)$ and consider $\psi \in H^2(\Omega)$. Then, $\psi|_\Gamma \in H^1(\Gamma)$ and $\nabla \psi \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$. So we have

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \psi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} \stackrel{\text{Def. 3.1}}{=} - \int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \psi = - \int_\Gamma \boldsymbol{\lambda} \cdot \nabla \psi = -\langle \boldsymbol{\lambda}, \nabla \psi \rangle_\Gamma,$$

which gives us (6.2). In contrast, to recover Definition 3.1 from Definition 6.3 for $\mathbf{L}_\tau^2(\Gamma)$ -fields, we start with $\mu \in H^1(\Gamma)$. From Lemma 3.3, there exists a sequence $(\psi_k)_k \subset H^2(\Omega)$ such that $\psi_k|_\Gamma$ converges to μ in $H^1(\Gamma)$. Since $\operatorname{div}_\Gamma \boldsymbol{\lambda} \in H^{-1}(\Gamma)$, we have

$$\begin{aligned} \langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \mu \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} &= \lim_{k \rightarrow \infty} \langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \psi_k \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} \stackrel{\text{Def. 6.3}}{=} - \lim_{k \rightarrow \infty} \int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \psi_k \\ &= - \int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \mu, \end{aligned}$$

which recovers Definition 3.1.

Consider the two spaces

$$\begin{aligned} \gamma_\tau(\mathbf{H}(\operatorname{curl}, \Omega)) &:= \{\mathbf{u} \times \boldsymbol{\nu} : \mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)\}, \\ \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) &:= \{\boldsymbol{\lambda} \in \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma) : \operatorname{div}_\Gamma \boldsymbol{\lambda} \in H^{-1}(\Gamma)\}. \end{aligned}$$

Theorem 6.5. It holds $\gamma_\tau(\mathbf{H}(\operatorname{curl}, \Omega)) = \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$.

Proof. We start by showing the inclusion $\gamma_\tau(\mathbf{H}(\operatorname{curl}, \Omega)) \subseteq \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$.

Let $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ and $\chi \in H^1(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\Omega) \cap \mathbf{H}(\operatorname{curl}, \Omega)$, with $\boldsymbol{\varphi}|_\Gamma \in \mathbf{L}_\tau^2(\Gamma)$, be the functions given by the decomposition (4.7). By Lemma 3.5 we have $\boldsymbol{\varphi} \times \boldsymbol{\nu} \in$

$\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$. Furthermore, since $\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, we have $\nabla \chi \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and we obtain

$$\langle \nabla \chi \times \boldsymbol{\nu}, \nabla \psi \rangle_\Gamma \stackrel{(1.1)}{=} \int_\Omega (\nabla \chi \cdot \operatorname{curl} \nabla \psi - \nabla \psi \cdot \operatorname{curl} \nabla \chi) = 0$$

for all $\psi \in H^2(\Omega)$. That means that $\nabla \chi \times \boldsymbol{\nu} \in \mathbf{H}_\tau^{-\frac{1}{2}}(\Gamma)$ and by Definition 6.3 we have $\operatorname{div}_\Gamma(\nabla \chi \times \boldsymbol{\nu}) = 0$, i.e., also $\nabla \chi \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$. Hence,

$$\mathbf{u} \times \boldsymbol{\nu} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma).$$

Conversely, let $\boldsymbol{\lambda} \in \mathbf{H}_\tau^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$. Then, by [16, Thm. 2] we find a $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ such that $\boldsymbol{\lambda} = \mathbf{u} \times \boldsymbol{\nu}$ on Γ . $\square$

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REFERENCES

- [1] Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.* **21**, no. 9, (1998), pp. 823–864.
- [2] Alonso, A., Valli, A.: Some remarks on the characterization of the space of tangential traces of $H(\operatorname{rot}; \Omega)$ and the construction of an extension operator, *Manuscr. Math.* **89**, no. 2 (1996), pp. 159–178.
- [3] Amrouche, C., Moussaoui, M.: The Dirichlet problem for the Laplacian in Lipschitz domains revisited, arXiv: 2509.08543, (2026).
- [4] Assous, F., Ciarlet, P., Labrunie, S.: *Mathematical foundations of computational electromagnetism*, Applied Mathematical Sciences **198**, Springer, (2018).
- [5] Ben Belgacem, F., Bernardi, C., Costabel, M., Dauge, M.: Un résultat de densité pour les équations de Maxwell, *C. R. Acad. Sci., Paris, Sér. I* **324**, no. 6, (1997), pp. 731–736.
- [6] Buffa, A.: Trace theorems on non-smooth boundaries for functional spaces related to Maxwell equations: An overview, *Proceedings of the GAMM workshop on computational electromagnetics*, Kiel, Germany, January 26–28, 2001. Berlin: Springer, Lect. Notes Comput. Sci. Eng. **28** (2003), pp. 23–34.
- [7] Buffa, A., Costabel, M., Sheen, D.: On traces for $\mathbf{H}(\operatorname{curl}, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.* **276**, no. 2, (2002), pp. 845–867.
- [8] Chen, Z., Du, Q., Zou, J.: Finite element methods with matching and nonmatching meshes for Maxwell equations with discontinuous coefficients, *SIAM J. Numer. Anal.* **37**, no. 5 (2000), pp. 1542–1570.
- [9] Costabel, M.: A remark on the regularity of solutions of Maxwell’s equations on Lipschitz domains, *Math. Methods Appl. Sci.* **12**, no. 4, (1990), pp. 365–368.
- [10] Girault, V., Raviart, P.-A.: *Finite element methods for Navier-Stokes equations. Theory and algorithms*. Springer Series in Computational Mathematics **5**, Berlin, Springer-Verlag, (1986).
- [11] Hiptmair, R.: Finite elements in computational electromagnetism, *Acta Numerica* **11** (2002), pp. 237–339.
- [12] Hiptmair, R., Li, J., Zou, J.: Universal extension for Sobolev spaces of differential forms and applications, *J. Funct. Anal.* **263**, no. 2 (2012), pp. 364–382.
- [13] Jerison, D.S., Kenig, C.E.: The Neumann problem on Lipschitz domains, *Bull. Am. Math. Soc., New Ser.* **4** (1981), pp. 203–207.
- [14] Nečas, J.: *Les méthodes directes en théorie des équations elliptiques*, Paris: Masson et Cie; Prague: Academia, (1967).
- [15] Paquet, L.: Problèmes mixtes pour le système de Maxwell, *Ann. Fac. Sci. Toulouse, V. Ser., Math.* **4** (1982), pp. 103–141.

- [16] Tartar, L: On the characterization of traces of a Sobolev space used for Maxwell's equation, *Meeting held in Bordeaux on November 6–7, (1997), in honour of Michel Artola, Alain-Yves Le Roux, Sep 1997, Bordeaux, France.* [⟨hal-04552466⟩](#)

(Aissa Aibeche) LABORATOIRE DE MATHÉMATIQUES APPLIQUÉES, UNIVERSITÉ FERHAT ABBAS, SÉTIF 1,
CAMPUS EL BEZ, 19137 SÉTIF, ALGERIA
Email address: `aibeche@univ-setif.dz`

(Chérif Amrouche) LABORATOIRE DE MATHÉMATIQUES ET LEURS APPLICATIONS, UNIVERSITÉ DE PAU ET
DES PAYS DE L'ADOUR, AVENUE DE L'UNIVERSITÉ, 64000 PAU, FRANCE
Email address: `cherif.amrouche@univ-pau.fr`

(Peter Lewintan) FACULTY OF MATHEMATICS, UNIVERSITY OF DUISBURG-ESSEN, THEA-LEYMANN-STR. 9,
45127 ESSEN, GERMANY
Email address: `peter.lewintan@uni-due.de`