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Locally Conformally Flat Manifolds with Positive Scalar Curvature: Kleinian Groups, Moduli Spaces, and Euclidean Rigidity

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Abstract

We study locally conformally flat (LCF) Riemannian manifolds with nonnegative or positive scalar curvature (PSC), using the conformal boundary of the developing image. For closed oriented LCF n-manifolds with PSC and infinite fundamental group, $n \geq 5$, we bound the macroscopic dimension of their Riemannian universal covers by $\lfloor (n-1)/2 \rfloor$, establish the existence of a nontrivial homotopy group above the middle dimension, and, under mild additional hypotheses, bound the Hausdorff dimension of the limit sets of their Kleinian groups in the interval $(1, (n-2)/2)$. In particular, no closed aspherical manifold admits an LCF metric with PSC. If the scalar curvature is at least $n(n-1)$ and the manifold is not isometric to the round sphere, then every smooth nonzero-degree map to the sphere expands somewhere when its Kleinian group is elementary, while the developing map expands somewhere whenever the fundamental group is infinite. We prove that the space of LCF metrics with PSC and its moduli space are contractible in dimension three for finite fundamental group and for $S^2 \times S^1$, and that the moduli space is empty or contractible for smooth manifolds homeomorphic to spherical space forms but not diffeomorphic to S^n in dimensions $n \geq 4$. For complete open simply connected LCF manifolds of nonnegative scalar curvature, we obtain Euclidean rigidity under additional topological hypotheses at infinity (and, in dimension three, from vanishing second homology alone). We also show that, in dimensions $n \geq 4$, the Euclidean conclusion can fail when neither of the two additional topological hypotheses is assumed: we construct complete contractible examples of PSC that are not homeomorphic to $\mathbb{R}^n$.

Contents

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1 Introduction

This paper is the second in a project [Den25] devoted to the geometry of locally conformally flat (LCF) Riemannian manifolds with positive scalar curvature (PSC). By Kuiper's theorem [Kui49], every closed, oriented, simply connected LCF n-manifold, $n \geq 3$, is conformally diffeomorphic to the round sphere (S^n, g_{st}). The case of nontrivial fundamental group is considerably more flexible. Whereas [Den25] treats LCF 4-manifolds with negative scalar curvature, the present paper treats LCF n-manifolds in all dimensions $n \geq 3$, under nonnegative or positive scalar curvature assumptions, and develops their geometric, topological, and metric consequences. The results of this paper are driven by a single mechanism: scalar curvature imposes quantitative control on the conformal boundary of the developing image, and this control is translated into global rigidity, dimension bounds, and topological restrictions by means of Kleinian group theory, shape theory, potential theory, capacity, and topology. The control itself appears through Hausdorff dimension bounds for the limit set, boundary behavior of conformal factors, and Patterson–Sullivan theory.

In dimensions $n \geq 4$, Schoen and Yau [SY88] prove that if an LCF manifold (M^n, g) has PSC, then its developing map into (S^n, g_{st}) is injective, without any assumption on the fundamental group. With its image denoted by $\Omega \subset S^n$, the associated holonomy representation identifies $\pi_1(M)$ with a Kleinian group Γ, yielding a realization $M^n \cong \Omega/\Gamma$. With $\Lambda(\Gamma) := S^n \setminus \Omega$, the limit set of Γ, the Schoen–Yau theorem further gives

$$\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) \leq \frac{n-2}{2}. \tag{1}$$

These results naturally lead to a problem posed by Yau in his 1990 problem list [Yau93, Problem 2], which asks for a description of the possible algebraic and geometric structures of the fundamental group $\pi_1(M)$ of a closed LCF manifold. They are also related to another problem in Yau's problem list [Yau93, Problem 36], which asks whether one can characterize the domain Ω. Yau remarks that, in the case where the scalar curvature can be made constant, the limit set $\Lambda(\Gamma)$ coincides with the singular set of the natural Yamabe equation.

On the other hand, using the Gauss–Bonnet–Chern formula, Gursky [Gur94] shows that if $n = 4$ or 6 and $H_1(M; \mathbb{Z}) = 0$, then (M^n, g) is conformally equivalent to the round sphere (S^n, g_{st}). He asks whether this rigidity phenomenon persists in other dimensions $n \geq 3$. In higher dimensions, however, the Gauss–Bonnet–Chern integrand becomes less directly effective for extracting scalar curvature rigidity. We therefore take a different route: Patterson–Sullivan theory for the Kleinian group Γ converts the size of the limit set into algebraic information about Γ, and this yields a mechanism available in dimensions $n \geq 5$.

When the fundamental group is finite, we answer Gursky's question affirmatively when $n \not\equiv 3 \pmod 4$ and negatively when $n \equiv 3 \pmod 4$; see Theorem 2.1. In the four-dimensional case, the author [Den25, p. 12] gives an alternative proof of Gursky's result using the classification of closed, LCF 4-manifolds with PSC.

The most intriguing cases of Yau's two problems and Gursky's question arise when the fundamental group is infinite and $n \geq 5$. Our main result in this direction is the following theorem.

Theorem A. *Let (M^n, g), $n \geq 5$, be a closed, connected, oriented, locally conformally flat Riemannian manifold with positive scalar curvature and infinite fundamental group. Then there exists an integer $i \geq \lfloor n/2 \rfloor + 1$ such that $\pi_i(M^n) \neq 0$, and its universal Riemannian cover $(\widetilde{M}, \widetilde{g})$ has macroscopic dimension at most $\lfloor \frac{n-1}{2} \rfloor$.*

Let $M^n \cong \Omega/\Gamma$ be the Kleinian realization given by the developing map. If, in addition, $H_1(M^n; \mathbb{Z}) = 0$, then Γ is non-elementary. Under these additional assumptions, if Γ is torsion-free, then the Hausdorff dimension of its limit set satisfies

$$1 \; < \; \dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) \; < \; \frac{n-2}{2}. \tag{2}$$

Theorem A gives a partial answer to Yau's Problems 2 and 36 and to Gursky's question. The lower bound in (2) is the new content: combined with (1), it traps $\dim_{\mathcal{H}}(\Lambda(\Gamma))$ in the interval $(1, (n-2)/2)$ under the hypotheses $H_1(M; \mathbb{Z}) = 0$ and Γ torsion-free. This constrains the geometry of Ω (addressing Problem 36) and exhibits nontrivial algebraic structure of $\pi_1(M)$ (addressing Problem 2). In particular, since $\dim_{\mathcal{H}}(\Lambda(\Gamma)) > 1$ implies $\Lambda(\Gamma) \neq \varnothing$ and hence $M \not\cong S^n$ conformally, Gursky's rigidity fails when $\pi_1(M)$ is infinite. Using Schoen and Yau's Liouville theorem [LUY24b], the proof can be generalized to the case of nonnegative scalar curvature, see Corollary 2.15.

Theorem A also verifies Gromov's macroscopic dimension conjecture [Gro96] for LCF manifolds. It asserts that if (M^n, g) is a closed Riemannian manifold with PSC, then the macroscopic dimension of its universal cover is at most $n-2$. The conjecture is established for 3-manifolds by Bolotov [Bol03], while in higher dimensions only partial results are available; see, for example, [Dra13, DH25].

The first conclusion follows by combining Schoen–Yau's low degree homotopy vanishing and limit set estimate [SY88] with Izeki's convex cocompactness theorem [Ize02] and Kapovich's formula for cohomological dimension [Kap09, Proposition 9.6]. Non-elementarity of Γ under $H_1(M^n; \mathbb{Z}) = 0$ is established by a separate algebraic argument (Lemma 2.8 in the body). For the Hausdorff dimension bound, Sullivan's theorem [Sul84] gives $\delta(\Gamma) = \dim_{\mathcal{H}}(\Lambda(\Gamma))$ for non-elementary convex cocompact groups. To obtain the strict lower bound, set $d = \dim_{\mathrm{top}} \Lambda(\Gamma)$. If $d = 0$, Kapovich's formula [Kap09, Proposition 9.6] gives $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 1$, and the Stallings–Swan theorem, in the form recorded by Kapovich [Kap09, Theorem 2.5], then implies that Γ is free. If $d = 1$ and $\dim_{\mathcal{H}} \Lambda(\Gamma) \leq 1$, Kapovich's rigidity theorem [Kap09, Theorem 1.3] shows that the limit set is a round circle; torsion-freeness and convex cocompactness then identify Γ with a closed hyperbolic surface group. Both alternatives contradict $H_1(M^n; \mathbb{Z}) = 0$. Nayatani's criterion [Nay97, Corollary 3.4] supplies the strict upper bound. For the macroscopic dimension, the proof realizes $\pi_1(M)$ as a convex cocompact Kleinian group whose limit set, by PSC, has strictly bounded Hausdorff dimension, the floor function arising from integrality of topological dimension. The geometric action on the convex hull identifies this limit set with the Gromov boundary, so a boundary formula computes the asymptotic dimension, which quasi-isometry invariance transfers to the universal cover and which bounds the macroscopic dimension.

An immediate topological consequence of Theorem A, which also plays a role in the proof of Theorem C below, is the following.

Corollary B (Corollary 2.7)**.** *A closed, connected, oriented, aspherical manifold of dimension $n \geq 5$ does not admit a locally conformally flat metric with positive scalar curvature.*

Beyond these topological consequences, conformally flat geometries also arise as backgrounds for rigid supersymmetry in gauged-supergravity constructions [KR13]. This connection provides additional motivation for studying their global rigidity. We next investigate pointwise Lipschitz properties of natural maps associated with LCF manifolds of PSC.

Theorem C. *Let (M^n, g), $n \geq 4$, be a closed, connected, oriented, locally conformally flat manifold with scalar curvature $\mathrm{Sc}_g \geq n(n-1)$, and assume that (M^n, g) is not isometric to (S^n, g_{st}). View its fundamental group $\pi_1(M)$ as a Kleinian group.*

(1) If $\pi_1(M)$ is elementary, then every smooth map of nonzero degree $f \colon (M^n, g) \to (S^n, g_{st})$ has a point $p \in M$ at which the pointwise Lipschitz constant of f is strictly greater than 1.

(2) If $\pi_1(M)$ is infinite (whether elementary or non-elementary), then for the developing map $\mathrm{dev} \colon (\widetilde{M}, \tilde{g}) \to (S^n, g_{st})$, there exists a point $x \in \widetilde{M}$ such that the pointwise Lipschitz constant satisfies $\mathrm{Lip}_x(\mathrm{dev}) > 1$.

For assertion (1), suppose that $\pi_1(M)$ is elementary. If the pointwise Lipschitz constant were everywhere bounded above by 1, then Proposition 3.3 would imply that (M^n, g) is isometric to (S^n, g_{st}), a contradiction. For assertion (2), suppose that $\pi_1(M)$ is infinite. Injectivity of the developing map identifies $\widetilde{M}$ conformally with a proper domain $\Omega \subsetneq S^n$. With $(\mathrm{dev}^{-1})^* \tilde{g} = u^{4/(n-2)} g_{st}$ on Ω, Proposition 3.5 gives $\min_\Omega u < 1$. Since the pointwise Lipschitz constant of dev at the point corresponding to $x \in \Omega$ is $u(x)^{-2/(n-2)}$, the conclusion follows.

The developing-map framework used in Theorems A and C also provides information about the topology of spaces and moduli spaces of locally conformally flat metrics with positive scalar curvature. Questions concerning the topology and rigidity of such moduli spaces have recently arisen in dimension three. Moroianu, Pino Carmona, and Shahbazi[1] [MPS26] show that every closed three-dimensional heterotic soliton with vanishing torsion which is LCF and has constant scalar curvature is rigid, in the sense that it is an isolated point of the corresponding moduli space. In a different direction, Bamler and Kleiner [BK19] prove, using Ricci flow through singularities, that if M^3 is a closed, connected, oriented smooth three-manifold, then the space of positive scalar curvature metrics on M^3 is either empty or contractible. They ask in Question 1.8 whether the same conclusion holds for the space of conformally flat PSC metrics.

In dimension 3, the notion of conformal flatness used in their paper coincides with the notion of LCF used in our work: a Riemannian metric is LCF if and only if its Cotton tensor vanishes, equivalently, if its Schouten tensor satisfies the Cotton–York condition; in dimensions $n \geq 4$ this is equivalent to the vanishing of the Weyl tensor. Ricci flow does not preserve local conformal flatness in dimension 3, so the Bamler–Kleiner approach is unavailable in the LCF setting. They also note that such a contractibility statement would imply Theorem 1.6 and Corollary 1.7 of their paper.

Theorem D. *Let M^n be a connected, closed, oriented smooth n-manifold. Denote by $X(M^n)$ the space of locally conformally flat metrics with positive scalar curvature on M^n, and by $\mathcal{M}(M^n)$ its moduli space. Then:*

(i) If $n = 3$ and either $\pi_1(M^3)$ is finite or M^3 is diffeomorphic to $S^2 \times S^1$, then both $X(M^3)$ and $\mathcal{M}(M^3)$ are contractible.

(ii) If $n \geq 4$ and M^n is homeomorphic to a spherical space form but not diffeomorphic to S^n, then $\mathcal{M}(M^n)$ is either empty or contractible.

Theorem D(i) gives a positive answer to Bamler and Kleiner's Question 1.8 for 3-manifolds with finite fundamental group, and extends it to the case $M^3 \cong S^2 \times S^1$. The empty alternative in Theorem D(ii) is essential. For example, if M is an exotic sphere, then Kuiper's theorem implies that $X(M) = \varnothing$, since an LCF metric on M would force M to be diffeomorphic to S^n. The author shows [Den21c] that the conformal class of a metric is dense in the moduli space in the Gromov–Hausdorff topology; whether $\mathcal{M}(M^n)$ is contractible in that topology remains open.

For $n = 3$ with finite fundamental group, applying the normalization argument of Bamler and Kleiner [BK19, Lemma 9.2(a)] on the universal cover and then descending the unique minimizing conformal factor to M produces a canonical $\mathrm{Diff}(M)$-equivariant normalized round representative in each conformal class. Interpolation of the corresponding conformal factors gives a strong deformation retraction, and their Theorem 1.2 makes its target contractible. Equivariance allows the deformation to descend to the moduli space, whose round locus is a point by de Rham rigidity [dR50]. For $M \cong S^2 \times S^1$, the same interpolation retracts onto the locally cylindrical metrics, whose moduli space is identified explicitly with $(0, \infty) \times [0, \pi]$. For $n \geq 4$, nonemptiness of $X(M)$ and Kuiper's theorem, followed by a Cartan fixed-point argument, identify M diffeomorphically with a spherical space form. Marques' result

[1] I thank Carlos Shahbazi for bringing their beautiful paper to my attention.

[Mar12, Corollary 3.2] supplies a positive constant sectional curvature representative in each conformal class, whereas Obata's theorem [Oba71] makes its curvature one normalization unique. The implicit function theorem and the strict Lichnerowicz–Obata spectral gap give continuity of this canonical projection. Interpolation then yields the equivariant retraction, and de Rham rigidity completes the proof.

Theorems A, C, and D use the developing map to study closed LCF manifolds with PSC. We next turn to complete noncompact manifolds. In the simply connected case, the holonomy is trivial, and the developing map realizes M itself as a domain $\Omega \subset S^n$. The central object is then the conformal boundary $\Lambda := S^n \setminus \Omega$. Since no nontrivial Kleinian group remains, the topology of M must be read directly from Λ. Shape theory provides the mechanism for doing this without assuming any regularity of Λ.

This point of view is related to a classical rigidity problem for open three-manifolds. Gromov and Lawson [GL83, Corollary 10.9], by minimal surface methods, show that a complete noncompact 3-manifold with uniformly positive scalar curvature and finitely generated fundamental group is simply connected at infinity. Since an open contractible 3-manifold which is simply connected at infinity is homeomorphic to $\mathbb{R}^3$ [ha], it follows that any complete open contractible 3-manifold with uniformly positive scalar curvature is homeomorphic to $\mathbb{R}^3$; see also the index-theoretic proof in [CWY10, Theorem 1]. Motivated by the work of Gromov–Lawson, the following conjecture has circulated in the field for decades, but has only recently begun to receive attention: A complete, connected, open, contractible 3-manifold with nonnegative scalar curvature is diffeomorphic to $\mathbb{R}^3$.

Under additional geometric or topological assumptions, the conjecture is partially verified in recent work; see [Car26, p. 2]. The LCF case requires new methods: the conformal boundary Λ may be a wild compact set with no smoothness or rectifiability, so the minimal surface and index theoretic methods do not apply. We introduce shape-theoretic methods to extract topological information on Λ directly from the global topology of M, without imposing any regularity on Λ. To the best of our knowledge, this is the first use of shape theory to study developing map limit sets in the scalar curvature setting. Furthermore, the dictionary relating properties (algebraic, coarse, etc.) of Kleinian groups to the global and local shape of their limit sets, as well as to the homology and end topology of the associated domains of discontinuity, will be developed in a subsequent paper in this project [Den26].

Theorem E. *Let (M^n, g), $n \geq 3$, be a complete, open, simply connected, locally conformally flat n-manifold with nonnegative scalar curvature. Let $\Phi : M \to \Phi(M) =: \Omega \subset S^n$ be the developing map, and set $\Lambda := S^n \setminus \Omega$. Assume that one of the following holds:*

(i) $n = 3$ and $H_2(M; \mathbb{Z}) = 0$;

(ii) $n \geq 4$, M is $\lfloor \frac{n-2}{2} \rfloor$-connected at infinity, and Λ satisfies the small loops condition in S^n.

Then M is homeomorphic to $\mathbb{R}^n$.

The proof proceeds as follows. By the Liouville theorem of Schoen and Yau, M^n embeds as a domain $\Omega \subset S^n$. The Ma–Qing estimate [MQ25, Theorem 1.3] yields

$\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2}{2}$ and $\dim(\Lambda) \leq \lfloor \frac{n-2}{2} \rfloor =: d$. When $n = 3$, Čech–Alexander duality and the hypothesis $H_2(M;\mathbb{Z}) = 0$ imply that Λ is connected; hence it is a single point. Suppose that $n \geq 4$. The d-connectedness at infinity implies that Ω is one-ended. The non-separation theorem then identifies the ends of Ω with the components of Λ, so Λ is connected. The connectivity at infinity, combined with the small loops condition and the dimension bound, gives the vanishing of the homotopy pro-groups of Λ. Morita's finite-dimensional Whitehead theorem [Mor75, Theorem 1.2, p. 394] therefore implies that Λ has the shape of a point and hence has Property UV^∞. The same pro-triviality at infinity also yields the cellularity criterion. Repovš's theorem [Rep87, Theorem, p. 564] for $n = 4$, and McMillan's cellularity criterion in its cell-like form [DV09, Theorem 3.2.3, p. 107] for $n \geq 5$, originating in [McM64, Theorem 1, p. 327], then show that Λ is cellular in S^n. Consequently, collapsing Λ to a point produces a space homeomorphic to S^n, and hence Ω is homeomorphic to $\mathbb{R}^n$.

Theorem E(i) settles the conjecture above for three-dimensional LCF manifolds with nonnegative scalar curvature. In dimension four, Theorem 5.5 gives a differentiable strengthening in which the hypothesis in Theorem E(ii) is replaced by $H_3(M;\mathbb{R}) = 0$, bounded geometry, and uniformly positive scalar curvature, and M is shown to be diffeomorphic to the standard $\mathbb{R}^4$. Beyond the topological hypotheses, Corollary 5.4 identifies five analytic and asymptotic-geometric conditions, each independently implying Euclidean rigidity, thereby partially answering Yau's Problem 36 [Yau93, Problem 36]. On the other hand, the Euclidean conclusion in Theorem E(ii) can fail when neither of its additional topological hypotheses is assumed, as Corollary 6.20 makes precise; the following theorem supplies the underlying examples.

Theorem F. *For every $n \geq 4$, there exists a complete open contractible locally conformally flat n-manifold with positive scalar curvature that is not homeomorphic to $\mathbb{R}^n$. Moreover, for every $n \geq 6$, there exists such a manifold with uniformly positive scalar curvature.*

Karakhanyan's theorem [Kar24, Theorem 1.1] motivates the metric construction. It characterizes the compact sets $K \subset S^n$ for which $S^n \setminus K$ admits a complete scalar-flat metric conformal to the round metric: this occurs if and only if the corresponding spherical Bessel capacity vanishes, in the sense of [Kar24, Section 2.2].

The proof has complementary topological and analytic parts. For $n \geq 4$, we construct an increasingly thin nested tower whose bonding maps kill homology while preserving fundamental-group information. Its limit is a Čech-acyclic compactum whose complement is contractible but not simply connected at infinity, and hence is not Euclidean. Analytically, the tower is chosen so that the limit set has zero critical Bessel capacity. Potential theory then provides a conformal factor whose blow-up along the limit set yields a complete scalar-flat metric; constant shifts of this factor produce complete LCF metrics with PSC. For $n \geq 6$, we use Newman's construction [New48] (see also [Gui16, Example 3.2.2]) to obtain an acyclic piecewise linear (PL) 2-complex whose complement is contractible but not homeomorphic to $\mathbb{R}^n$, and whose area measure is Ahlfors 2-regular. A conformal potential of this measure is then chosen with the homogeneity dictated by the conformal Laplacian:

its blow-up gives completeness, while the corresponding potential estimates yield scalar curvature bounded between two positive constants. The restriction $n \geq 6$ is precisely the condition that the 2-dimensional spine lie at or below the critical dimension $(n-2)/2$.

Organization of the Paper. In Section 2, for manifolds with finite fundamental group, we answer Gursky's question affirmatively when $n \not\equiv 3 \pmod 4$ and negatively when $n \equiv 3 \pmod 4$, and we obtain a strict lower bound for the Hausdorff dimension of the limit set of some infinite Kleinian group. In Section 3, we show that, for certain maps, the pointwise Lipschitz constant is strictly greater than one at some point. In Section 4, we prove contractibility of both the space of metrics and its moduli space in the specified three-dimensional cases, and prove the empty-or-contractible alternative for the moduli spaces of the higher-dimensional spherical-space-form manifolds appearing in Theorem D. In Section 5, we prove Euclidean rigidity for complete open LCF manifolds with nonnegative scalar curvature under additional topological, analytic, or bounded geometry hypotheses. In Section 6, we construct complete LCF metrics with PSC on contractible manifolds.

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Disclosure on AI assistance. The author developed the proof structure of Theorem F, while ChatGPT-5.6 Sol was used to assist in elaborating the detailed arguments. The author also used ChatGPT-5.6 Sol to polish and copy-edit this manuscript, including refining the presentation of some mathematical arguments, improving mathematical exposition and language, checking grammar and style, and providing typesetting assistance. The author assumes full responsibility for the final manuscript, including all mathematical claims and source attributions.

2 Limit sets of Kleinian groups

The following theorem gives a sharp answer to Gursky's question when the fundamental group is finite: rigidity holds if $n \not\equiv 3 \pmod 4$, whereas counterexamples exist in every dimension $n \equiv 3 \pmod 4$.

Theorem 2.1. *Let (M^n, g) be a closed, connected, oriented, smooth Riemannian manifold of dimension $n \geq 3$, which is locally conformally flat with positive scalar curvature. Assume further that $H_1(M^n; \mathbb{Z}) = 0$ and that the fundamental group $\pi_1(M)$ is finite.*

(i) If n is even or $n \equiv 1 \pmod 4$, then (M^n, g) is conformally equivalent to the standard sphere (S^n, g_{st}).

(ii) If $n \equiv 3 \pmod 4$, then $\pi_1(M)$ is trivial or isomorphic to the binary icosahedral group $I^*_{120} \cong \mathrm{SL}_2(\mathbb{F}_5)$; accordingly, (M^n, g) is conformally equivalent to (S^n, g_{st}) or to a spherical space form S^n/Γ with $\Gamma \cong I^*_{120}$. The congruence restriction is optimal: for every $n \equiv 3 \pmod 4$, both alternatives occur.

Proof outline. Kuiper's theorem and the Cartan fixed-point theorem identify M conformally with a spherical space form S^n/Γ. Since $H_1(M;\mathbb{Z}) = 0$, the group Γ is perfect, so Wolf's classification in Allcock's reformulation gives $\Gamma = \{e\}$ or $\Gamma \cong I^*_{120}$; in the latter case, restriction to $Q_8 \subset I^*_{120}$ equips $\mathbb{R}^{n+1}$ with a quaternionic vector-space structure and forces $4 \mid n+1$. Conversely, the diagonal action of $I^*_{120} \subset \mathrm{Sp}(1)$ on $S^{4r-1} \subset \mathbb{H}^r$ realizes the nontrivial alternative in every admissible dimension, while the round sphere realizes the trivial one.

Proof. Let $p\colon \widetilde{M} \to M$ be the universal covering, let $\widetilde{g} := p^*g$, and denote the deck transformation group by Π. Since $\Pi \cong \pi_1(M)$ is finite, p is finite-sheeted; hence $\widetilde{M}$ is closed. Moreover, $(\widetilde{M}, \widetilde{g})$ is simply connected and locally conformally flat. Kuiper's theorem, recalled in the Introduction, therefore gives a conformal diffeomorphism $D\colon (\widetilde{M}, \widetilde{g}) \to (S^n, g_{st})$. Each $\delta \in \Pi$ is an isometry of $\widetilde{g}$, because $p \circ \delta = p$. Thus $\rho(\delta) := D \circ \delta \circ D^{-1}$ defines an injective homomorphism $\rho\colon \Pi \to \mathrm{Conf}(S^n)$. Its image is finite and acts freely on S^n: if $\rho(\delta)$ fixes $x \in S^n$, then δ fixes $D^{-1}(x)$, and hence $\delta = e$.

We next conjugate this conformal action to an isometric action. Boundary extension gives a group isomorphism $\mathrm{Conf}(S^n) \cong \mathrm{Isom}(\mathbb{H}^{n+1})$: every conformal diffeomorphism of $S^n = \partial_\infty \mathbb{H}^{n+1}$ extends uniquely to a hyperbolic isometry, and every hyperbolic isometry induces a conformal boundary map. Recall also the Cartan fixed-point theorem: if a compact group acts isometrically on a complete, simply connected Riemannian manifold of nonpositive sectional curvature, then it has a common fixed point. Applying this theorem to the finite group corresponding to $\rho(\Pi)$ in $\mathrm{Isom}(\mathbb{H}^{n+1})$, we obtain a common fixed point $x_0 \in \mathbb{H}^{n+1}$.

Choose $A \in \mathrm{Isom}(\mathbb{H}^{n+1})$ such that $A(x_0) = 0$, where 0 is the origin in the Poincaré ball model, and let $a \in \mathrm{Conf}(S^n)$ be its boundary map. Set

$$\rho_0(\delta) := a\rho(\delta)a^{-1}, \qquad \Gamma := \rho_0(\Pi), \qquad F := a \circ D.$$

The hyperbolic extensions of the elements of Γ fix 0. The stabilizer of 0 in the isometry group of the Poincaré ball is naturally identified with $\mathrm{O}(n+1)$, and its boundary action is precisely $\mathrm{Isom}(S^n, g_{st})$. Hence $\Gamma \subset \mathrm{Isom}(S^n, g_{st})$. The map $\rho_0\colon \Pi \to \Gamma$ is an isomorphism, the action of Γ remains free, and $F \circ \delta = \rho_0(\delta) \circ F$ for every $\delta \in \Pi$. Consequently, F descends to a diffeomorphism $\overline{F}\colon M = \widetilde{M}/\Pi \to S^n/\Gamma$.

We verify explicitly that $\overline{F}$ is conformal. Write $F^*g_{st} = e^{2u}\widetilde{g}$ for some $u \in C^\infty(\widetilde{M})$. For every $\delta \in \Pi$,

$$\begin{aligned} e^{2u\circ\delta}\widetilde{g} &= \delta^*(F^*g_{st}) = (F \circ \delta)^*g_{st} \\ &= (\rho_0(\delta) \circ F)^*g_{st} = F^*g_{st} = e^{2u}\widetilde{g}. \end{aligned}$$

Thus $u \circ \delta = u$, so $u = \overline{u} \circ p$ for some $\overline{u} \in C^\infty(M)$. Let $\pi_\Gamma \colon S^n \to S^n/\Gamma$ be the quotient map and let g_Γ be the quotient round metric, characterized by $\pi_\Gamma^* g_\Gamma = g_{st}$. Since $\pi_\Gamma \circ F = \overline{F} \circ p$, we have

$$p^*(\overline{F}^* g_\Gamma) = F^*(\pi_\Gamma^* g_\Gamma) = F^* g_{st} = e^{2u}\widetilde{g} = p^*(e^{2\overline{u}} g).$$

Because p is a surjective local diffeomorphism, $\overline{F}^* g_\Gamma = e^{2\overline{u}} g$. Therefore, (M, g) is conformally equivalent to the spherical space form $(S^n/\Gamma, g_\Gamma)$, and $\pi_1(M) \cong \Pi \cong \Gamma$.

The Hurewicz theorem in degree one gives $H_1(M; \mathbb{Z}) \cong \pi_1(M)^{\mathrm{ab}} \cong \Gamma^{\mathrm{ab}}$. Hence $H_1(M; \mathbb{Z}) = 0$ implies that Γ is perfect. We now use Wolf's classification in Allcock's intrinsic reformulation [All18, Theorem 1.1 and Remark 1.3, pp. 5563–5564]. At the beginning of § 3, Allcock notes that the trivial group is the perfect base case of type (I) and that the only other perfect group in Theorem 1.1 is the binary icosahedral group $2A_5$ [All18, p. 5569]. Hence

$$\Gamma = \{e\} \qquad \text{or} \qquad \Gamma \cong I_{120}^* \cong 2A_5 \cong \mathrm{SL}_2(\mathbb{F}_5).$$

Suppose that $\Gamma \cong I_{120}^*$, and let

$$\sigma \colon \Gamma \longrightarrow \mathrm{O}(V), \qquad V := \mathbb{R}^{n+1},$$

be the orthogonal representation defining the action of Γ on S^n, where S^n is regarded as the unit sphere in V. Freeness of the action gives $\ker\big(\sigma(\gamma) - \mathrm{id}_V\big) = \{0\}$ for every $\gamma \in \Gamma \setminus \{e\}$.

We first exhibit a quaternion subgroup of I_{120}^*. Under the double covering $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, the binary icosahedral group $I_{120}^* = 2A_5$ is the inverse image of the icosahedral subgroup $A_5 \subset \mathrm{SO}(3)$ [All18, p. 5563]. Choose a tetrahedral subgroup $A_4 \subset A_5$. Its inverse image $2A_4$ is a subgroup of $2A_5$; in the standard quaternionic realization [All18, (3.1), p. 5573],

$$2A_4 = \left\{ \pm 1,\ \pm i,\ \pm j,\ \pm k,\ \frac{\pm 1 \pm i \pm j \pm k}{2} \right\},$$

where the four signs in the last term are chosen independently. Hence $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\} \subset 2A_4 \subset I_{120}^*$. After choosing an isomorphism $\Gamma \cong I_{120}^*$, we regard this Q_8 as a subgroup of Γ.

Let $z = -1 \in Q_8$. Since $z \neq e$, freeness implies $\ker\big(\sigma(z) - \mathrm{id}_V\big) = \{0\}$. Moreover, $z^2 = e$, so the minimal polynomial of $\sigma(z)$ divides $X^2 - 1 = (X-1)(X+1)$. The roots are distinct; hence $\sigma(z)$ is diagonalizable over $\mathbb{R}$ with eigenvalues in $\{1, -1\}$. Its 1-eigenspace is zero, and therefore $\sigma(z) = -\mathrm{id}_V$.

Set $\mathsf{I} := \sigma(i)$ and $\mathsf{J} := \sigma(j)$. Since $i^2 = j^2 = z$ and $ji = zij$ in Q_8, we obtain $\mathsf{I}^2 = \mathsf{J}^2 = -\mathrm{id}_V$ and $\mathsf{JI} = -\mathsf{IJ}$. By the defining relations of the quaternion algebra, there is consequently a unital $\mathbb{R}$-algebra homomorphism

$$\Phi \colon \mathbb{H} \longrightarrow \mathrm{End}_{\mathbb{R}}(V), \qquad \Phi(i) = \mathsf{I}, \quad \Phi(j) = \mathsf{J}.$$

Its kernel is a proper two-sided ideal of the division algebra $\mathbb{H}$; hence Φ is injective. Thus V is a finite-dimensional left $\mathbb{H}$-vector space. It therefore has an $\mathbb{H}$-basis, so $V \cong \mathbb{H}^m$ for some $m \geq 1$. Consequently, $n + 1 = \dim_{\mathbb{R}} V = 4m$, and hence $n \equiv 3 \pmod 4$.

It follows that if n is even or $n \equiv 1 \pmod 4$, then $\Gamma \ncong I^*_{120}$, and therefore $\Gamma = \{e\}$. The conformal equivalence constructed above then identifies (M, g) with (S^n, g_{st}), proving (i). If $n \equiv 3 \pmod 4$, the classification gives exactly the two alternatives in (ii).

It remains to establish optimality. Let $n = 4r - 1$, where $r \geq 1$, and realize $I^*_{120} \subset \mathrm{Sp}(1)$ as a group of unit quaternions [All18, p. 5563]. Let it act diagonally on $\mathbb{H}^r$ by $\xi \cdot (v_1, \ldots, v_r) := (\xi v_1, \ldots, \xi v_r)$. The quaternion norm is multiplicative, so this action preserves the Euclidean norm and restricts to an isometric action on $S^{4r-1} \subset \mathbb{H}^r$. It is free: if $\xi \cdot (v_1, \ldots, v_r) = (v_1, \ldots, v_r)$, choose an index ℓ with $v_\ell \neq 0$. Then $(\xi - 1)v_\ell = 0$, and multiplication on the right by v_ℓ^{-1} gives $\xi = 1$.

The action is orientation preserving. Indeed, the determinant of the diagonal action defines a continuous map $\mathrm{Sp}(1) \to \{\pm 1\}$. Since $\mathrm{Sp}(1)$ is connected and the determinant equals 1 at the identity, this map is identically 1. Therefore, $N_r := S^{4r-1}/I^*_{120}$ is a closed, connected, oriented spherical space form. Its quotient round metric has constant sectional curvature 1; hence it is locally conformally flat and has scalar curvature $(4r - 1)(4r - 2) > 0$. Since $4r - 1 \geq 3$, the sphere S^{4r-1} is simply connected. Thus $\pi_1(N_r) \cong I^*_{120} \neq \{e\}$. Because I^*_{120} is perfect, $H_1(N_r; \mathbb{Z}) \cong \pi_1(N_r)^{\mathrm{ab}} \cong (I^*_{120})^{\mathrm{ab}} = 0$. Hence N_r realizes the nontrivial alternative in every dimension $n \equiv 3 \pmod 4$, while the standard round sphere realizes the trivial-group alternative. For $r = 1$, N_1 is the Poincaré homology sphere $P^3 = S^3/I^*_{120}$. □

Remark 2.2. The PSC hypothesis is retained only to formulate the result as an answer to Gursky's question; it is not used in the classification argument. Thus the conclusions of Theorem 2.1 remain valid without any assumption on the scalar curvature. If, moreover, M is an integral homology sphere, then the binary-icosahedral alternative can occur only when $n = 3$: for $n = 4r - 1 \geq 7$, a spherical space form with fundamental group I^*_{120} has $H_3(M; \mathbb{Z}) \cong H_3(BI^*_{120}; \mathbb{Z}) \cong \mathbb{Z}/120$. Consequently, an LCF integral homology sphere with finite fundamental group is conformally equivalent to S^n when $n \geq 4$, while in dimension three it is conformally equivalent either to S^3 or, up to orientation, to the Poincaré homology sphere.

2.1 Properties of Kleinian Groups

In light of Theorem 2.1, we now focus on closed, LCF Riemannian n-manifolds with PSC for $n \geq 5$ and infinite fundamental group. Additional hypotheses on $H_1(M; \mathbb{Z})$ will be imposed only where explicitly stated.

Let $B^{n+1} := \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$ be the Poincaré ball model of $\mathbb{H}^{n+1}$, endowed with the hyperbolic metric

$$g_H = 4(1 - |x|^2)^{-2} \sum_{i=1}^{n+1} (dx^i)^2.$$

Every element of $\mathrm{Conf}(S^n)$ extends uniquely to a diffeomorphism of the closed ball $\overline{B^{n+1}} := B^{n+1} \cup S^n$ whose restriction to B^{n+1} is an isometry of g_H. Conversely, every isometry of (B^{n+1}, g_H) extends continuously to the boundary S^n, where it induces a conformal transformation of (S^n, g_{st}). Therefore, boundary restriction induces a natural group isomorphism $\mathrm{Isom}(B^{n+1}, g_H) \cong \mathrm{Conf}(S^n)$.

Let Γ be a discrete subgroup of the conformal group $\mathrm{Conf}(S^n)$ of the standard sphere (S^n, g_{st}); that is, Γ is a Kleinian group. Fix $o \in B^{n+1}$. Its limit set is

$$\Lambda(\Gamma) := \overline{\Gamma \cdot o}^{B^{n+1} \cup S^n} \cap S^n,$$

and this definition is independent of the choice of $o \in B^{n+1}$.

Proposition 2.3. *Let Γ be a Kleinian group and let $\Gamma' \leq \Gamma$ be a finite-index subgroup. Then $\Lambda(\Gamma') = \Lambda(\Gamma)$.*

Proof. Fix $o \in B^{n+1}$. Since $\Gamma' \subseteq \Gamma$, every accumulation point of the Γ'-orbit of o is an accumulation point of the Γ-orbit of o. Thus $\Lambda(\Gamma') \subseteq \Lambda(\Gamma)$.

Let $\xi \in \Lambda(\Gamma)$. Then there exists a sequence of distinct elements $\{\gamma_k\}_{k\geq 1} \subset \Gamma$ such that $\gamma_k o \to \xi$ in $B^{n+1} \cup S^n$. Because $[\Gamma : \Gamma'] < \infty$, there is a disjoint right-coset decomposition $\Gamma = \bigsqcup_{i=1}^m \Gamma' g_i$. By the pigeonhole principle, some coset $\Gamma' g_j$ contains infinitely many γ_k. Passing to that subsequence (still denoted γ_k), write $\gamma_k = \gamma_k' g_j$, $\gamma_k' \in \Gamma'$. The γ_k' are distinct: if $\gamma_k' = \gamma_\ell'$, then $\gamma_k = \gamma_\ell$, contrary to the choice of distinct γ_k.

Set $o' := g_j o \in B^{n+1}$. Then $\gamma_k o = \gamma_k'(g_j o) = \gamma_k' o' \to \xi$. Thus ξ is an accumulation point of the Γ'-orbit of o', so $\xi \in \Lambda(\Gamma')$ by the independence of the base point. Therefore $\Lambda(\Gamma) \subseteq \Lambda(\Gamma')$, and the two inclusions prove the claim. $\square$

Let (M^n, g) be a closed, connected, oriented, smooth, LCF Riemannian manifold with $n \geq 4$ and PSC. Schoen and Yau [SY88, Theorem 4.5] prove that the developing map of $(\widetilde{M}, \widetilde{g})$ is injective and that its holonomy representation $\rho\colon \pi_1(M) \to \mathrm{Conf}(S^n)$ is faithful and has discrete image. Set $\Gamma := \rho(\pi_1(M))$. The image of the developing map is an open subset $\Omega \subset S^n$ on which Γ acts properly discontinuously, and M is diffeomorphic to the quotient Ω/Γ.

Moreover, $\Omega = \Omega(\Gamma) := S^n \setminus \Lambda(\Gamma)$. Indeed, the maximality of the domain of discontinuity gives $\Omega \subseteq \Omega(\Gamma)$ [Apa91], and hence $\Lambda(\Gamma) \subseteq S^n \setminus \Omega$. Since $M \cong \Omega/\Gamma$ is compact, there exists a compact set $K \subset \Omega$ such that $\Gamma K = \Omega$. Suppose that $x \in \partial\Omega \cap \Omega(\Gamma)$. Choose $y_k \in \Omega$ with $y_k \to x$, and write $y_k = \gamma_k z_k$, where $z_k \in K$. Choose a compact neighborhood C of x contained in $\Omega(\Gamma)$. Proper discontinuity implies that $\{\gamma \in \Gamma : \gamma K \cap C \neq \varnothing\}$ is finite. For all sufficiently large k, $y_k \in C$; after passing to a subsequence, $\gamma_k = \gamma$ is therefore constant. Since γK is compact and hence closed, $x \in \gamma K \subset \Omega$, contradicting $x \in \partial\Omega$. Thus Ω is closed in $\Omega(\Gamma)$. Furthermore, Schoen and Yau's estimate for the complement of the developing image [SY88, Theorem 4.7] and the inclusion $\Lambda(\Gamma) \subseteq S^n \setminus \Omega$ give

$$\dim_{\mathrm{top}} \Lambda(\Gamma) \leq \dim_{\mathcal{H}} \Lambda(\Gamma) \leq \frac{n-2}{2} < n-1.$$

The non-separation theorem for compact subsets of the sphere now implies that $S^n \setminus \Lambda(\Gamma)$ is connected [HW41, p. 48, Corollary 1 to Theorem IV.4]. Thus $\Omega(\Gamma)$ is connected, and the nonempty subset Ω, being both open and closed in $\Omega(\Gamma)$, equals $\Omega(\Gamma)$.

The following standard facts may be found in [Apa91]; for the structure of elementary groups, see also [Rat19, § 5.5, especially Lemma 1 and Theorem 5.5.9]. A Kleinian group Γ is called *elementary* if its limit set $\Lambda(\Gamma)$ has at most two points;

otherwise, it is called *non-elementary.* If $\Lambda(\Gamma)$ is empty, then Γ is finite. If $\Lambda(\Gamma)$ consists of a single point, then Γ contains an abelian subgroup of finite index of rank k with $1 \leq k \leq n$. If $\Lambda(\Gamma)$ consists of two points, then Γ contains an infinite cyclic subgroup of finite index. Conversely, every abelian Kleinian group is elementary. Together with Proposition 2.3, these facts show that a Kleinian group is elementary if and only if it is virtually abelian. If Γ is non-elementary, then $\Lambda(\Gamma)$ is the unique minimal nonempty closed Γ-invariant subset of S^n.

For a Kleinian group Γ, the *critical exponent* $\delta(\Gamma)$ is defined by

$$\delta(\Gamma) := \inf \left\{ s > 0 \;\middle|\; \sum_{\gamma \in \Gamma} \exp\bigl(-s\, d_H(x, \gamma y)\bigr) < \infty \right\},$$

where $x, y \in B^{n+1}$ and d_H denotes the hyperbolic distance induced by g_H. The value is independent of the choices of x and y. For a finite group, the definition gives $\delta(\Gamma) = 0$. If Γ is non-elementary, then $0 < \delta(\Gamma) \leq n$. If $\Gamma' \leq \Gamma$, then $\delta(\Gamma') \leq \delta(\Gamma)$.

A Kleinian group Γ is said to be *convex cocompact* if the quotient $(\Omega(\Gamma) \cup B^{n+1})/\Gamma$ is compact. If $\#\Lambda(\Gamma) \geq 2$, this is equivalent to the compactness of $\mathrm{CH}(\Gamma)/\Gamma$, where $\mathrm{CH}(\Gamma) \subset B^{n+1}$ is the closed convex hull of the union of all complete geodesics with distinct ideal endpoints in $\Lambda(\Gamma)$. Equivalently, it is the smallest closed convex subset whose closure in $\overline{B^{n+1}}$ contains $\Lambda(\Gamma)$ [Bow93]. Indeed, any closed convex subset with this boundary-closure property contains every such complete geodesic and hence contains the closed convex hull above. For $\#\Lambda(\Gamma) \leq 1$, we set $\mathrm{CH}(\Gamma) = \varnothing$ by convention.

A Kleinian group Γ is said to be *geometrically finite* if there exists a uniform bound on the orders of its finite subgroups and the ε-neighborhood of $\mathrm{CH}(\Gamma)/\Gamma$ in B^{n+1}/Γ has finite volume for some $\varepsilon > 0$.

Proposition 2.4. *Let Γ be a geometrically finite (resp. convex cocompact) Kleinian group, and let $\Gamma' \leq \Gamma$ be a subgroup of finite index. Then Γ' is also geometrically finite (resp. convex cocompact).*

Proof. By Proposition 2.3, the limit set $\Lambda(\Gamma')$ coincides with $\Lambda(\Gamma)$. Consequently, the hyperbolic convex hull $\mathrm{CH}(\Gamma') \subset B^{n+1}$ equals $\mathrm{CH}(\Gamma)$. The quotient $\mathrm{CH}(\Gamma')/\Gamma'$ is a finite-sheeted orbifold covering of $\mathrm{CH}(\Gamma)/\Gamma$. Likewise, $(\Omega(\Gamma') \cup B^{n+1})/\Gamma'$ is a finite-sheeted orbifold covering of $(\Omega(\Gamma) \cup B^{n+1})/\Gamma$, which proves convex cocompactness when applicable. Moreover, the ε-neighborhood of $\mathrm{CH}(\Gamma')/\Gamma'$ has finite volume, since finiteness of volume is preserved under finite-sheeted orbifold coverings.

In addition, any finite subgroup of Γ' is also a finite subgroup of Γ, so the orders of finite subgroups of Γ' are uniformly bounded. This proves geometric finiteness. $\square$

Convex cocompact Kleinian groups are geometrically finite. Conversely, geometrically finite Kleinian groups without parabolic elements are precisely the convex cocompact ones [Bow93]. If Γ is a non-elementary convex cocompact Kleinian group, then Sullivan's theorem [Sul84] states that

$$\Gamma \text{ non-elementary and convex cocompact} \quad \Longrightarrow \quad \delta(\Gamma) = \dim_{\mathcal{H}} \Lambda(\Gamma),$$

where $\dim_{\mathcal{H}}$ denotes Hausdorff dimension.

2.2 The Hausdorff Dimension of the Limit Set

Let $q\colon \Omega \to M \cong \Omega/\Gamma$ be the quotient map, and continue to write g for the pullback metric q^*g on Ω. Schoen and Yau's estimate [SY88, Theorem 4.7] gives $\dim_{\mathcal{H}} \Lambda(\Gamma) \le \frac{n-2}{2}$. For non-elementary Γ, Proposition 2.5 below sharpens this inequality to a strict one. If Γ is elementary, then, because Γ is infinite, its limit set has one or two points and hence $\dim_{\mathcal{H}} \Lambda(\Gamma) = 0$. Under the additional hypotheses of Theorem 2.6, torsion-freeness yields a strict lower bound. We first record the critical-exponent estimate used repeatedly below.

Proposition 2.5. *Let (M^n, g), $n \ge 4$, be a closed, connected, oriented, locally conformally flat manifold with positive scalar curvature and infinite fundamental group, and write $M = \Omega/\Gamma$ as above. Then Γ is Gromov-hyperbolic and convex cocompact. If Γ is non-elementary, then*

$$0 < \dim_{\mathcal{H}} \Lambda(\Gamma) = \delta(\Gamma) < \frac{n-2}{2}.$$

Proof. The manifold M^n is not finitely covered by a torus, so Izeki's results [Ize02, Theorems 2 and 3] show that Γ is Gromov-hyperbolic and convex cocompact. The first assertion follows. For the remaining assertion, suppose that Γ is non-elementary. Because $\Gamma \cong \pi_1(M)$ is finitely generated and $\mathrm{Conf}(S^n) \cong \mathrm{PO}(n+1,1)$ has a faithful finite-dimensional linear representation (for example, its adjoint representation), Selberg's lemma [Sel60] provides a torsion-free subgroup $\Gamma_0 \le \Gamma$ of finite index. Then Γ_0 is non-elementary by Proposition 2.3 and convex cocompact by Proposition 2.4. Let g_0 be the pullback of g to $M_0 := \Omega/\Gamma_0$. Then $\Lambda(\Gamma_0) = \Lambda(\Gamma)$ by Proposition 2.3, so $\Omega = \Omega(\Gamma_0)$. Choose a right-coset decomposition $\Gamma = \bigsqcup_{i=1}^m \Gamma_0 g_i$. For $s > 0$,

$$\sum_{\gamma\in\Gamma} e^{-s d_H(x,\gamma y)} = \sum_{i=1}^{m} \sum_{\gamma_0\in\Gamma_0} e^{-s d_H(x,\gamma_0 g_i y)}.$$

By base-point independence, every inner series has critical exponent $\delta(\Gamma_0)$. Since the outer sum is finite, $\delta(\Gamma) = \delta(\Gamma_0)$.

The manifold M_0 is closed, and g_0 is locally conformally flat with positive scalar curvature; in particular, M_0 is not finitely covered by a torus. We have already shown that Γ_0 is torsion-free, non-elementary, and convex cocompact. Therefore Nayatani's criterion [Nay97, Corollary 3.4] gives $\delta(\Gamma_0) < \frac{n-2}{2}$. Finally, Sullivan's theorem [Sul84] gives $\dim_{\mathcal{H}} \Lambda(\Gamma) = \delta(\Gamma)$, while the standard critical-exponent inequality recalled above gives $\delta(\Gamma) > 0$. □

Theorem 2.6. *Let (M^n, g), $n \ge 5$, be a closed, connected, oriented, locally conformally flat Riemannian manifold with positive scalar curvature and infinite fundamental group. Then M can be realized as a quotient Ω/Γ, as described above, with $\Gamma \cong \pi_1(M)$. There exists an integer $i \ge \left\lfloor \frac{n}{2} \right\rfloor + 1$ such that $\pi_i(M^n) \ne 0$.*

If, in addition, $H_1(M^n;\mathbb{Z}) = 0$, then Γ is non-elementary.

If, moreover, Γ is torsion-free, then the Hausdorff dimension of the limit set satisfies

$$1 < \dim_{\mathcal{H}} \Lambda(\Gamma) < \frac{n-2}{2}.$$

Theorem 2.6 immediately yields a partial result toward the long-standing conjecture that a closed, oriented aspherical manifold does not admit a Riemannian metric of PSC. For other partial results related to this conjecture, see [Den21b].

Corollary 2.7. *Let M^n be a closed, connected, oriented, aspherical manifold with $n \geq 5$. Then M^n does not admit a locally conformally flat Riemannian metric with positive scalar curvature.*

Proof. The fundamental group $\pi_1(M)$ is infinite. Indeed, if it were finite, then the universal cover would be both a closed oriented manifold and a contractible space, contradicting its nonzero top-dimensional homology. If such a metric existed, Theorem 2.6 would imply that there exists an integer $i \geq \lfloor n/2 \rfloor + 1$ such that $\pi_i(M^n) \neq 0$, which contradicts the asphericity of M. $\square$

The proof of Theorem 2.6 uses Izeki's convex cocompactness theorem, Nayatani's critical-exponent criterion, and Kapovich's cohomological-dimension and rigidity theorems. We first record the elementary algebraic lemma needed to exclude elementary holonomy.

Lemma 2.8. *Let G and F be groups fitting into a short exact sequence*

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\iota} G \xrightarrow{\pi} F \longrightarrow 1,$$

where F is finite. Then the abelianization $G^{\mathrm{ab}} := G/[G,G]$ is nontrivial.

Proof. Let $H := \iota(\mathbb{Z})$ and $a := \iota(1)$. Since ι is injective, $H \cong \mathbb{Z}$. We write H multiplicatively inside G, so $a^k := \iota(k)$ for $k \in \mathbb{Z}$. Exactness gives $\ker \pi = H$, hence $H \trianglelefteq G$.

Let $\pi_0 \colon G \to G/H$ be the quotient map. Then there exists an isomorphism $\bar{\pi} \colon G/H \xrightarrow{\sim} F$ such that $\pi = \bar{\pi} \circ \pi_0$. We henceforth identify F with G/H via $\bar{\pi}$. Thus $m := [G : H] = |G/H| = |F| < \infty$. Whenever we refer to an element of F, or to a coset $gH \in G/H$, this identification is tacitly used.

Because H is normal, for every $g \in G$ and $x \in H$ we have $gxg^{-1} \in H$ (indeed, $\pi(gxg^{-1}) = \pi(g)\pi(x)\pi(g)^{-1} = 1$, so $gxg^{-1} \in \ker \pi = H$). Define $\varepsilon \colon G \to \mathrm{Aut}(H)$ by $\varepsilon(g)(x) := gxg^{-1}$ for $x \in H$. For $g_1, g_2 \in G$ and $x \in H$,

$$\varepsilon(g_1g_2)(x) = (g_1g_2)x(g_1g_2)^{-1} = g_1\big(g_2xg_2^{-1}\big)g_1^{-1} = \varepsilon(g_1)(\varepsilon(g_2)(x)),$$

so $\varepsilon(g_1g_2) = \varepsilon(g_1) \circ \varepsilon(g_2)$ and ε is a group homomorphism.

Since $H \cong \mathbb{Z}$ is infinite cyclic, fix the generator $a = \iota(1)$ of H. Any automorphism of H must send a to a generator of H, i.e., a or a^{-1}; conversely, either choice extends uniquely by $a^k \mapsto a^{\pm k}$ for all $k \in \mathbb{Z}$. If a' is another generator, then $a' = a^{\pm 1}$, hence the sign is independent of the chosen generator. Therefore $\mathrm{Aut}(H) = \{\pm \mathrm{id}_H\} \cong \{\pm 1\}$. Under the identification $\mathrm{Aut}(H) \cong \{\pm 1\}$, write $\varepsilon(g) \in \{\pm 1\}$ so that $gag^{-1} = a^{\varepsilon(g)}$. Then for each $k \in \mathbb{Z}$, $ga^kg^{-1} = (gag^{-1})^k = a^{\varepsilon(g)k}$, so the single sign $\varepsilon(g)$ determines the whole automorphism $\varepsilon(g) \in \mathrm{Aut}(H)$.

For $h \in H$, H is abelian, so $hah^{-1} = a$; hence $\varepsilon(h) = 1$ and $H \subseteq \ker \varepsilon$.

Define $\bar{\varepsilon} : G/H \to \{\pm 1\}$, $\quad \bar{\varepsilon}(gH) := \varepsilon(g)$. This map is well defined: if $gH = g'H$, then $g^{-1}g' \in H \subseteq \ker \varepsilon$, so $\varepsilon(g') = \varepsilon(g)\varepsilon(g^{-1}g') = \varepsilon(g)$. Moreover, for $g_1H, g_2H \in G/H$ one has

$$\bar{\varepsilon}(g_1H \cdot g_2H) = \bar{\varepsilon}(g_1g_2H) = \varepsilon(g_1g_2) = \varepsilon(g_1)\varepsilon(g_2) = \bar{\varepsilon}(g_1H)\bar{\varepsilon}(g_2H),$$

so $\bar{\varepsilon}$ is a homomorphism. Via our identification $F \simeq G/H$, we view $\bar{\varepsilon}$ as a homomorphism $F \to \{\pm 1\}$ such that $\varepsilon = \bar{\varepsilon} \circ \pi$.

Case 1: ε (equivalently $\bar{\varepsilon}$) is nontrivial. Choose $g \in G$ with $\varepsilon(g) = -1$. Since $\{\pm 1\}$ is abelian, $\varepsilon([x, y]) = 1 \qquad (x, y \in G)$, so $[G, G] \subseteq \ker \varepsilon$.

We recall the universal property of abelianization: let $q : G \to G^{\mathrm{ab}} := G/[G, G]$ be the quotient map. If $f : G \to A$ is a homomorphism with A abelian and $[G, G] \subseteq \ker f$, then there exists a *unique* homomorphism $\widehat{f} : G^{\mathrm{ab}} \to A$ with $f = \widehat{f} \circ q$.

Hence ε factors through the abelianization: applying this with $f = \varepsilon$ gives a unique $\widetilde{\varepsilon} : G^{\mathrm{ab}} \to \{\pm 1\}$ such that $\varepsilon = \widetilde{\varepsilon} \circ q$. Explicitly, $\widetilde{\varepsilon}(g[G, G]) = \varepsilon(g)$; this is well defined because $[G, G] \subseteq \ker \varepsilon$. Since $\widetilde{\varepsilon}(g[G, G]) = -1 \neq 1$, the map $\widetilde{\varepsilon}$ is nontrivial. The only homomorphism from the trivial group is the trivial map, so G^{ab} cannot be trivial. Thus $G^{\mathrm{ab}} \neq 0$.

Case 2: ε is trivial. Then $\varepsilon(g) = 1$ for all $g \in G$, i.e., $gag^{-1} = a$. For $h = a^k \in H$, $ghg^{-1} = (gag^{-1})^k = a^k = h$, so $H \subseteq Z(G)$.

Fix a set of right coset representatives $\mathcal{T} = \{t_1, \ldots, t_m\} \subset G$, so that $G = \bigsqcup_{i=1}^{m} Ht_i$, and fix the order $1 < 2 < \cdots < m$. For each $g \in G$ and $1 \leq i \leq m$ there exist unique elements $h_i(g) \in H$ and $\sigma_g(i) \in \{1, \ldots, m\}$ such that

$$t_i g = h_i(g)\, t_{\sigma_g(i)}. \tag{3}$$

The existence and uniqueness of such elements can be established as follows. The right cosets $\{Ht_j\}_{j=1}^{m}$ partition G, so $t_i g$ lies in exactly one Ht_j; set $\sigma_g(i) := j$ and define $h_i(g) \in H$ by (3). If $t_i g = ht_j = h' t_{j'}$ with $h, h' \in H$, then $Ht_j = Ht_{j'}$, so $j = j'$ because the cosets Ht_j are pairwise disjoint. With j fixed, $ht_j = h' t_j$ implies $h = h'$ since right multiplication by t_j^{-1} is injective in a group.

Define the transfer map (also called the Verlagerung)

$$\mathrm{Ver} : G \to H, \qquad \mathrm{Ver}(g) := \prod_{i=1}^{m} h_i(g),$$

with the factors taken in the fixed order $1, \ldots, m$. Because H is abelian and the product is finite, any permutation of the factors yields the same element of H; fixing an order just makes the definition explicit.

The map Ver is a group homomorphism. Fix $g_1, g_2 \in G$. For each i there exist $h_i^{(1)} \in H$ and $\sigma_{g_1}(i)$ with $t_i g_1 = h_i^{(1)}\, t_{\sigma_{g_1}(i)}$. Likewise, for each i there exist $h_{\sigma_{g_1}(i)}^{(2)} \in H$ and $\sigma_{g_2}(\sigma_{g_1}(i))$ with $t_{\sigma_{g_1}(i)} g_2 = h_{\sigma_{g_1}(i)}^{(2)}\, t_{\sigma_{g_2} \circ \sigma_{g_1}(i)}$. Multiplying gives

$$t_i(g_1 g_2) = h_i^{(1)} h_{\sigma_{g_1}(i)}^{(2)}\, t_{\sigma_{g_2} \circ \sigma_{g_1}(i)}.$$

By the uniqueness in (3), $h_i(g_1 g_2) = h_i^{(1)} h_{\sigma_{g_1}(i)}^{(2)}$. Hence

$$\mathrm{Ver}(g_1 g_2) = \prod_{i=1}^{m} h_i^{(1)} h_{\sigma_{g_1}(i)}^{(2)} = \Big(\prod_{i=1}^{m} h_i^{(1)}\Big)\Big(\prod_{i=1}^{m} h_{\sigma_{g_1}(i)}^{(2)}\Big). \tag{4}$$

The map σ_{g_1} is a permutation. If $\sigma_{g_1}(i) = \sigma_{g_1}(j)$, then

$$Ht_i g_1 = Ht_{\sigma_{g_1}(i)} = Ht_{\sigma_{g_1}(j)} = Ht_j g_1.$$

Right-multiplying by g_1^{-1} gives $Ht_i = Ht_j$, hence $i = j$ by disjointness. Thus the map σ_{g_1} is injective. Fix $k \in \{1, \dots, m\}$. Since the cosets Ht_r partition G, there exists a unique r with $t_k g_1^{-1} \in Ht_r$, say $t_k g_1^{-1} = h' t_r$ with $h' \in H$. Multiplying on the right by g_1 yields $t_k = h' t_r g_1$, so $Ht_k = Ht_r g_1$, hence $\sigma_{g_1}(r) = k$. Thus σ_{g_1} is bijective (a permutation). Therefore the multiset $\{h^{(2)}_{\sigma_{g_1}(i)} : 1 \leq i \leq m\}$ is just a reordering of $\{h_i^{(2)} : 1 \leq i \leq m\}$, and since H is abelian, $\prod_{i=1}^m h^{(2)}_{\sigma_{g_1}(i)} = \prod_{i=1}^m h_i^{(2)}$. Applying this to (4) gives $\mathrm{Ver}(g_1 g_2) = \mathrm{Ver}(g_1)\mathrm{Ver}(g_2)$.

For any $x, y \in G$,

$$\mathrm{Ver}([x, y]) = \mathrm{Ver}(x)\mathrm{Ver}(y)\mathrm{Ver}(x)^{-1}\mathrm{Ver}(y)^{-1} = 1,$$

because H is abelian. Hence $[G, G] \subseteq \ker \mathrm{Ver}$. By the universal property of abelianization, there is a unique homomorphism $\overline{\mathrm{Ver}} : G^{\mathrm{ab}} \to H$ such that $\mathrm{Ver} = \overline{\mathrm{Ver}} \circ q$, where $q : G \to G^{\mathrm{ab}}$ is the quotient map.

Since $H \subseteq Z(G)$, $\mathrm{Ver}(h) = h^m$ for all $h \in H$. Indeed, if $h \in H$, then $t_i h = h t_i$ for every i. Comparing with (3) and invoking the uniqueness of $h_i(\cdot)$ and $\sigma_{\cdot}(\cdot)$ yields $\sigma_h(i) = i$ and $h_i(h) = h$ for all i. Consequently, $\mathrm{Ver}(h) = \prod_{i=1}^m h = h^m$.

Let $j : H \hookrightarrow G$ denote the inclusion and let $j_* : H \to G^{\mathrm{ab}}$, $j_* := q \circ j$ be the homomorphism induced by inclusion. Since $\mathrm{Ver}(h) = h^m$ for all $h \in H \subseteq Z(G)$, we obtain $(\overline{\mathrm{Ver}} \circ j_*)(h) = \mathrm{Ver}(h) = h^m$, $(h \in H)$. Since $H \cong \mathbb{Z}$, it is torsion-free: if $h^m = 1$ with $h = a^p$, then $a^{mp} = 1$ in H, hence $mp = 0$ in $\mathbb{Z}$, so $p = 0$ and $h = 1$. Thus $h \mapsto h^m$ is injective, so the composite $\overline{\mathrm{Ver}} \circ j_*$ is nontrivial.

If $G^{\mathrm{ab}} = 0$, then every homomorphism from G^{ab} is trivial. In particular, $\overline{\mathrm{Ver}}$ is trivial, and hence $\overline{\mathrm{Ver}} \circ j_*$ is trivial, contradicting the previous paragraph. Therefore $G^{\mathrm{ab}} \neq 0$. If $F = 1$, then $m = 1$ and $G = H \cong \mathbb{Z}$, so $G^{\mathrm{ab}} \cong \mathbb{Z} \neq 0$; this is already encompassed by Case 2. □

Remark 2.9. The infinite cyclic group $\mathbb{Z}$ in Lemma 2.8 cannot be replaced by a free group of rank at least two. For instance, let A_5 denote the alternating group on five letters. Since A_5 is perfect, the free product $G = A_5 * A_5$ is also perfect, as the abelianization of a free product is the direct sum of the abelianizations of its factors.

Consider the natural folding homomorphism $\pi : G \to A_5$, which restricts to the identity on each factor. Hence $\ker \pi$ meets every conjugate of either factor trivially, and the Kurosh subgroup theorem implies that $\ker \pi$ is free. Using the rational Euler characteristic of virtually free groups and its multiplicativity under passage to finite-index subgroups, we obtain

$$\chi(G) = \frac{1}{60} + \frac{1}{60} - 1 = -\frac{29}{30}, \qquad \chi(\ker \pi) = 60\chi(G) = -58,$$

so $\ker \pi$ has rank $1 - (-58) = 59$. Thus $\ker \pi \cong F_{59}$, yielding the short exact sequence

$$1 \longrightarrow F_{59} \longrightarrow G \xrightarrow{\pi} A_5 \longrightarrow 1.$$

Thus, G admits a free normal subgroup of rank 59 and finite index while remaining perfect, showing that Lemma 2.8 fails if $\mathbb{Z}$ is replaced by a free group of rank at least two.

Lemma 2.10. *Let $\Gamma \leq \mathrm{Isom}(\mathbb{H}^{n+1})$ be a torsion-free, non-elementary, convex cocompact Kleinian group. If $\Gamma^{\mathrm{ab}} = 0$, then $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) > 1$.*

Proof. Suppose, for contradiction, that $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) \le 1$, and put $d := \dim_{\mathrm{top}}\big(\Lambda(\Gamma)\big)$. The limit set is a nonempty compact metric space, and topological dimension does not exceed Hausdorff dimension. Hence $d \in \{0, 1\}$.

Because Γ is convex cocompact, it is geometrically finite and has no parabolic elements. Thus the peripheral family Π' in [Kap09, Proposition 9.6] is empty, and that proposition gives $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = d + 1$. If $d = 0$, then $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 1$. We use the following form of the Stallings–Swan theorem, recorded by Kapovich [Kap09, Theorem 2.5]: if G is a torsion-free group satisfying $\mathrm{cd}_R(G) \le 1$, then G is free. Applying this theorem with $R = \mathbb{Z}$ and $G = \Gamma$, we conclude that Γ is free. Since Γ is non-elementary, it is a free group of rank at least two and therefore has nontrivial abelianization, contradicting $\Gamma^{\mathrm{ab}} = 0$.

It remains to consider $d = 1$. Our assumption then gives $\dim_{\mathrm{top}}\big(\Lambda(\Gamma)\big) = \dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) = 1$. By [Kap09, Theorem 1.3], the limit set is a round circle C and Γ preserves the totally geodesic plane $P \cong \mathbb{H}^2 \subset B^{n+1}$ bounded by C. Consider the restriction homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(P)$. Its kernel is a discrete subgroup of the compact pointwise stabilizer of P, hence is finite and therefore trivial because Γ is torsion-free.

Set $\Gamma_P := \rho(\Gamma)$. Since ρ is injective, the proper discontinuity of the action of Γ on $\mathbb{H}^{n+1}$ implies that Γ_P acts properly discontinuously on P. Moreover, $\mathrm{CH}(\Gamma) = P$, so convex cocompactness implies that P/Γ_P is compact. Since Γ_P is torsion-free, $\Sigma := P/\Gamma_P$ is a closed hyperbolic surface, possibly nonorientable. In either case $H_1(\Sigma; \mathbb{Z}) \ne 0$. Therefore $\Gamma^{\mathrm{ab}} \cong \Gamma_P^{\mathrm{ab}} \cong H_1(\Sigma; \mathbb{Z}) \ne 0$, again contradicting $\Gamma^{\mathrm{ab}} = 0$.

Both possibilities are impossible. Therefore $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) > 1$. □

Remark 2.11. Torsion-freeness is used essentially in Lemma 2.10: it is needed both for the cohomological-dimension-one conclusion and to eliminate the finite kernel of the action on the invariant hyperbolic plane. Passing to a finite-index torsion-free subgroup by Selberg's lemma does not remove this issue, because trivial abelianization need not pass to finite-index subgroups.

Proof of Theorem 2.6. By Proposition 2.5, Γ is Gromov-hyperbolic and its Kleinian action is convex cocompact, hence geometrically finite.

Suppose that $\pi_i(M^n) = 0$ for all $i \ge \lfloor n/2 \rfloor + 1$. We first verify the required low-degree vanishing. If Γ is elementary, then, since Γ is infinite, $\Lambda(\Gamma)$ is nonempty and hence consists of one or two points. Accordingly, $\Omega(\Gamma)$ is diffeomorphic to either $\mathbb{R}^n$ or $S^{n-1} \times \mathbb{R}$, and hence

$$\pi_i(M) = \pi_i(\Omega(\Gamma)) = 0, \qquad 2 \le i \le n-2.$$

If Γ is non-elementary, Proposition 2.5 gives $\dim_{\mathcal{H}} \Lambda(\Gamma) < \frac{n-2}{2}$. For every $i \le \lfloor n/2 \rfloor$, this implies $(i+1) + \dim_{\mathcal{H}} \Lambda(\Gamma) < n$. The relative general-position argument in the proof of Schoen and Yau [SY88, Theorem 4.6(ii)] now applies: it extends a map $S^i \to \Omega(\Gamma)$ to $B^{i+1} \to S^n$ and deforms the extension, relative to its boundary, away from $\Lambda(\Gamma)$ under the strict inequality above. Therefore,

$$\pi_i(M) = \pi_i(\Omega(\Gamma)) = 0, \qquad 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, under the supposition above, all higher homotopy groups of M vanish. The simply connected CW complex $\widetilde{M}$ is therefore weakly contractible and hence contractible by Whitehead's theorem. Consequently, M is aspherical and is a finite n-dimensional model for $B\Gamma$. In particular, Γ has finite integral cohomological dimension and is torsion-free. If $C_*(\widetilde{M})$ denotes the cellular chain complex, then the free cocompact action and the contractibility of $\widetilde{M}$ give the cochain identification

$$\mathrm{Hom}_{\mathbb{Z}\Gamma}\big(C_*(\widetilde{M}), \mathbb{Z}\Gamma\big) \cong C_c^*(\widetilde{M}; \mathbb{Z}).$$

Consequently, $H^n(\Gamma; \mathbb{Z}\Gamma) \cong H_c^n(\widetilde{M}; \mathbb{Z})$, and compact-support Poincaré duality on the oriented manifold $\widetilde{M}$ gives $H_c^n(\widetilde{M}; \mathbb{Z}) \cong H_0(\widetilde{M}; \mathbb{Z}) \cong \mathbb{Z}$. Since M is an n-dimensional model for $B\Gamma$, it follows that $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = n$.

If Γ were elementary, then it would be virtually abelian; being infinite and Gromov-hyperbolic, it would therefore be virtually cyclic. Since it is torsion-free, the classification of infinite virtually cyclic groups implies that $\Gamma \cong \mathbb{Z}$, which would give $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 1 < n$. Thus Γ is non-elementary. Since Γ is convex cocompact, it is geometrically finite without parabolic elements, and Kapovich's formula [Kap09, Proposition 9.6], together with Proposition 2.5, yields

$$n = \mathrm{cd}_{\mathbb{Z}}(\Gamma) = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1 \leq \dim_{\mathcal{H}} \Lambda(\Gamma) + 1 < \frac{n}{2} < n,$$

a contradiction. Therefore there exists an integer $i \geq \lfloor n/2 \rfloor + 1$ such that $\pi_i(M^n) \neq 0$.

For the remaining conclusions, assume that $H_1(M^n; \mathbb{Z}) = 0$.

Suppose, for contradiction, that the infinite Kleinian group Γ is elementary. Then Γ is virtually abelian. Since abelian subgroups of Gromov-hyperbolic groups are finite or virtually cyclic, Γ must be virtually cyclic. Taking the core of a finite-index infinite cyclic subgroup gives a normal infinite cyclic subgroup of finite index, and hence a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow F \longrightarrow 1,$$

where F is finite. By Lemma 2.8, such a group has nontrivial abelianization, contradicting $H_1(M^n; \mathbb{Z}) \cong \Gamma^{\mathrm{ab}} = 0$. Hence Γ is non-elementary.

Suppose now that Γ is torsion-free. Since the holonomy representation is faithful, $\Gamma^{\mathrm{ab}} \cong H_1(M^n; \mathbb{Z}) = 0$. Thus Γ is perfect, and Lemma 2.10 gives $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) > 1$.

For the upper bound, Proposition 2.5 gives $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) < (n-2)/2$. Combining this with the strict lower bound, we obtain $1 < \dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) < \frac{n-2}{2}$. □

Remark 2.12. Lemma 2.10 excludes the equality $\dim_{\mathcal{H}} \Lambda(\Gamma) = 1$ in every dimension under the additional hypotheses $H_1(M; \mathbb{Z}) = 0$ and Γ torsion-free in Theorem 2.6. In particular, no congruence condition on n is needed for the strict lower bound.

Remark 2.13. More generally, let Γ be a non-elementary geometrically finite Kleinian group. If $\dim_{\mathrm{top}} \Lambda(\Gamma) = \dim_{\mathcal{H}} \Lambda(\Gamma) = k$, then Kapovich's rigidity theorem [Kap09, Theorem 1.3] implies that $\Lambda(\Gamma)$ is a round k-sphere. Under the additional hypotheses $H_1(M; \mathbb{Z}) = 0$ and Γ torsion-free in Theorem 2.6, the case $k = 1$ is impossible by Lemma 2.10.

Remark 2.14. The proof of Theorem 2.6 also yields further information about the Hausdorff dimension of $\Lambda(\Gamma)$. For example, suppose that Γ is elementary. Since Γ is infinite, $\#\Lambda(\Gamma) \in \{1, 2\}$. If $\#\Lambda(\Gamma) = 1$, then $\Omega \cong \mathbb{R}^n$, so M is aspherical, contradicting Corollary 2.7. Hence $\#\Lambda(\Gamma) = 2$, and therefore $\dim_{\mathcal{H}} \Lambda(\Gamma) = 0$. Moreover, Γ contains an infinite cyclic subgroup of finite index generated by a loxodromic element, Ω is conformally equivalent to the round cylinder $S^{n-1} \times \mathbb{R}$, and M is finitely covered by $S^{n-1} \times S^1$.

Corollary 2.15. *Let (M^n, g) be a closed, connected, oriented, locally conformally flat Riemannian manifold of dimension $n \geq 7$ with nonnegative scalar curvature. Suppose (M^n, g) is not finitely covered by a torus, $H_1(M^n; \mathbb{Z}) = 0$, and $\Gamma := \pi_1(M)$ is infinite. Then Γ is non-elementary. Furthermore, if Γ is torsion-free, then*

$$1 < \dim_{\mathcal{H}} \Lambda(\Gamma) \leq \frac{n-2}{2}.$$

Proof. Since $n \geq 7$, $\mathrm{Sc}_g \geq 0$, and M^n is not finitely covered by a torus, Izeki's theorem [Ize02, Theorem 2(2)] shows that the developing map is injective and that the holonomy image, identified with $\Gamma = \pi_1(M)$, is a convex cocompact Kleinian group. Izeki's Theorem 4 identifies the developing image with $\Omega(\Gamma)$, so $M \cong \Omega(\Gamma)/\Gamma$. Izeki's Theorem 3 shows that Γ is Gromov-hyperbolic. If Γ were elementary, then it would be virtually abelian. Since it is infinite and Gromov-hyperbolic, it would therefore be virtually cyclic. Taking the core of a finite-index infinite cyclic subgroup yields a normal infinite cyclic subgroup of finite index and hence a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow F \longrightarrow 1,$$

where F is finite. Lemma 2.8 would then imply $\Gamma^{\mathrm{ab}} \neq 0$, contradicting $H_1(M^n; \mathbb{Z}) = 0$. Hence Γ is non-elementary.

Assume now that Γ is torsion-free. Since $\Gamma^{\mathrm{ab}} = 0$, Lemma 2.10 gives $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) > 1$. Izeki's result [Ize02, Theorem 4] identifies the Schoen–Yau invariant with the critical exponent and gives $\delta(\Gamma) \leq (n-2)/2$. Since Γ is non-elementary and convex cocompact, Sullivan's theorem [Sul84] gives $\dim_{\mathcal{H}}\big(\Lambda(\Gamma)\big) = \delta(\Gamma) \leq \frac{n-2}{2}$, which proves the assertion. $\square$

Remark 2.16. In Corollary 2.15 the two cases separate. Let $L_g = -\frac{4(n-1)}{n-2}\Delta_g + \mathrm{Sc}_g$ be the conformal Laplacian and $\lambda_1(L_g)$ its first eigenvalue, whose sign is a conformal invariant. If $\mathrm{Sc}_g \not\equiv 0$, then $\lambda_1(L_g) > 0$: indeed, the Rayleigh quotient gives $\lambda_1(L_g) \geq 0$, while equality would let a positive first eigenfunction u satisfy $L_g u = 0$. In that case,

$$0 = \int_M u L_g u \, dV_g = \frac{4(n-1)}{n-2} \int_M |\nabla u|^2 \, dV_g + \int_M \mathrm{Sc}_g u^2 \, dV_g.$$

Both terms are nonnegative, so u is constant and $\mathrm{Sc}_g \equiv 0$, a contradiction. Thus the conformal class of g contains a metric of positive scalar curvature with the same holonomy group. Proposition 2.5 therefore gives $\dim_{\mathcal{H}} \Lambda(\Gamma) < \frac{n-2}{2}$. If $\mathrm{Sc}_g \equiv 0$, then $\lambda_1(L_g) = 0$. Choose a torsion-free finite-index subgroup $\Gamma_0 \leq \Gamma$ by Selberg's lemma and pull g back to $M_0 := \Omega(\Gamma_0)/\Gamma_0$. Here $\Lambda(\Gamma_0) = \Lambda(\Gamma)$ and $\Omega(\Gamma_0) = \Omega(\Gamma)$ by Proposition 2.3; moreover, Γ_0 is non-elementary and convex cocompact by Propositions 2.3 and 2.4. The pullback metric is scalar-flat and lies in the canonical conformal class on $\Omega(\Gamma_0)/\Gamma_0$. The finite-coset argument in the proof of Proposition 2.5 gives $\delta(\Gamma_0) = \delta(\Gamma)$. If $\delta(\Gamma) < (n-2)/2$, Nayatani's criterion [Nay97, Corollary 3.4] would therefore imply that this conformal class contains a metric of positive scalar curvature, contradicting the conformal invariance of the sign of λ_1. Since $\delta(\Gamma) \le (n-2)/2$ by [Ize02, Theorem 4], it follows that $\delta(\Gamma) = \dim_{\mathcal{H}} \Lambda(\Gamma) = \frac{n-2}{2}$. No torsion-freeness assumption on Γ is needed for this equality.

Remark 2.17. Suppose, in the setting of Corollary 2.15 but with the torus-cover exclusion removed, that M is finitely covered by a torus T^n. Then M is flat. Indeed, scalar-curvature rigidity for the torus [GL83] implies that the pullback of g to T^n is flat, and hence g is flat. After the developing map is postcomposed with a Möbius transformation, the classical Euclidean developing description identifies $\widetilde{M}$ with $\mathbb{R}^n = S^n \setminus \{\infty\}$ and Γ with its Bieberbach deck group. It has a finite-index translation subgroup $L \cong \mathbb{Z}^n$. In the isometric upper-half-space model of $\mathbb{H}^{n+1}$, with $o = (0,1)$ and $T_v(x,t) = (x+v,t)$, one has

$$d_H(o, T_v o) = 2 \operatorname{arsinh}\left(\frac{|v|}{2}\right) = 2 \log |v| + O(1) \qquad \text{as } |v| \to \infty.$$

Consequently, the Poincaré series of L has the same convergence behavior as $\sum_{v \in \mathbb{Z}^n} (1 + |v|)^{-2s}$, whose critical exponent is $n/2$. Finite-index invariance therefore gives $\delta(\Gamma) = n/2$. The orbit of the translation lattice L accumulates only at ∞, so $\Lambda(L) = \{\infty\}$. Proposition 2.3 then gives $\Lambda(\Gamma) = \{\infty\}$ and $\dim_{\mathcal{H}} \Lambda(\Gamma) = 0$. Thus the equality between the critical exponent and the Hausdorff dimension of the limit set does not extend to elementary groups.

Remark 2.18. By Bieberbach's theorem, every closed flat manifold is finitely covered by a torus. Thus the hypothesis that M is not finitely covered by a torus in Corollary 2.15 excludes the flat case described above.

On a closed manifold, positive scalar curvature implies $\mathrm{Sc}_g \ge c > 0$ for some constant c. The results of Schoen and Yau [SY88] used above hold in dimensions at least four under this assumption. In that paper, they also remark that their result [SY88, Proposition 4.4′] would remain valid under the assumption of nonnegative scalar curvature, provided the positive energy theorem can be extended to the case of complete manifolds—namely, when M has one asymptotically flat end and other ends that are merely complete. Since every orientable 3-manifold is spin, Witten's proof of the positive mass theorem applies to the orientable 3-dimensional case as well. Consequently, Schoen and Yau's result [SY88] also holds for orientable 3-manifolds; see the discussion in [CH06, Appendix A, p. 805] for further details.

Recently, Lesourd, Unger, and Yau [LUY24b, Theorem 1.2] use minimal hypersurfaces to prove that there does not exist a complete Riemannian metric with PSC on $T^3 \# X$, where X is an arbitrary (possibly noncompact) manifold. Furthermore, they show that the nonexistence of a complete smooth metric with PSC on $T^n \# X$ implies the following Liouville theorem [LUY24b, Theorem 1.7], see also [CL24, Corollary 4]:

Theorem 2.19 (Liouville theorem)**.** *Let (M^n, g), $n \ge 3$, be a complete, locally conformally flat manifold with nonnegative scalar curvature. If $\Phi\colon M^n \to S^n$ is a conformal map, then Φ is injective and the boundary $\partial\Phi(M)$ has zero Newtonian capacity (i.e., 2-capacity).*

In another paper, Lesourd, Unger, and Yau [LUY24a, Theorem 1.2] use ν-buddles to prove the above Schoen–Yau conjecture and obtain a new proof of Liouville theorem, which is completely in the spirit of the approach outlined in [SY88].

Remark 2.20. Corollary 2.15 also holds for $n \geq 5$, because the Liouville theorem implies that Izeki's theorem [Ize02, Theorem 3] holds under the assumptions $\dim M \geq 3$ and nonnegative scalar curvature. Namely, if (M^n, g') $(n \geq 3)$ is a compact, connected, locally conformally flat manifold that is not covered by a torus, and if there exists a metric $g \in [g']$ with nonnegative scalar curvature, then $\pi_1(M)$ is a hyperbolic group. Indeed, the fact that $\partial\Phi(\widetilde{M})$ has zero Newtonian capacity implies that $\Phi(\widetilde{M})$ is dense in S^n, and that $\Omega(\Gamma) = \Phi(\widetilde{M})$, where Γ is the image of the holonomy representation. Thus $(M, g') = \Omega(\Gamma)/\Gamma$. Since the scalar curvature of g in g' is nonnegative, one has $\delta(\Gamma) \leq \frac{n-2}{2}$. Hence Γ is a convex cocompact Kleinian group. Moreover, every convex cocompact Kleinian group is hyperbolic [Ize02, Proposition 7] [2].

2.3 The Macroscopic Dimension of the Universal Riemannian Cover

The notion of macroscopic dimension was introduced by Gromov to study manifolds with positive scalar curvature [Gro96]. He conjectured that if (M^n, g) is a closed Riemannian manifold with positive scalar curvature, then its universal cover satisfies $\dim_{mc}(\widetilde{M}) \leq n-2$. Here, $\dim_{mc}(X) \leq k$ means that there exist a k-dimensional simplicial complex K and a continuous map $f : X \to K$ such that $\operatorname{diam}\big(f^{-1}(y)\big) \leq b$ for all $y \in K$ and for some constant $b > 0$. We call such maps *uniformly cobounded.* The macroscopic dimension of X, denoted by $\dim_{mc}(X)$, is the least integer k such that there exists a continuous uniformly cobounded map $f \colon X \to K$, where K is a k-dimensional simplicial complex.

Gromov's conjecture is proved for 3-manifolds by Bolotov [Bol03]. In higher dimensions, some partial results are known; see, for example, [Dra13, DH25]. As an application of the results established in Subsection 2.2, we verify Gromov's conjecture for LCF manifolds.

Theorem 2.21 (Macroscopic dimension)**.** *Let (M^n, g), $n \geq 5$, be a closed, connected, oriented, locally conformally flat Riemannian manifold with positive scalar curvature and infinite fundamental group. Then its universal Riemannian cover $(\widetilde{M}, \widetilde{g})$ has macroscopic dimension at most $\lfloor \frac{n-1}{2} \rfloor$, that is,*

$$\dim_{mc}(\widetilde{M}, \widetilde{g}) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \leq n-2.$$

Strategy of the proof. Izeki's theorem makes $\Gamma = \pi_1(M)$ a convex cocompact, Gromov-hyperbolic Kleinian group. If Γ is non-elementary, then Nayatani's criterion [Nay97, Corollary 3.4] and Sullivan's theorem [Sul84] give $\dim_{\mathcal{H}} \Lambda(\Gamma) < \frac{n-2}{2}$; if Γ is elementary, its limit set has topological dimension zero. In either case, the integrality of topological dimension gives the required floor function. For Γ

[2] I thank H. Izeki for explaining this argument to me.

non-elementary, the Buyalo–Lebedeva theorem [BL08] applied to the convex hull $C = \mathrm{CH}(\Gamma)$, whose Gromov boundary is identified with $\Lambda(\Gamma)$ in Lemmas 2.23 and 2.24, computes $\dim_{as} C = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1$, which the Milnor–Švarc lemma transports to $(\widetilde{M}, \widetilde{g})$ and the comparison $\dim_{mc} \leq \dim_{as}$ converts into the stated bound; the elementary case is settled directly in Lemma 2.22 by the proper uniformly cobounded map $x \mapsto \log|x|$ on $\widetilde{M} \cong \mathbb{R}^n \setminus \{0\}$.

Lemma 2.22. *Let (M^n, g) satisfy the hypotheses of Theorem 2.21, and identify $\Gamma = \pi_1(M)$ with its Kleinian holonomy group. If Γ is elementary, then $\#\Lambda(\Gamma) = 2$ and $\dim_{mc}(\widetilde{M}, \widetilde{g}) = 1$.*

Proof. By Proposition 2.5, Γ is Gromov-hyperbolic and convex cocompact. Since Γ is infinite and discrete, $\Lambda(\Gamma) \neq \emptyset$. Suppose $\Lambda(\Gamma) = \{\xi\}$. Normalizing $\xi = \infty$, the group Γ consists of Möbius transformations of S^n fixing ∞, that is, of Euclidean similarities $x \mapsto \lambda A x + b$ with $\lambda > 0$ and $A \in \mathrm{O}(n)$. If some $\gamma \in \Gamma$ had $\lambda \neq 1$, then $I - \lambda A$ would be invertible, so γ would have a unique fixed point in $\mathbb{R}^n$. Together with ∞, this would give the two distinct fixed points of a loxodromic element. Both fixed points of a loxodromic element belong to its limit set and hence to $\Lambda(\Gamma)$, contradicting $\Lambda(\Gamma) = \{\infty\}$. Therefore every element has $\lambda = 1$, and consequently $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$. The developing map identifies $\widetilde{M}$ with $\Omega(\Gamma) = \mathbb{R}^n$, so $M = \mathbb{R}^n/\Gamma$ is closed and Γ is crystallographic and hence virtually $\mathbb{Z}^n$ by Bieberbach's theorem. For $n \geq 2$ such a group is not Gromov-hyperbolic. Thus $\Lambda(\Gamma)$ is nonempty and is not a singleton; elementarity therefore gives $\#\Lambda(\Gamma) = 2$.

Normalize $\Lambda(\Gamma) = \{0, \infty\}$. The developing map then identifies $\widetilde{M}$ conformally with $\Omega(\Gamma) = S^n \setminus \{0, \infty\} = \mathbb{R}^n \setminus \{0\}$. Under this identification, define $\tau \colon \widetilde{M} \to \mathbb{R}$, $\tau(x) = \log|x|$. This map is continuous and surjective. Every $\gamma \in \Gamma$ preserves the unordered pair $\{0, \infty\}$. If γ fixes both points, then

$$\gamma(x) = \lambda A x, \qquad \lambda > 0, \quad A \in \mathrm{O}(n),$$

whereas if γ interchanges them, then

$$\gamma(x) = \frac{\lambda A x}{|x|^2}, \qquad \lambda > 0, \quad A \in \mathrm{O}(n).$$

Define $\rho(\gamma) \in \mathrm{Isom}(\mathbb{R})$ by

$$\rho(\gamma)(t) = \begin{cases} t + \log\lambda, & \text{if } \gamma(x) = \lambda A x, \\ \log\lambda - t, & \text{if } \gamma(x) = \dfrac{\lambda A x}{|x|^2}. \end{cases}$$

Then $\tau \circ \gamma = \rho(\gamma) \circ \tau$. Moreover, for $\gamma, \gamma' \in \Gamma$, $\rho(\gamma\gamma') \circ \tau = \tau \circ (\gamma\gamma') = \rho(\gamma) \circ \rho(\gamma') \circ \tau$. Since τ is surjective, it follows that $\rho(\gamma\gamma') = \rho(\gamma) \circ \rho(\gamma')$. Thus $\rho \colon \Gamma \to \mathrm{Isom}(\mathbb{R})$ is a homomorphism. Because $M = \widetilde{M}/\Gamma$ is compact, there exists a compact set $F \subset \widetilde{M}$ such that $\Gamma F = \widetilde{M}$. Choose a compact interval $J = [a, b] \subset \mathbb{R}$ containing $\tau(F)$. We claim that $\rho(\Gamma) J = \mathbb{R}$. Indeed, given $t \in \mathbb{R}$, choose $x \in \widetilde{M}$ with $\tau(x) = t$. Write $x = \gamma y$ for some $\gamma \in \Gamma$ and $y \in F$. Then $t = \tau(\gamma y) = \rho(\gamma)\tau(y), \qquad \tau(y) \in J$, which proves the claim.

The map τ is proper. Indeed, if $L \subset \mathbb{R}$ is compact and $L \subset [r, s]$, then $\tau^{-1}(L) \subset \{x \in \mathbb{R}^n : e^r \leq |x| \leq e^s\}$. The left-hand side is closed, while the annulus on the right is compact in $\mathbb{R}^n \setminus \{0\}$. Hence $\tau^{-1}(L)$ is compact.

In particular, $\tau^{-1}(J) = \{x \in \mathbb{R}^n : e^a \leq |x| \leq e^b\}$ is compact. Set $B := \operatorname{diam}_{\widetilde{g}} \tau^{-1}(J) < \infty$. Given $t \in \mathbb{R}$, choose $\gamma \in \Gamma$ and $t' \in J$ such that $t = \rho(\gamma)t'$. Equivariance, applied also to γ^{-1}, gives $\tau^{-1}(t) = \gamma\, \tau^{-1}(t')$. Because every deck transformation is a $\widetilde{g}$-isometry,

$$\operatorname{diam}_{\widetilde{g}} \tau^{-1}(t) = \operatorname{diam}_{\widetilde{g}} \tau^{-1}(t') \leq \operatorname{diam}_{\widetilde{g}} \tau^{-1}(J) = B.$$

Thus τ is uniformly cobounded. Endowing $\mathbb{R}$ with its standard locally finite simplicial structure with vertex set $\mathbb{Z}$, we obtain a continuous uniformly cobounded map from $\widetilde{M}$ to a 1-dimensional simplicial complex. Therefore $\dim_{mc}(\widetilde{M}, \widetilde{g}) \leq 1$. Suppose $\varphi\colon \widetilde{M} \to K^0$ were a uniformly cobounded continuous map to a 0-dimensional simplicial complex. Such a complex is discrete and $\widetilde{M}$ is connected, so φ is constant and its single nonempty fiber is all of $\widetilde{M}$; uniform coboundedness would give $\operatorname{diam}_{\widetilde{g}} \widetilde{M} < \infty$. But $(\widetilde{M}, \widetilde{g})$ is complete, being the Riemannian universal cover of a closed manifold, so by Hopf–Rinow a bounded $\widetilde{M}$ would be compact. A properly discontinuous deck group acting on a compact universal cover is finite, contradicting the infiniteness of $\pi_1(M)$. Hence $\dim_{mc}(\widetilde{M}, \widetilde{g}) = 1$. □

We next establish the asymptotic-dimension identity used in the proof of Theorem 2.21. Let Γ be a non-elementary convex cocompact Kleinian group and put $C := \mathrm{CH}(\Gamma)$. We identify precisely the boundary that appears in [BL08, Theorem 6.4]. For a closed subset $A \subset \mathbb{H}^{n+1}$, write $\partial^i_\infty A := \overline{A}^{\mathbb{H}^{n+1} \cup S^n} \cap S^n$ for its ideal boundary, and write $\partial^G_\infty X$ for the sequence-defined Gromov boundary of a hyperbolic space X, as in [BL08, § 6.1]. We shall show that

$$\partial^G_\infty C \cong \partial^i_\infty C = \Lambda(\Gamma).$$

Lemma 2.23. *Let Γ be a non-elementary convex cocompact Kleinian group, and let $C := \mathrm{CH}(\Gamma)$ be the closed convex hull of $\Lambda(\Gamma)$. Then C is a nonempty closed convex subset of $\mathbb{H}^{n+1}$. Equipped with its intrinsic metric, C is a proper geodesic $\mathrm{CAT}(-1)$ space, hence a Gromov-hyperbolic space, and Γ acts on C properly, cocompactly, and isometrically. Moreover, $\partial^i_\infty C = \Lambda(\Gamma)$.*

Proof. The set C is closed and convex by construction. Since Γ is non-elementary, we may choose distinct $\xi, \eta \in \Lambda(\Gamma)$; the complete hyperbolic geodesic with ideal endpoints ξ and η lies in C, so $C \neq \varnothing$.

Convexity implies that the ambient hyperbolic geodesic joining any two points of C lies in C. Thus C is geodesic and $d_C = d_{\mathbb{H}^{n+1}}\big|_{C \times C}$. Since C is closed in the proper space $\mathbb{H}^{n+1}$, it is proper. Since it is convex in the $\mathrm{CAT}(-1)$ space $\mathbb{H}^{n+1}$, it is itself $\mathrm{CAT}(-1)$ and hence Gromov-hyperbolic.

The limit set is Γ-invariant, so Γ preserves C and acts on it isometrically. Since Γ is discrete, it acts properly on $\mathbb{H}^{n+1}$, and therefore properly on the closed invariant subset C. Its action on C is cocompact by convex cocompactness.

Fix $o \in C$. Since C is Γ-invariant, $\Gamma o \subset C$. Conversely, cocompactness supplies a compact $K \subset C$ with $\Gamma K = C$; writing $R := \max_{k \in K} d_{\mathbb{H}^{n+1}}(o, k) < \infty$, we see that every $x \in C$ is of the form $x = \gamma k$ with $\gamma \in \Gamma$, $k \in K$, whence $d_{\mathbb{H}^{n+1}}(x, \gamma o) = d_{\mathbb{H}^{n+1}}(k, o) \le R$. Thus, C lies in the R-neighborhood of the orbit Γo, that is,

$$\Gamma o \subset C \subset N_R(\Gamma o). \tag{5}$$

We use that hyperbolic R-balls are Euclidean-small near the sphere. In the ball model, if $t = \tanh(R/2)$, then $B_{\mathbb{H}^{n+1}}(x, R)$ is a Euclidean ball of radius

$$r_E(x, R) = \frac{t(1 - |x|^2)}{1 - t^2|x|^2}.$$

Consequently,

$$\operatorname{diam}_{\text{eucl}} B_{\mathbb{H}^{n+1}}(x, R) \le \frac{4t}{1 - t^2}(1 - |x|).$$

Thus two sequences at hyperbolic distance at most R, one of which tends to the sphere, have the same ideal limit.

Let $\xi \in \partial^i_\infty C$ and choose $x_j \in C$ with $x_j \to \xi$ in $\mathbb{H}^{n+1} \cup S^n$; as $\xi \in S^n$ we have $|x_j| \to 1$. By (5) pick $\gamma_j \in \Gamma$ with $d_{\mathbb{H}^{n+1}}(x_j, \gamma_j o) \le R$; by the previous paragraph $\gamma_j o \to \xi$, so $\xi \in \Lambda(\Gamma)$. Conversely, by definition, $\Lambda(\Gamma)$ is the set of ideal accumulation points of the orbit Γo, independently of the choice of base point. Since $\Gamma o \subset C$, every such accumulation point belongs to $\overline{C}^{\mathbb{H}^{n+1} \cup S^n} \cap S^n = \partial^i_\infty C$. Therefore $\partial^i_\infty C = \Lambda(\Gamma)$. $\square$

Fix a point $o \in C$. We write the Gromov product as $(x \mid y)_o := \frac{1}{2}\big(d(o, x) + d(o, y) - d(x, y)\big) \ge 0$. Every CAT$(-1)$ space is δ-hyperbolic for a universal $\delta \ge 0$; by Lemma 2.23 both $\mathbb{H}^{n+1}$ and C are CAT(-1), so we fix one such δ valid for both, and both then satisfy $(x \mid y)_o \ge \min\{(x \mid z)_o, (z \mid y)_o\} - \delta$.

For a δ-hyperbolic space X, the product extends to $\partial^G_\infty X$ by

$$(\xi \mid \eta)_o := \inf \liminf_j (x_j \mid y_j)_o,$$

where the infimum ranges over representing sequences $(x_j) \in \xi$ and $(y_j) \in \eta$. For *any* such pair,

$$(\xi \mid \eta)_o \;\le\; \liminf_{j \to \infty} (x_j \mid y_j)_o \;\le\; (\xi \mid \eta)_o + 2\delta. \tag{6}$$

The topology of $\partial^G_\infty X$ is that in which $\xi_k \to \xi$ iff $(\xi_k \mid \xi)_o \to \infty$, equivalently the one induced by any visual metric ρ, i.e., one satisfying $c_1 a^{-(\xi|\eta)_o} \le \rho(\xi, \eta) \le c_2 a^{-(\xi|\eta)_o}$ for some $a > 1$, $c_1, c_2 > 0$.

We use ball-model coordinates with o at the center, normalize $d_{\text{ch}}(\alpha, \beta) = |\alpha - \beta|$ on $S^n \subset \mathbb{R}^{n+1}$, and let $\mathcal{I} \colon \partial^G_\infty \mathbb{H}^{n+1} \to S^n$ send the class of a Gromov sequence to its ideal limit. Since $\mathbb{H}^{n+1}$ is a proper CAT(-1) space, Bourdon's boundary theory [Bou95, §§ 1.4 and 2.4–2.5] shows that, under this identification, every sequence representing α converges to $\mathcal{I}\alpha$ in $\mathbb{H}^{n+1} \cup S^n$, and that the Gromov product extends continuously to the complement of the boundary diagonal in $(\mathbb{H}^{n+1} \cup S^n)^2$. Consequently, if $\alpha \ne \beta$ and $(x_j) \in \alpha$, $(y_j) \in \beta$ are any representing sequences, then $\lim_{j \to \infty}(x_j \mid y_j)_o = (\alpha \mid \beta)_o$ exists and is independent of the representatives. To compute this value, let θ be the round angle between $\mathcal{I}\alpha$ and $\mathcal{I}\beta$, and use the unit-speed radial representatives

$$x_t = \tanh(t/2)\, \mathcal{I}\alpha, \qquad y_t = \tanh(t/2)\, \mathcal{I}\beta.$$

The hyperbolic law of cosines gives

$$\cosh d(x_t, y_t) = \cosh^2 t - \sinh^2 t \cos\theta = 1 + 2\sinh^2 t \sin^2(\theta/2).$$

Hence

$$d(x_t, y_t) = 2t + 2\log\sin(\theta/2) + o(1), \qquad t \to \infty,$$

and therefore $(\alpha \mid \beta)_o = -\log\sin(\theta/2)$. Since the chordal distance is $2\sin(\theta/2)$, we obtain

$$d_{\mathrm{ch}}\big(\mathcal{I}\alpha, \mathcal{I}\beta\big) = 2e^{-(\alpha|\beta)_o^{\mathbb{H}^{n+1}}}. \tag{7}$$

Lemma 2.24. *The inclusion $C \hookrightarrow \mathbb{H}^{n+1}$ induces $\iota_\infty \colon \partial_\infty^G C \to \partial_\infty^G \mathbb{H}^{n+1}$, and $\mathcal{I} \circ \iota_\infty$ is a homeomorphism of $\partial_\infty^G C$ onto $\Lambda(\Gamma)$ with its round subspace topology.*

Proof. The map ι_∞ is well defined and injective. Since d_C is the restriction of $d_{\mathbb{H}^{n+1}}$, the product $(x \mid y)_o$ of points of C depends only on three mutual distances that agree in either space, so

$$(x \mid y)_o^C = (x \mid y)_o^{\mathbb{H}^{n+1}} \qquad (x, y \in C). \tag{8}$$

Hence a sequence in C is a Gromov sequence for C iff it is one for $\mathbb{H}^{n+1}$, and two such are equivalent for C iff they are equivalent for $\mathbb{H}^{n+1}$. The first gives well-definedness, the second injectivity.

Let $\xi \neq \eta$ in $\partial_\infty^G C$, with representing sequences $(x_j) \in \xi$, $(y_j) \in \eta$. Then the same pair represents $\iota_\infty\xi, \iota_\infty\eta$, and the estimate (6) holds in both spaces with the same δ. When this estimate is applied in both C and $\mathbb{H}^{n+1}$ to the same pair, both enclosing intervals contain $\liminf_j (x_j \mid y_j)_o$, whence

$$\Big|(\xi \mid \eta)_o^C - (\iota_\infty\xi \mid \iota_\infty\eta)_o^{\mathbb{H}^{n+1}}\Big| \le 2\delta. \tag{9}$$

Substituting (9) into (7) gives, for $\xi \neq \eta$,

$$2e^{-2\delta}e^{-(\xi|\eta)_o^C} \;\le\; d_{\mathrm{ch}}\big(\mathcal{I}(\iota_\infty\xi), \mathcal{I}(\iota_\infty\eta)\big) \;\le\; 2e^{2\delta}e^{-(\xi|\eta)_o^C}. \tag{10}$$

Thus the pullback of d_{ch} is a visual metric on $\partial_\infty^G C$ with parameter $a = e$ and constants $c_1 = 2e^{-2\delta}$, $c_2 = 2e^{2\delta}$. It therefore induces the topology of $\partial_\infty^G C$. Hence $\mathcal{I} \circ \iota_\infty$ is a homeomorphism onto its image for the topology induced by d_{ch}, which is the round subspace topology, d_{ch} and the round metric being bi-Lipschitz equivalent on S^n.

We now prove $\mathrm{im}(\mathcal{I} \circ \iota_\infty) = \partial_\infty^i C$. Let $\xi \in \partial_\infty^G C$ be represented by $(x_j) \subset C$. Viewed as a sequence in $\mathbb{H}^{n+1}$, it represents $\iota_\infty\xi$. By the standard ball-model identification used to define $\mathcal{I}$, $x_j \to \mathcal{I}(\iota_\infty\xi) \in S^n$. Since $x_j \in C$ for every j, that limit lies in $\overline{C}^{\mathbb{H}^{n+1} \cup S^n} \cap S^n = \partial_\infty^i C$.

Let $\zeta \in \partial_\infty^i C$ and choose $x_j \in C$ with $x_j \to \zeta$ in $\mathbb{H}^{n+1} \cup S^n$. In the ball model with o at the center one has $e^{-d(o,x)} = \frac{1-|x|}{1+|x|}$ and $\sinh^2\!\big(\frac{d(x,y)}{2}\big) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$, so $e^{d(x,y)} \le 4\sinh^2(d(x,y)/2) + 2$ gives

$$e^{-2(x|y)_o} = e^{d(x,y)}e^{-d(o,x)}e^{-d(o,y)} \le 4|x-y|^2 + 2\big(1-|x|\big)\big(1-|y|\big).$$

As $i, j \to \infty$, we have $|x_i - x_j| \to 0$ and $(1-|x_i|)(1-|x_j|) \to 0$. Hence $(x_i \mid x_j)_o \to \infty$. Thus (x_j) is a Gromov sequence in $\mathbb{H}^{n+1}$, and its class has ideal limit ζ. By (8), (x_j)

is a Gromov sequence in C as well; let $\xi \in \partial_\infty^G C$ be its class there. By construction ι_∞ sends the C-class of a sequence to the $\mathbb{H}^{n+1}$-class of the same sequence, so $\iota_\infty \xi = [(x_j)]_{\mathbb{H}^{n+1}}$ and hence $\mathcal{I}(\iota_\infty \xi) = \zeta$.

Hence $\mathrm{im}(\mathcal{I} \circ \iota_\infty) = \partial_\infty^i C$, which equals $\Lambda(\Gamma)$ by Lemma 2.23. Therefore $\mathcal{I} \circ \iota_\infty$ is a homeomorphism of $\partial_\infty^G C$ onto $\Lambda(\Gamma)$ with its round subspace topology. □

In particular, there is a canonical homeomorphism $\partial_\infty^G C \cong \Lambda(\Gamma)$, where the limit set carries its round subspace topology. Thus, $\dim_{\mathrm{top}}(\partial_\infty^G C) = \dim_{\mathrm{top}} \Lambda(\Gamma)$.

The Buyalo–Lebedeva theorem [BL08, Theorem 6.4] says that the asymptotic dimension of every cobounded, Gromov-hyperbolic, proper, geodesic space X equals the topological dimension of its boundary at infinity plus 1. Here a metric space X is cobounded if there is a bounded subset $A \subset X$ such that the orbit of A under the isometry group of X covers X.

We verify that C satisfies the four hypotheses of the Buyalo–Lebedeva theorem. Recall from Lemma 2.23 that C is a proper, geodesic, Gromov-hyperbolic space. Moreover, the action of Γ on C is cocompact. Hence there exists a compact, and therefore bounded, subset $K \subset C$ such that $\Gamma K = C$. Since Γ acts by isometries, its image is contained in $\mathrm{Isom}(C)$, and the $\mathrm{Isom}(C)$-orbit of K also covers C. Thus the action of $\mathrm{Isom}(C)$ on C is cobounded. Therefore, the Buyalo–Lebedeva theorem [BL08, Theorem 6.4], together with Lemmas 2.23 and 2.24, yields

$$\dim_{as}(C) = \dim_{\mathrm{top}}(\partial_\infty^G C) + 1 = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1.$$

By Lemma 2.23, C is proper and geodesic, hence a length space, and Γ acts on it properly, cocompactly, and isometrically. The Milnor–Švarc lemma applies: Γ is finitely generated, and for any $o \in C$ the orbit map $\gamma \mapsto \gamma o$ is a quasi-isometry from Γ to C for the word metric associated with a finite generating set supplied by the Milnor–Švarc lemma. Word metrics of any two finite generating sets are quasi-isometric, so for any finite generating set S, (Γ, d_S) is quasi-isometric to C.

A quasi-isometry is a coarse equivalence, and asymptotic dimension is invariant under coarse equivalence [BD08, Proposition 22]; thus,

$$\dim_{as}(\Gamma, d_S) = \dim_{as}(C) = \dim_{\mathrm{top}}(\partial_\infty^G C) + 1 = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1.$$

Proof of Theorem 2.21. By Proposition 2.5, $\Gamma = \pi_1(M)$ is Gromov-hyperbolic and its Kleinian action is convex cocompact, hence geometrically finite. The elementary case follows from Lemma 2.22, since $n \geq 5$.

Suppose now that Γ is non-elementary. Proposition 2.5 gives $\dim_{\mathcal{H}} \Lambda(\Gamma) = \delta(\Gamma) < \frac{n-2}{2}$. The limit set is a compact metric space, and hence $d := \dim_{\mathrm{top}} \Lambda(\Gamma) \leq \dim_{\mathcal{H}} \Lambda(\Gamma) < \frac{n-2}{2}$. The integer-valuedness of d now gives $d + 1 \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Indeed, if $n = 2m$, then $d < m - 1$, so $d \leq m - 2$; if $n = 2m + 1$, then $d < m - \frac{1}{2}$, so $d \leq m - 1$.

We use asymptotic dimension $\dim_{as}$, in the sense of [BD08], to complete the non-elementary case, since $\dim_{mc}(X) \leq \dim_{as}(X)$ for every metric space X [Dra13, p. 230]. The identity established above gives $\dim_{as}(\Gamma, d_S) = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1$, where d_S is a word metric. The deck transformation action of Γ on $(\widetilde{M}, \widetilde{g})$ is proper, cocompact, and isometric. Since M is closed, $(\widetilde{M}, \widetilde{g})$ is complete, proper, and geodesic. By the Milnor–Švarc lemma, $(\widetilde{M}, \widetilde{g})$ is quasi-isometric to Γ, equipped with any word metric. Since asymptotic dimension is invariant under quasi-isometry [BD08, Proposition 22], we obtain $\dim_{as}(\widetilde{M}, \widetilde{g}) = \dim_{as}(\Gamma, d_S)$. Therefore, for $n \geq 5$, we have

$$\dim_{mc}(\widetilde{M}, \widetilde{g}) \leq \dim_{as}(\widetilde{M}, \widetilde{g}) = \dim_{as}(\Gamma, d_S) = \dim_{\mathrm{top}} \Lambda(\Gamma) + 1 \leq \left\lfloor \frac{n-1}{2} \right\rfloor \leq n - 2.$$

□

Remark 2.25. The assumption of PSC in Theorem 2.21 cannot be weakened to nonnegative scalar curvature, since the flat torus T^n has universal cover $\mathbb{R}^n$, and $\dim_{mc}(\mathbb{R}^n) = n$. Furthermore, since $\dim_{mc}(\mathbb{H}^n) = n$, taking the metric product of a round sphere with a closed hyperbolic manifold shows that the upper bound is sharp. Indeed, every integer in the range $\{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ is realized as the upper bound for $\dim_{mc}$.

Remark 2.26. For a manifold M as in Theorem 2.21, we have $\dim_{mc}(\widetilde{M}, \widetilde{g}) \leq \lfloor \frac{n-1}{2} \rfloor \leq n - 2 < n$. Hence M is md-small in the sense of Dranishnikov [Dra13], and [Dra13, Theorem 5.4(2)] implies that the locally finite fundamental class of M maps to zero in the integral coarse homology group $HX_n(M; \mathbb{Z})$.

Proof of Theorem A. The macroscopic dimension estimate follows immediately from Theorem 2.21, while the remaining assertions are consequences of Theorem 2.6. □

3 Maps between LCF manifolds with PSC

In this section, we study the geometric and topological constraints imposed by nonzero-degree maps between closed LCF manifolds with PSC. We first show that the vanishing of the Hausdorff dimension of the holonomy limit set is preserved under such maps. We then exploit the dichotomy between elementary and non-elementary holonomy Kleinian groups to establish Llarull-type rigidity results. In the elementary case, smooth 1-Lipschitz maps of nonzero degree to the standard sphere are rigid; whenever the fundamental group is infinite—in particular, in the non-elementary case—the developing map expands some tangent vector at some point.

Proposition 3.1. *Let (M^n, g_M) and (N^n, g_N), $n \geq 4$, be closed, connected, oriented, locally conformally flat Riemannian manifolds with positive scalar curvature. Let Γ_M and Γ_N be their holonomy Kleinian groups. Assume that $\dim_{\mathcal{H}} \Lambda(\Gamma_M) = 0$. If there exists a continuous map $f : M^n \to N^n$ of nonzero degree, then Γ_N is elementary. In particular, $\dim_{\mathcal{H}} \Lambda(\Gamma_N) = 0$.*

Proof. For each $X \in \{M, N\}$, let $\widetilde{X}$ be the universal cover. Then the developing map $\mathrm{dev}_X : \widetilde{X} \to S^n$ is injective, with image a Γ_X-invariant domain $\Omega_X \subset S^n$. Its holonomy representation is an isomorphism, $\rho_X : \pi_1(X) \xrightarrow{\cong} \Gamma_X$, and the developing map gives an equivariant diffeomorphism $\widetilde{X} \cong \Omega_X$. Thus $X \cong \Omega_X / \Gamma_X$, and the action of Γ_X on Ω_X is free.

Choose basepoints and put

$$\varphi := \rho_N \circ f_\# \circ \rho_M^{-1} : \Gamma_M \longrightarrow \Gamma_N, \qquad H := \varphi(\Gamma_M).$$

Let $q\colon \Omega_N/H \to \Omega_N/\Gamma_N \cong N$ be the connected covering associated with the subgroup $\rho_N^{-1}(H) \le \pi_1(N)$. Since $\rho_N^{-1}(H) = f_\#\big(\pi_1(M)\big)$, the lifting criterion gives a map $\bar f\colon M\Omega_N/H$ with $q \circ \bar f = f$. If $[\Gamma_N : H] = \infty$, then q is infinite-sheeted. Its total space cannot be compact, since a covering of the compact manifold N with compact total space has finite fibers. Thus Ω_N/H is a connected noncompact n-manifold, and hence $H_n(\Omega_N/H;\mathbb{Z}) = 0$. Therefore $f_*[M] = q_*\bar f_*[M] = 0$, contradicting $f_*[M] = \deg(f)[N] \neq 0$. Therefore $[\Gamma_N : H] < \infty$.

We claim that Γ_M is elementary. If it were non-elementary, Proposition 2.5 would give $\dim_{\mathcal{H}} \Lambda(\Gamma_M) = \delta(\Gamma_M) > 0$, a contradiction. Hence Γ_M is elementary and therefore virtually abelian, by the structure of elementary Kleinian groups recalled above. Choose an abelian subgroup $A \le \Gamma_M$ of finite index. Then $\varphi(A)$ is abelian and $[H : \varphi(A)] \le [\Gamma_M : A] < \infty$. Consequently, $[\Gamma_N : \varphi(A)] = [\Gamma_N : H]\,[H : \varphi(A)] \le [\Gamma_N : H]\,[\Gamma_M : A] < \infty$.

Since $\varphi(A)$ is an abelian Kleinian group, it is elementary, so its limit set contains at most two points. Proposition 2.3, applied to the finite-index inclusion $\varphi(A) \le \Gamma_N$, gives $\Lambda(\Gamma_N) = \Lambda(\varphi(A))$. Hence $\Lambda(\Gamma_N)$ contains at most two points, so Γ_N is elementary, and consequently $\dim_{\mathcal{H}} \Lambda(\Gamma_N) = 0$. □

Remark 3.2. The Hausdorff dimension of the limit set is not determined by the abstract isomorphism type of a Kleinian group. Let Σ_h be a closed, orientable surface of genus $h \ge 2$, and let $\rho_0\colon \pi_1(\Sigma_h) \to \mathrm{PSL}(2,\mathbb{R})$ be a cocompact Fuchsian representation. Set $\Gamma_0 := \rho_0(\pi_1(\Sigma_h))$. The group Γ_0 preserves a totally geodesic copy of $\mathbb{H}^2 \subset \mathbb{H}^3$, and its limit set is the ideal boundary of this plane, hence a round circle. Therefore $\dim_{\mathcal{H}} \Lambda(\Gamma_0) = 1$. One can choose a quasi-Fuchsian representation $\rho_1\colon \pi_1(\Sigma_h) \to \mathrm{PSL}(2,\mathbb{C})$ which is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$, and set $\Gamma_1 := \rho_1(\pi_1(\Sigma_h))$. The limit set $\Lambda(\Gamma_1)$ is a quasicircle, and hence a Jordan curve in $\mathbb{CP}^1$; consequently, $1 \le \dim_{\mathcal{H}} \Lambda(\Gamma_1) \le 2$. Bowen's theorem [Bow79, Theorem 2], in the standard formulation recalled by Farre–Pozzetti–Viaggi [FPV24, § 1, p. 2], states that the limit set of a quasi-Fuchsian representation has Hausdorff dimension 1 if and only if the representation is Fuchsian, that is, conjugate into $\mathrm{PSL}(2,\mathbb{R})$. Since ρ_1 is non-Fuchsian, it follows that $\dim_{\mathcal{H}} \Lambda(\Gamma_1) > 1$.

Both ρ_0 and ρ_1 are faithful, so $\rho_1 \circ \rho_0^{-1}\colon \Gamma_0 \to \Gamma_1$ is an abstract group isomorphism. Finally, for every $n \ge 2$, the standard inclusion $\mathrm{Conf}(S^2) \hookrightarrow \mathrm{Conf}(S^n)$ regards Γ_0 and Γ_1 as Kleinian groups in $\mathrm{Conf}(S^n)$, without changing their limit sets or their Hausdorff dimensions.

Proposition 3.3. *Let (M^n, g), $n \ge 4$, be a closed, connected, oriented, locally conformally flat manifold with scalar curvature $\mathrm{Sc}_g \ge n(n-1)$. Identify $\Gamma := \pi_1(M)$ with its holonomy Kleinian group, and suppose that Γ is elementary. If there exists a smooth distance-nonincreasing (i.e., 1-Lipschitz) map $f\colon (M^n, g) \to (S^n, g_{st})$ of nonzero degree, then f is a Riemannian isometry.*

Proof. Identifying $\widetilde{M}$ with $\Omega := \mathrm{dev}(\widetilde{M}) \subset S^n$ and transporting the lifted metric to Ω, we write $\tilde g = u^{\frac{4}{n-2}} g_{st}\big|_\Omega$ for some positive smooth function $u\colon \Omega \to \mathbb{R}_{>0}$. By the Kleinian realization established in Section 2, $\Omega = \Omega(\Gamma)$, and hence $\partial\Omega = \Lambda(\Gamma)$. Since Γ is elementary, $\partial\Omega$ contains at most two points.

The boundary $\partial\Omega$ cannot consist of a single point. Indeed, if $\partial\Omega = \{\xi\}$, then $\widetilde{M} \cong \Omega = S^n \setminus \{\xi\} \cong \mathbb{R}^n$, so $\widetilde{M}$ is contractible and M is aspherical. If $n = 4$, this contradicts the Schoen–Yau obstruction to positive scalar curvature on closed aspherical 4-manifolds [SY87, Theorem 6]. If $n \geq 5$, it contradicts Corollary 2.7.

Assume that $\partial\Omega$ consists of exactly two points. Then $\widetilde{M}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$ and has two ends. Choose a finite CW structure on M and lift it to $\widetilde{M}$. The resulting complex is a path-connected free $\pi_1(M)$-CW complex with finite quotient, so [Geo08, Corollary 13.5.12] implies that $\pi_1(M)$ has two ends. It therefore contains a finite-index subgroup isomorphic to $\mathbb{Z}$ by [Geo08, Theorem 13.5.9]. Hence M admits a connected finite cover $\widehat{M}$ such that $\pi_1(\widehat{M}) \cong \mathbb{Z}$. Since M is closed and oriented, so is $\widehat{M}$.

We claim that $\widehat{M}$ is spin. Its universal cover satisfies $\widetilde{\widehat{M}} \cong S^{n-1} \times \mathbb{R} \simeq S^{n-1}$. Since $n \geq 4$, it follows that $\pi_2(\widehat{M}) = \pi_2(S^{n-1}) = 0$. The Hopf exact sequence

$$\pi_2(\widehat{M}) \longrightarrow H_2(\widehat{M};\mathbb{Z}) \longrightarrow H_2(\pi_1(\widehat{M});\mathbb{Z}) \longrightarrow 0$$

and the equality $H_2(\pi_1(\widehat{M});\mathbb{Z}) = H_2(\mathbb{Z};\mathbb{Z}) = 0$ imply that $H_2(\widehat{M};\mathbb{Z}) = 0$. The universal coefficient theorem gives the exact sequence

$$0 \longrightarrow \operatorname{Ext}_{\mathbb{Z}}\big(H_1(\widehat{M};\mathbb{Z}),\mathbb{Z}_2\big) \longrightarrow H^2(\widehat{M};\mathbb{Z}_2) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\big(H_2(\widehat{M};\mathbb{Z}),\mathbb{Z}_2\big) \longrightarrow 0.$$

Since $H_1(\widehat{M};\mathbb{Z}) \cong \pi_1(\widehat{M})^{\mathrm{ab}} \cong \mathbb{Z}$, we have $\operatorname{Ext}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2) = 0$, while $H_2(\widehat{M};\mathbb{Z}) = 0$ gives $\operatorname{Hom}_{\mathbb{Z}}(0,\mathbb{Z}_2) = 0$. Therefore $H^2(\widehat{M};\mathbb{Z}_2) = 0$. Thus $w_2(T\widehat{M}) = 0$, and since $w_1(T\widehat{M}) = 0$, the manifold $\widehat{M}$ is spin. In fact, for $n \geq 5$, one may choose $\widehat{M}$ to be diffeomorphic to $S^{n-1} \times S^1$; see [hrw].

Give $\widehat{M}$ the pullback orientation, let $p\colon \widehat{M} \to M$ denote the finite covering, and equip $\widehat{M}$ with the lifted metric $\widehat{g}$. Set $\widehat{f} := f \circ p\colon (\widehat{M},\widehat{g}) \to (S^n, g_{st})$. Then $\widehat{f}$ is smooth and distance-nonincreasing, and $\deg(\widehat{f}) = \deg(p)\deg(f) \neq 0$. Since $\mathrm{Sc}_{\widehat{g}} \geq n(n-1)$, Llarull's rigidity theorem [Lla98] implies that $\widehat{f}$ is a Riemannian isometry. This would make $\widehat{M}$ simply connected, contradicting $\pi_1(\widehat{M}) \cong \mathbb{Z}$. Thus $\partial\Omega$ cannot consist of two points.

Consequently, $\partial\Omega = \varnothing$, so $\Omega = S^n$. Thus $\widetilde{M}$ is compact and spin, and the universal covering $\pi\colon \widetilde{M} \to M$ is finite-sheeted. Give $\widetilde{M}$ the pullback orientation and set $\widetilde{f} := f \circ \pi\colon (\widetilde{M},\tilde{g}) \to (S^n, g_{st})$. Then $\widetilde{f}$ is smooth and distance-nonincreasing, and $\deg(\widetilde{f}) = \deg(\pi)\deg(f) \neq 0$. Llarull's rigidity theorem implies that $\widetilde{f}$ is a Riemannian isometry. Hence $(\widetilde{M},\tilde{g})$ is isometric to (S^n, g_{st}). Conjugation by $\widetilde{f}$ identifies $\Gamma = \pi_1(M)$ with a finite subgroup of $\mathrm{O}(n+1)$ acting freely on S^n. With this identification, $(M,g) \cong (S^n/\Gamma, g_{st})$.

Because the hypotheses do not guarantee that the spherical space form M is spin, we cannot apply Llarull's rigidity theorem directly to M. We instead use the degree formula to descend the rigidity conclusion to M. Since f is smooth and 1-Lipschitz, $\|df_p\| \leq 1$ and $|\operatorname{Jac} f(p)| \leq 1$ for every $p \in M$. The degree formula gives

$$\begin{aligned} |\deg(f)| \operatorname{Vol}_{g_{st}}(S^n) &= \left| \int_M f^*(d\operatorname{vol}_{g_{st}}) \right| \\ &\leq \int_M |\operatorname{Jac} f| \, d\operatorname{vol}_g \\ &\leq \operatorname{Vol}_g(M) = \frac{\operatorname{Vol}_{g_{st}}(S^n)}{|\Gamma|}. \end{aligned}$$

Since $|\deg(f)|$ and $|\Gamma|$ are positive integers, this forces $|\deg(f)| = |\Gamma| = 1$. Thus Γ is trivial. Hence the universal covering $\pi\colon (\widetilde{M}, \tilde{g}) \to (M, g)$ is one-sheeted and is therefore a Riemannian isometry. Since $\widetilde{f} = f \circ \pi$ is a Riemannian isometry, so is $f = \widetilde{f} \circ \pi^{-1}$. □

Remark 3.4. Proposition 3.3 also holds in dimension three without assuming either local conformal flatness or that $\pi_1(M)$ is elementary. Indeed, in 2022 Lee and Tam [LT26] extend Cecchini, Hanke, and Schick's result [CHS26] from even dimensions to all dimensions, and every oriented 3-manifold is spin. These two results do not require local conformal flatness, and the smoothness assumptions on the metric and the map can be relaxed to less than C^2. The weighted Llarull rigidity theorem for weighted Riemannian manifolds is established by the author in [Den21a, Theorem 1.3]. These results rely on the spin condition, and the proofs follow Llarull's index-theoretic rigidity argument, adapted to the present geometric setting.

Even without assuming local conformal flatness or the spin condition, in 2021 the author proves a Llarull-type rigidity theorem (see [Den22, Theorem A]) under the additional hypothesis that the map is harmonic and satisfies Condition C.

We next record the conformal minimum estimate used below.

Proposition 3.5. *Let $\Omega \subsetneq S^n$, $n \geq 3$, be a connected open subset carrying a complete metric $\tilde{g} = u^{4/(n-2)} g_{st}\big|_{\Omega}$, where $u\colon \Omega \to \mathbb{R}_{>0}$ is smooth. Then $\sup_\Omega u = +\infty$. If, moreover, $\mathrm{Sc}_{\tilde{g}} \geq n(n-1)$, then u attains its minimum on Ω, and $\min_\Omega u < 1$.*

Proof. We first show that completeness forces $\sup_\Omega u = +\infty$. Suppose, to the contrary, that $A := \sup_\Omega u < \infty$. Fix $y_0 \in \Omega$. Since $\Omega \subsetneq S^n$ is open, $K := S^n \setminus \Omega$ is nonempty and compact. Choose $p \in K$ such that $\ell := d_{g_{st}}(y_0, p) = d_{g_{st}}(y_0, K) > 0$. Let $\gamma\colon [0, \ell] \to S^n$ be a unit-speed minimizing g_{st}-geodesic from y_0 to p. The minimality of ℓ implies that $\gamma([0, \ell)) \subset \Omega$. Hence, for $0 \leq s < t < \ell$,

$$d_{\tilde{g}}\big(\gamma(s), \gamma(t)\big) \leq \int_s^t u(\gamma(r))^{\frac{2}{n-2}}\, dr \leq A^{\frac{2}{n-2}}(t - s).$$

Thus, if $t_j \uparrow \ell$, then $\big(\gamma(t_j)\big)$ is $d_{\tilde{g}}$-Cauchy. By completeness and the Hopf–Rinow theorem, it converges to some point of Ω. Since the topology induced by $d_{\tilde{g}}$ is the manifold topology, this would also be its limit in S^n. On the other hand, $\gamma(t_j) \to p \notin \Omega$ in S^n, a contradiction. Therefore $\sup_\Omega u = +\infty$.

Now assume that $\mathrm{Sc}_{\tilde{g}} \geq n(n-1)$. By hypothesis, $(\Omega, \tilde{g})$ is complete. Write

$$\tilde{g} = e^{2\psi} g_{st}\big|_{\Omega}, \qquad \psi := \frac{2}{n-2}\log u.$$

We use the following result of Ma and Qing [MQ22, Lemma 3.1]: if (X^n, h) is a compact Riemannian manifold, $U \subset X$ is a domain, and $\widehat{h} = e^{2\eta} h\big|_U$ is complete with

$$\mathrm{Sc}^-_{\widehat{h}} := \max\{-\mathrm{Sc}_{\widehat{h}}, 0\} \in L^p(U, \widehat{h})$$

for some $p > n/2$, then $\eta(x) \to +\infty$ as $x \to \partial U$.

In the present setting, $\mathrm{Sc}^-_{\tilde{g}} \equiv 0$, and hence $\mathrm{Sc}^-_{\tilde{g}} \in L^p(\Omega, \tilde{g})$ for every $p > n/2$. Applying the result with

$$(X, h) = (S^n, g_{st}), \qquad U = \Omega, \qquad \widehat{h} = \tilde{g}, \qquad \eta = \psi,$$

we obtain $\psi(x) \to +\infty$ as $x \to \partial\Omega$. Equivalently,

$$v := u^{-1} \longrightarrow 0 \qquad \text{as } x \to \partial\Omega.$$

Define $\bar{v} \colon \overline{\Omega} \to [0, \infty)$ by $\bar{v} := v$ on Ω and $\bar{v} := 0$ on $\partial\Omega$. This extension is continuous. Indeed, if $z_j \in \overline{\Omega}$ and $z_j \to p \in \partial\Omega$, then the terms lying in $\partial\Omega$ have value zero, while the subsequence of terms lying in Ω, if infinite, has values tending to zero by the preceding boundary limit. Since $\overline{\Omega}$ is compact, $\bar{v}$ attains its maximum. This maximum is positive and cannot occur on $\partial\Omega$; hence it occurs at some $x_{\min} \in \Omega$. Consequently, u attains its global minimum at $x_{\min}$.

It remains to prove that this minimum is strictly less than 1. For the remainder of the proof, we use the Laplacian $\Delta := -\operatorname{div}_{g_{st}} \nabla$. Thus Δ has nonnegative spectrum. Suppose, to the contrary, that $\min_\Omega u \geq 1$. The conformal scalar-curvature equation is

$$\frac{4(n-1)}{n-2} \Delta u + n(n-1)u = \mathrm{Sc}_{\tilde{g}}\, u^{\frac{n+2}{n-2}}.$$

It follows that

$$\Delta u \geq \frac{n(n-2)}{4} \left(u^{\frac{n+2}{n-2}} - u \right) \geq 0 \qquad \text{on } \Omega.$$

Equivalently, $\operatorname{div}_{g_{st}} \nabla u \leq 0$. Since u attains its global minimum at a point of the connected open set Ω, the strong minimum principle implies that u is constant. This contradicts $\sup_\Omega u = +\infty$. Therefore $\min_\Omega u < 1$. $\square$

For a continuous map $F \colon (X, g_X) \to (Y, g_Y)$ between Riemannian manifolds, define its pointwise Lipschitz constant by

$$\mathrm{Lip}_x(F) := \limsup_{y \to x,\, y \neq x} \frac{d_{g_Y}(F(y), F(x))}{d_{g_X}(y, x)}.$$

If F is smooth, then $\mathrm{Lip}_x(F) = \|dF_x\|$. In particular, if X is connected and $\mathrm{Lip}_x(F) \leq 1$ for every $x \in X$, then integrating $\|dF\| \leq 1$ along piecewise smooth curves and taking the infimum of their lengths shows that F is globally 1-Lipschitz. Consequently, Llarull's rigidity theorem implies that, if (X^n, g) is a closed, connected spin manifold with $\mathrm{Sc}_g \geq n(n-1)$ and is not isometric to the standard round sphere, then every smooth nonzero-degree map $f \colon (X^n, g) \to (S^n, g_{st})$ satisfies $\mathrm{Lip}_{x_0}(f) > 1$ at some point $x_0 \in X$. The next theorem gives the corresponding conclusion for developing maps.

Theorem 3.6. *Let (M^n, g), $n \geq 3$, be a closed, connected, locally conformally flat manifold satisfying $\mathrm{Sc}_g \geq n(n-1)$. Assume that $\pi_1(M)$ is infinite. Let $\mathrm{dev} \colon (\widetilde{M}, \tilde{g}) \to (S^n, g_{st})$ denote a developing map, where $\tilde{g}$ is the lifted metric. Then there exists $x \in \widetilde{M}$ such that $\mathrm{Lip}_x(\mathrm{dev}) > 1$.*

Proof. Since M is closed, the lifted metric $\tilde{g}$ is complete, and $\mathrm{Sc}_{\tilde{g}} \geq n(n-1)$. By Theorem 2.19, the developing map is injective. Hence, with $\Omega := \mathrm{dev}(\widetilde{M})$, the map $\varphi := \mathrm{dev} \colon \widetilde{M} \to \Omega$ is a conformal diffeomorphism. The connected domain Ω is proper: if $\Omega = S^n$, then $\widetilde{M} \cong S^n$ would be compact, so its properly discontinuous deck-transformation group $\pi_1(M)$ would be finite, a contradiction.

Transporting $\tilde{g}$ by φ, write $g_\Omega := (\varphi^{-1})^* \tilde{g} = u^{\frac{4}{n-2}} g_{st}\big|_\Omega$ for some smooth $u \colon \Omega \to (0, \infty)$. Then φ is an isometry, (Ω, g_Ω) is complete, and $\mathrm{Sc}_{g_\Omega} \geq n(n-1)$.

The developing map decomposes as $\mathrm{dev} = \iota\circ\varphi$, where $\iota : \Omega \hookrightarrow S^n$ is the inclusion. We compute the pointwise operator norm of $d(\mathrm{dev})_p : (T_p\widetilde{M}, \tilde{g}) \to (T_{\mathrm{dev}(p)}S^n, g_{st})$. Since φ is an isometry $(\widetilde{M}, \tilde{g}) \cong (\Omega, g_\Omega)$, one has $\|d(\mathrm{dev})_p\|_{(\tilde{g}\to g_{st})} = \|d\iota_{\varphi(p)}\|_{(g_\Omega\to g_{st})}$. Thus it suffices to study the inclusion $\iota : (\Omega, g_\Omega) \hookrightarrow (S^n, g_{st})$.

Fix $x \in \Omega$ and $v \in T_x\Omega$. Since $d\iota_x$ is the identity on $T_x\Omega$, its operator norm is

$$\|d\iota_x\|_{(g_\Omega\to g_{st})} = \sup_{v\neq 0} \frac{|v|_{g_{st}}}{|v|_{g_\Omega}} = u(x)^{-\frac{2}{n-2}}.$$

Since g_Ω is complete and $\mathrm{Sc}_{g_\Omega} \geq n(n-1)$, Proposition 3.5 implies that there exists a point $x_0 \in \Omega$ such that $u(x_0) = \min_\Omega u < 1$. Thus, $\|d\iota_{x_0}\| = u_{\min}^{-\frac{2}{n-2}} > 1$.

Setting $p_0 := \varphi^{-1}(x_0) \in \widetilde{M}$, we conclude that $\|d(\mathrm{dev})_{p_0}\|_{(\tilde{g}\to g_{st})} = \|d\iota_{x_0}\|_{(g_\Omega\to g_{st})} = u_{\min}^{-\frac{2}{n-2}} > 1$. This shows that the developing map strictly expands lengths at p_0, completing the proof.

□

Proof of Theorem C. Suppose first that $\pi_1(M)$ is elementary and that the conclusion fails for a smooth nonzero-degree map $f\colon (M^n, g) \to (S^n, g_{st})$. Then $\mathrm{Lip}_p(f) \leq 1$ for every $p \in M$. Since f is smooth, $\|df_p\| = \mathrm{Lip}_p(f) \leq 1$ for every $p \in M$.

Fix distinct points $x, y \in M$, set $\ell := d_g(x, y)$, and let $\gamma\colon [0, \ell] \to M$ be a unit-speed minimizing geodesic from x to y. Then

$$\begin{aligned} d_{g_{st}}(f(x), f(y)) &\leq L_{g_{st}}(f\circ\gamma) \\ &= \int_0^\ell |df_{\gamma(t)}(\dot\gamma(t))|_{g_{st}}\, dt \\ &\leq \int_0^\ell |\dot\gamma(t)|_g\, dt = \ell = d_g(x, y). \end{aligned}$$

Thus f is globally 1-Lipschitz. Proposition 3.3 therefore implies that f is a Riemannian isometry, contradicting the assumption that (M^n, g) is not isometric to (S^n, g_{st}).

If $\pi_1(M)$ is infinite, the conclusion follows from Theorem 3.6. □

4 Moduli spaces of LCF metrics with PSC

In this section, we answer the question of Bamler and Kleiner [BK19, Question 1.8] affirmatively for closed, oriented 3-manifolds with finite fundamental group and for $S^2 \times S^1$. Retaining their notation, the question is:

> Is $\mathrm{Met}_{\mathrm{PSC}}(M) \cap \mathrm{Met}_{\mathrm{CF}}(M)$ always empty or contractible? Equivalently, is the space of conformally flat metrics with positive Yamabe constant always empty or contractible?

We then extend the method to smooth n-manifolds, $n \geq 4$, that are homeomorphic to spherical space forms but not diffeomorphic to S^n. Throughout, $\mathrm{Met}(M^n)$ and $\mathrm{Diff}(M^n)$ are equipped with their C^∞-topologies, and all subspaces of $\mathrm{Met}(M^n)$ carry the subspace topology. We write g_{st} for the round metric of sectional curvature 1 on the sphere whose dimension is clear from context. We use the sign convention $\Delta_g = \operatorname{div}_g \nabla$, so that $-\Delta_g$ has nonnegative spectrum. Let

$$X(M^n) := \Big\{ g \in \mathrm{Met}(M^n) \ \Big| \ g \text{ is locally conformally flat and } \mathrm{Sc}_g > 0 \Big\},$$

equipped with the natural action of $\mathrm{Diff}(M^n)$ given by $\psi \cdot g := (\psi^{-1})^* g$. Thus, in the notation of the quotation above, $X(M^3) = \mathrm{Met}_{\mathrm{PSC}}(M^3) \cap \mathrm{Met}_{\mathrm{CF}}(M^3)$. We define the moduli space $\mathcal{M}(M^n) := X(M^n)/\mathrm{Diff}(M^n)$, endowed with the quotient topology.

We shall repeatedly use the following fact. Let (N, h) be a closed, connected Riemannian 3-manifold locally isometric to $(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$, where $g_{\mathrm{cyl}} := g_{st} + dt^2$, and let $(\widetilde{N}, \widetilde{h})$ be its universal Riemannian covering. Then:

(i) $(\widetilde{N}, \widetilde{h})$ is isometric to $(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$;

(ii) (N, h) is isometric to a quotient of $(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$ by a group of cylinder isometries.

Indeed, since N is closed, (N, h) is complete; hence $(\widetilde{N}, \widetilde{h})$ is complete and simply connected. The Ricci endomorphism of $\widetilde{h}$ is parallel, because this is true in every cylindrical chart. Its eigendistributions corresponding to the eigenvalues 1 and 0 have ranks 2 and 1, respectively, and are parallel. The de Rham decomposition theorem therefore gives an isometric splitting

$$(\widetilde{N}, \widetilde{h}) \cong (\Sigma^2, h_\Sigma) \times (\mathbb{R}, dt^2),$$

where (Σ, h_Σ) is complete, simply connected, and has constant sectional curvature 1. By the Killing–Hopf theorem, (Σ, h_Σ) is isometric to (S^2, g_{st}), proving (i). Given an isometry $J\colon (\widetilde{N}, \widetilde{h}) \to (S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$, the group $J\,\mathrm{Deck}(\widetilde{N}/N)\,J^{-1} \subset \mathrm{Isom}(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$ acts freely and properly discontinuously, and J descends to the quotient isometry asserted in (ii).

Proposition 4.1. *Let M^3 be a closed, connected, oriented smooth 3-manifold with finite fundamental group. Then both $X(M^3)$ and $\mathcal{M}(M^3)$ are contractible.*

Proof. By the elliptization theorem of Perelman [Per02, Per03], M is diffeomorphic to a spherical space form S^3/Γ, where $\Gamma < \mathrm{SO}(4)$ is finite and acts freely on S^3. In particular, M carries a metric of constant sectional curvature 1.

Let

$$\mathrm{Met}^{CC}(M) := \Big\{ h \in \mathrm{Met}(M) \ \Big| \ h \text{ is locally isometric to } (S^3, g_{st}) \Big\}.$$

In particular, the curvature scale in $\mathrm{Met}^{CC}(M)$ is fixed: every metric in this space has constant sectional curvature exactly 1. Moreover, $\mathrm{Met}^{CC}(M) \neq \varnothing$ and $\mathrm{Met}^{CC}(M) \subset X(M)$, because every metric locally isometric to (S^3, g_{st}) is locally conformally flat and has scalar curvature 6.

We use the superscript CC for the space of normalized locally spherical metrics above, whereas $\mathrm{Met}_{CC}(M)$ in [BK19, Theorem 1.2] consists of metrics locally isometric to either the unit round sphere (S^3, g_{st}) or the unit round cylinder $(S^2 \times$

$\mathbb{R}, g_{st} + dt^2)$. On the present manifold the two spaces coincide. Indeed, if M carried a locally cylindrical metric, part (i) of the local-to-global cylinder observation would identify its universal cover with $S^2 \times \mathbb{R}$, which is noncompact. Since $\pi_1(M)$ is finite, however, the universal cover is a finite-sheeted cover of the compact manifold M and is therefore compact, a contradiction. Thus the cylindrical alternative cannot occur. Hence [BK19, Theorem 1.2] implies that $\mathrm{Met}^{CC}(M)$ is contractible.

For each $g \in X(M)$, define

$$V_g := \Big\{ \phi \in C^\infty(M) \;\Big|\; e^{2\phi} g \in \mathrm{Met}^{CC}(M) \Big\}.$$

Fix the universal covering

$$p \colon \widetilde{M} \longrightarrow M, \qquad \mathcal{D} := \mathrm{Deck}(p), \qquad |\mathcal{D}| = |\pi_1(M)|,$$

and write $\widetilde{g} := p^* g$. Since M is a spherical space form, $\widetilde{M}$ is diffeomorphic to S^3. Define

$$\widetilde{V}_{\widetilde{g}} := \Big\{ \widetilde{\phi} \in C^\infty(\widetilde{M}) \;\Big|\; (\widetilde{M}, e^{2\widetilde{\phi}} \widetilde{g}) \text{ is isometric to } (S^3, g_{st}) \Big\}.$$

Pullback identifies the downstairs admissible set with the deck-invariant part of the upstairs admissible set:

$$p^*(V_g) = \widetilde{V}_{\widetilde{g}} \cap C^\infty(\widetilde{M})^{\mathcal{D}}.$$

Indeed, if $\phi \in V_g$, then $p^*(e^{2\phi} g) = e^{2p^*\phi} \widetilde{g}$ is a complete metric of constant sectional curvature 1 on the simply connected manifold $\widetilde{M}$; by the Killing–Hopf theorem it is isometric to (S^3, g_{st}). Conversely, every $\mathcal{D}$-invariant $\widetilde{\phi} \in \widetilde{V}_{\widetilde{g}}$ descends to a unique $\phi \in C^\infty(M)$, and the descended metric $e^{2\phi} g$ is locally isometric to (S^3, g_{st}), so $\phi \in V_g$.

Apply the proof of [BK19, Lemma 9.2(a)] to $(\widetilde{M}, \widetilde{g})$. It gives a unique minimizer $\widetilde{\phi}_{\widetilde{g}} \in \widetilde{V}_{\widetilde{g}}$ of

$$\widetilde{\mathcal{F}}_{\widetilde{g}}(\widetilde{\phi}) := \int_{\widetilde{M}} e^{-\widetilde{\phi}} \, d\mu_{\widetilde{g}}.$$

For every $\kappa \in \mathcal{D}$, one has $\kappa^* \widetilde{g} = \widetilde{g}$ and hence $e^{2(\widetilde{\phi}_{\widetilde{g}} \circ \kappa)} \widetilde{g} = \kappa^* \Big(e^{2\widetilde{\phi}_{\widetilde{g}}} \widetilde{g} \Big)$, so $\widetilde{\phi}_{\widetilde{g}} \circ \kappa \in \widetilde{V}_{\widetilde{g}}$; indeed, one precomposes the associated isometry to (S^3, g_{st}) with κ. Moreover, $\widetilde{\mathcal{F}}_{\widetilde{g}}(\widetilde{\phi}_{\widetilde{g}} \circ \kappa) = \widetilde{\mathcal{F}}_{\widetilde{g}}(\widetilde{\phi}_{\widetilde{g}})$. Thus $\widetilde{\phi}_{\widetilde{g}} \circ \kappa$ is another minimizer, and uniqueness implies that $\widetilde{\phi}_{\widetilde{g}}$ is $\mathcal{D}$-invariant. It therefore descends to a function $\phi(g) \in V_g$. Define the downstairs functional by $\mathcal{F}_g(\phi) := \int_M e^{-\phi} \, d\mu_g$. Because $\widetilde{g} = p^* g$, the Riemannian volume densities satisfy $d\mu_{\widetilde{g}} = p^*(d\mu_g)$. Hence, for every $\phi \in V_g$, the covering-integration formula gives

$$\widetilde{\mathcal{F}}_{\widetilde{g}}(p^*\phi) = \int_{\widetilde{M}} e^{-p^*\phi} \, d\mu_{\widetilde{g}} = \int_{\widetilde{M}} p^*\big(e^{-\phi} \, d\mu_g\big) = |\mathcal{D}| \int_M e^{-\phi} \, d\mu_g = |\mathcal{D}| \, \mathcal{F}_g(\phi).$$

Since the unique upstairs minimizer is deck-invariant, this identity shows that $\phi(g)$ minimizes $\mathcal{F}_g$ on V_g. If $\phi \in V_g$ were another minimizer, then $p^*\phi$ would also minimize $\widetilde{\mathcal{F}}_{\widetilde{g}}$; upstairs uniqueness would give $p^*\phi = \widetilde{\phi}_{\widetilde{g}} = p^*\phi(g)$, and hence $\phi = \phi(g)$.

Finally, the map $g \mapsto p^* g$ is continuous in the C^∞-topology, and the proof of [BK19, Lemma 9.2(a)] gives the C^∞-continuity of the upstairs minimizer $\widetilde{g} \mapsto \widetilde{\phi}_{\widetilde{g}}$. The pullback map $p^* \colon C^\infty(M) \to C^\infty(\widetilde{M})^{\mathcal{D}}$ is a linear homeomorphism for the Fréchet C^∞-topologies; continuity of its inverse follows, for example, from a finite collection of evenly covered coordinate charts. Therefore $g \longmapsto \phi(g) = (p^*)^{-1}(\widetilde{\phi}_{p^*g})$ is continuous in the C^∞-topology. Define $r(g) := e^{2\phi(g)}g \in \mathrm{Met}^{CC}(M)$. Then $r\colon X(M) \to \mathrm{Met}^{CC}(M)$ is continuous.

We claim that $r(g) = g$ for every $g \in \mathrm{Met}^{CC}(M)$. Fix such a metric g, and let $\phi \in V_g$. Since both g and $e^{2\phi}g$ belong to $\mathrm{Met}^{CC}(M)$, both are metrics of constant sectional curvature 1 on the same manifold M. Their universal covers are therefore isometric to (S^3, g_{st}), and the covering degree is $|\pi_1(M)|$ in both cases. Hence

$$\mathrm{Vol}(M,g) = \mathrm{Vol}(M, e^{2\phi}g) = \frac{\mathrm{Vol}(S^3, g_{st})}{|\pi_1(M)|}.$$

Since $d\mu_{e^{2\phi}g} = e^{3\phi}\, d\mu_g$, it follows that $\int_M e^{3\phi}\, d\mu_g = \mathrm{Vol}(M, e^{2\phi}g) = \mathrm{Vol}(M,g)$.

Now consider the function $f(s) := 3e^{-s} + e^{3s} - 4$, $s \in \mathbb{R}$. We have $f''(s) = 3e^{-s} + 9e^{3s} > 0$, so f is strictly convex, and $f'(s) = -3e^{-s} + 3e^{3s} = 0 \iff s = 0$. Therefore $f(s) \geq 0$ for all s, with equality if and only if $s = 0$. Applying this pointwise to $s = \phi(x)$ and integrating, we obtain

$$\int_M \big(3e^{-\phi} + e^{3\phi}\big)\, d\mu_g \geq 4\ \mathrm{Vol}(M,g),$$

with equality if and only if $\phi \equiv 0$. Using the preceding identity for $\int_M e^{3\phi}\, d\mu_g$, we get $3\mathcal{F}_g(\phi) + \mathrm{Vol}(M,g) \geq 4\ \mathrm{Vol}(M,g)$, hence $\mathcal{F}_g(\phi) \geq \mathrm{Vol}(M,g) = \mathcal{F}_g(0)$, with equality if and only if $\phi \equiv 0$. Thus 0 is the unique minimizer of $\mathcal{F}_g$ on V_g. By the uniqueness established above, we conclude that $\phi(g) \equiv 0$ and $r(g) = g$ for every $g \in \mathrm{Met}^{CC}(M)$.

We now construct a strong deformation retraction of $X(M)$ onto $\mathrm{Met}^{CC}(M)$. For a metric $g \in X(M)$, set $u(g) := e^{\phi(g)/2} > 0$, so that $r(g) = u(g)^4\, g$. Let $L_g := -8\Delta_g + \mathrm{Sc}_g$ be the conformal Laplacian. In dimension three, if $h = u^4 g$, then $\mathrm{Sc}_h = u^{-5} L_g(u)$. For $t \in [0,1]$, define

$$u_t(g) := (1-t) + t\, u(g), \qquad H(g,t) := u_t(g)^4\, g.$$

Since $g \mapsto \phi(g)$ is continuous in the C^∞-topology, the same is true for $g \mapsto u(g)$. Affine interpolation, pointwise powers, and tensor multiplication are continuous in the C^∞ Fréchet topology. Hence the preceding formula defines a jointly continuous map $H\colon X(M) \times [0,1] \to \mathrm{Met}(M)$. We claim that its image is contained in $X(M)$. First, local conformal flatness is a conformal invariant, and $H(g,t)$ is conformal to g, so $H(g,t)$ is LCF. Second, since $g \in X(M)$, $L_g(1) = \mathrm{Sc}_g > 0$. Since $r(g) \in \mathrm{Met}^{CC}(M)$, it has PSC, and hence $L_g(u(g)) = u(g)^5\, \mathrm{Sc}_{r(g)} > 0$ pointwise. By linearity of L_g, $L_g(u_t(g)) = (1-t)L_g(1) + tL_g(u(g)) > 0$ pointwise for all $t \in [0,1]$. Therefore $\mathrm{Sc}_{H(g,t)} = u_t(g)^{-5} L_g(u_t(g)) > 0$. Thus $H(g,t) \in X(M)$ for all (g,t).

Consequently, $H\colon X(M) \times [0,1] \to X(M)$ is continuous, $H(g,0) = g$, and $H(g,1) = r(g)$.

Finally, if $g \in \mathrm{Met}^{CC}(M)$, then $r(g) = g$, hence $u(g) \equiv 1$, so $u_t(g) \equiv 1$ and therefore $H(g,t) = g$ for all $t \in [0,1]$. It follows that H is a strong deformation retraction of $X(M)$ onto $\mathrm{Met}^{CC}(M)$. Since $\mathrm{Met}^{CC}(M)$ is contractible, so is $X(M)$.

It remains to prove that $\mathcal{M}(M) = X(M)/\mathrm{Diff}(M)$ is contractible. We first verify that H is $\mathrm{Diff}(M)$-equivariant. Let $\psi \in \mathrm{Diff}(M)$, and write $\psi \cdot g := (\psi^{-1})^* g$. If $\phi \in V_g$, then $e^{2(\phi\circ\psi^{-1})}(\psi \cdot g) = (\psi^{-1})^*(e^{2\phi}g) \in \mathrm{Met}^{CC}(M)$, so $\phi \circ \psi^{-1} \in V_{\psi\cdot g}$.

Moreover,

$$\mathcal{F}_{\psi\cdot g}(\phi\circ\psi^{-1}) = \int_M e^{-(\phi\circ\psi^{-1})}\, d\mu_{(\psi^{-1})^*g} = \int_M e^{-\phi}\, d\mu_g = \mathcal{F}_g(\phi).$$

Thus the bijection $V_g \to V_{\psi\cdot g}$, $\phi \longmapsto \phi\circ\psi^{-1}$, preserves the functional $\mathcal{F}$. By uniqueness of the minimizer, $\phi(\psi\cdot g) = \phi(g)\circ\psi^{-1}$. Consequently, $r(\psi\cdot g) = \psi\cdot r(g)$. Since $u(\psi\cdot g) = u(g)\circ\psi^{-1}$, it follows that $u_t(\psi\cdot g) = u_t(g)\circ\psi^{-1}$ and hence $H(\psi\cdot g, t) = \psi\cdot H(g,t)$ for all $(g,t)\in X(M)\times[0,1]$.

Let $q_M\colon X(M)\to\mathcal{M}(M) := X(M)/\mathrm{Diff}(M)$ be the quotient map. Since H is $\mathrm{Diff}(M)$-equivariant, the map $(g,t)\mapsto q_M(H(g,t))$ is constant on the fibers of $q_M\times\mathrm{id}_{[0,1]}$. Indeed, if $(q_M\times\mathrm{id}_{[0,1]})(g,t) = (q_M\times\mathrm{id}_{[0,1]})(g',t')$, then $t=t'$ and $g' = \psi\cdot g$ for some $\psi\in\mathrm{Diff}(M)$. Hence, by $\mathrm{Diff}(M)$-equivariance of H,

$$q_M(H(g',t')) = q_M(H(\psi\cdot g,t)) = q_M(\psi\cdot H(g,t)) = q_M(H(g,t)).$$

Since $\mathrm{Diff}(M)$ acts on $X(M)$ by homeomorphisms, q_M is open. Indeed, for every open set $U\subset X(M)$, $q_M^{-1}(q_M(U)) = \bigcup_{\psi\in\mathrm{Diff}(M)}\psi\cdot U$ is open. Hence $q_M\times\mathrm{id}_{[0,1]}$ is an open continuous surjection, and therefore a quotient map. Thus $(g,t)\mapsto q_M(H(g,t))$ factors uniquely through a continuous map

$$\overline{H}\colon \mathcal{M}(M)\times[0,1]\longrightarrow\mathcal{M}(M), \qquad \overline{H}([g],t) = [H(g,t)].$$

Because $\mathrm{Met}^{CC}(M)$ is $\mathrm{Diff}(M)$-invariant, the equivariant retraction $r\colon X(M)\to\mathrm{Met}^{CC}(M)$ induces a continuous map on orbit spaces, denoted by $\overline{r}$, with $\overline{r}([g]) = [r(g)]$. The inclusion of $\mathrm{Met}^{CC}(M)$ in $X(M)$ induces a continuous injection on orbit spaces, denoted by j, with $j([h]) = [h]$. Moreover, $\overline{r}\circ j = \mathrm{id}$. Hence j is a topological embedding, and we identify $\mathrm{Met}^{CC}(M)/\mathrm{Diff}(M)$ with its image in $\mathcal{M}(M)$. Since H fixes $\mathrm{Met}^{CC}(M)$ pointwise, $\overline{H}$ is a strong deformation retraction of $\mathcal{M}(M)$ onto $\mathrm{Met}^{CC}(M)/\mathrm{Diff}(M)$.

We now show that $\mathrm{Met}^{CC}(M)/\mathrm{Diff}(M)$ is a point. Let $h_1,h_2\in\mathrm{Met}^{CC}(M)$. Then (M,h_1) and (M,h_2) are closed spherical space forms on the same smooth manifold M, hence they are diffeomorphic. By de Rham rigidity for spherical space forms [dR50, pp. 66–67], diffeomorphic spherical space forms are isometric, so there exists an isometry $\Psi\colon(M,h_1)\to(M,h_2)$. In particular, $\Psi\in\mathrm{Diff}(M)$ and $h_1 = \Psi^*h_2$. Thus any two elements of $\mathrm{Met}^{CC}(M)$ lie in the same $\mathrm{Diff}(M)$-orbit. Therefore $\mathrm{Met}^{CC}(M)/\mathrm{Diff}(M)$ is a singleton. Since $\overline{H}$ is a strong deformation retraction onto a point, $\mathcal{M}(M)$ is contractible. □

Proposition 4.2. *Both $X(S^2\times S^1)$ and $\mathcal{M}(S^2\times S^1)$ are contractible.*

The proof follows the strategy of Proposition 4.1, except that the case of the moduli space $\mathcal{M}(S^2\times S^1)$ requires the independent classification in Lemma 4.3, whose proof follows the proposition.

Proof. Write $X := X(S^2\times S^1)$. Following [BK19], let $\mathrm{Met}^{CC}(S^2\times S^1)\subset\mathrm{Met}(S^2\times S^1)$ denote the subspace of metrics locally isometric to the standard round cylinder $(S^2\times\mathbb{R}, g_{\mathrm{cyl}})$. Thus the sectional curvature of the spherical factor is normalized to be 1. In [BK19, Theorem 1.2], the notation Met_{CC} also includes locally spherical metrics. On $S^2 \times S^1$ the spherical alternative cannot occur, since a closed locally spherical manifold has finite fundamental group, whereas $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$. Thus the space used here coincides with the space in that theorem. Identify $S^1 = \mathbb{R}/\mathbb{Z}$, and let $g_{\mathbb{R}/\mathbb{Z}}$ denote the metric on S^1 induced by dt^2. The product metric $g_{st} + g_{\mathbb{R}/\mathbb{Z}}$ is locally isometric to the round cylinder; hence $\mathrm{Met}^{CC}(S^2 \times S^1) \neq \varnothing$. Moreover, every such metric is LCF and has PSC, so $\mathrm{Met}^{CC}(S^2 \times S^1) \subset X$. By definition, $\mathrm{Met}^{CC}(S^2 \times S^1)$ is $\mathrm{Diff}(S^2 \times S^1)$-invariant.

Since every $g \in X(S^2 \times S^1)$ is conformally flat and $S^2 \times S^1$ is diffeomorphic to the quotient $(S^2 \times \mathbb{R})/\langle (x,t) \mapsto (x, t+1)\rangle$, [BK19, Lemma 9.2(b)] applies to any $g \in X(S^2 \times S^1)$ and yields a unique function $\phi(g) \in C^\infty(S^2 \times S^1)$ such that $e^{2\phi(g)}g$ is isometric to a quotient of the standard round cylinder. Every such quotient metric is locally isometric to $(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$ and therefore belongs to $\mathrm{Met}^{CC}(S^2 \times S^1)$. Moreover, the assignment $g \mapsto \phi(g)$ is continuous in the C^∞-topology.

Hence, we obtain a continuous map $r\colon X \to \mathrm{Met}^{CC}(S^2 \times S^1) \subset X$, $r(g) = e^{2\phi(g)}\, g$. If $g \in \mathrm{Met}^{CC}(S^2 \times S^1)$, part (ii) of the local-to-global cylinder observation shows that g is itself isometric to a quotient of the standard round cylinder. Thus $\phi \equiv 0$ has the property required in [BK19, Lemma 9.2(b)]; uniqueness gives $\phi(g) \equiv 0$, and hence $r(g) = g$. Thus $r|_{\mathrm{Met}^{CC}(S^2 \times S^1)} = \mathrm{id}$.

Following the argument of Proposition 4.1, we construct a strong deformation retraction of X onto $\mathrm{Met}^{CC}(S^2 \times S^1)$. Fix $g \in X$ and set $u(g) := e^{\phi(g)/2} > 0$, so that $r(g) = e^{2\phi(g)}g = u(g)^4\, g$. Let $L_g := -8\Delta_g + \mathrm{Sc}_g$ be the conformal Laplacian in dimension 3; then the conformal change formula gives $\mathrm{Sc}_{u^4 g} = u^{-5} L_g u$. Since $g \in X$, we have $L_g(1) = \mathrm{Sc}_g > 0$, and since $r(g) \in \mathrm{Met}^{CC}(S^2 \times S^1) \subset X$, we have $\mathrm{Sc}_{r(g)} > 0$, hence $L_g(u(g)) = u(g)^5\, \mathrm{Sc}_{r(g)} > 0$ pointwise on $S^2 \times S^1$. For $t \in [0,1]$, define

$$u_t(g) := (1-t)\cdot 1 + t \cdot u(g) \;>\; 0, \qquad H(g,t) := u_t(g)^4\, g.$$

Linearity of L_g gives $L_g(u_t(g)) = (1-t)L_g(1) + tL_g(u(g)) > 0$, so $\mathrm{Sc}_{H(g,t)} = u_t(g)^{-5} L_g(u_t(g)) > 0$ for all t. Moreover, local conformal flatness is preserved under conformal changes, so $H(g,t)$ is LCF. Since $g \mapsto \phi(g)$, and hence $g \mapsto u(g)$, is continuous in the C^∞-topology, and since affine interpolation, pointwise powers, and tensor multiplication are continuous in the C^∞ Fréchet topology, the map $(g,t) \mapsto u_t(g)^4 g$ is jointly continuous. Thus $H\colon X \times [0,1] \to X$ is a continuous homotopy with $H(g,0) = g$ and $H(g,1) = r(g)$. If $g \in \mathrm{Met}^{CC}(S^2 \times S^1)$, then $u(g) \equiv 1$, hence $H(g,t) = g$ for all t. Therefore H is a strong deformation retraction of X onto $\mathrm{Met}^{CC}(S^2 \times S^1)$.

By [BK19, Theorem 1.2], $\mathrm{Met}^{CC}(S^2 \times S^1)$ is contractible. Since X strongly deformation retracts onto $\mathrm{Met}^{CC}(S^2 \times S^1)$, it follows that X is contractible.

We claim that r and H are $\mathrm{Diff}(S^2 \times S^1)$-equivariant, hence descend to the quotient $\mathcal{M}(S^2 \times S^1) = X/\mathrm{Diff}(S^2 \times S^1)$. Let $\psi \in \mathrm{Diff}(S^2 \times S^1)$ and $\widehat{g} := \psi \cdot g = (\psi^{-1})^* g$. Define $\widehat{\phi} := \phi(g) \circ \psi^{-1} \in C^\infty(S^2 \times S^1)$. We compute

$$\begin{aligned} e^{2\widehat{\phi}}\widehat{g} &= e^{2(\phi(g)\circ\psi^{-1})}(\psi^{-1})^* g \\ &= (\psi^{-1})^*\big(e^{2\phi(g)}g\big) \\ &= \psi \cdot \big(e^{2\phi(g)}g\big). \end{aligned}$$

Since $e^{2\phi(g)}g$ is isometric to a quotient of the standard round cylinder, so is its pullback $e^{2\widehat{\phi}}\widehat{g}$. Thus $\widehat{\phi}$ has the property required in [BK19, Lemma 9.2(b)] for the metric $\widehat{g} = \psi \cdot g$.

The same lemma asserts that such a function is unique. Therefore $\widehat{\phi} = \phi(\widehat{g})$, i.e. $\phi(\psi \cdot g) = \phi(g) \circ \psi^{-1}$. Hence,

$$\begin{aligned}\psi \cdot r(g) &= (\psi^{-1})^*(e^{2\phi(g)}g) \\ &= e^{2(\phi(g)\circ\psi^{-1})}(\psi \cdot g) \\ &= e^{2\phi(\psi\cdot g)}(\psi \cdot g) = r(\psi \cdot g).\end{aligned}$$

It also follows that $u(\psi \cdot g) = u(g) \circ \psi^{-1}$. Hence $u_t(\psi \cdot g) = u_t(g) \circ \psi^{-1}$ and therefore $H(\psi \cdot g, t) = \psi \cdot H(g, t)$. Let $q_M \colon X \to \mathcal{M}(S^2 \times S^1)$ be the quotient map. Since $\mathrm{Diff}(S^2 \times S^1)$ acts on X by homeomorphisms, q_M is open: for every open $U \subset X$, $q_M^{-1}(q_M(U)) = \bigcup_{\psi\in\mathrm{Diff}(S^2\times S^1)} \psi \cdot U$ is open. Hence $q_M \times \mathrm{id}_{[0,1]}$ is an open continuous surjection, and therefore a quotient map. Thus the equivariant homotopy H descends to a continuous strong deformation retraction $\overline{H} \colon \mathcal{M}(S^2 \times S^1) \times [0, 1] \to \mathcal{M}(S^2 \times S^1)$ of $\mathcal{M}(S^2 \times S^1)$ onto the image of $\mathrm{Met}^{CC}(S^2 \times S^1)$ in the quotient. As in the proof of Proposition 4.1, the equivariant retraction r shows that the natural map

$$\mathcal{M}^{CC}(S^2 \times S^1) := \mathrm{Met}^{CC}(S^2 \times S^1)/\mathrm{Diff}(S^2 \times S^1) \longrightarrow \mathcal{M}(S^2 \times S^1)$$

is a topological embedding. We therefore identify its image with $\mathcal{M}^{CC}(S^2 \times S^1)$.

Lemma 4.3 shows that $\mathcal{M}^{CC}(S^2 \times S^1)$ is homeomorphic to $(0, \infty) \times [0, \pi]$, which is convex and hence contractible. Since $\mathcal{M}(S^2 \times S^1)$ deformation retracts onto this subspace, $\mathcal{M}(S^2 \times S^1)$ is contractible. □

Lemma 4.3. *$\mathcal{M}^{CC}(S^2 \times S^1)$ is homeomorphic to $(0, \infty) \times [0, \pi]$.*

Proof. We construct continuous maps $Q \colon (0, \infty) \times [0, \pi] \to \mathcal{M}^{CC}(S^2 \times S^1)$ and $\overline{P} \colon \mathcal{M}^{CC}(S^2 \times S^1) \to (0, \infty) \times [0, \pi]$ and prove that they are inverse homeomorphisms.

Fix once and for all the universal covering

$$\varpi \colon S^2 \times \mathbb{R} \longrightarrow S^2 \times S^1, \qquad \varpi(x, s) = (x, [s]),$$

with deck generator

$$\tau \colon S^2 \times \mathbb{R} \longrightarrow S^2 \times \mathbb{R}, \qquad \tau(x, s) = (x, s + 1).$$

Thus $\mathrm{Deck}(\varpi) = \langle\tau\rangle \cong \mathbb{Z}$.

Let $h \in \mathrm{Met}^{CC}(S^2 \times S^1)$, and write $\widetilde{h} := \varpi^*h$. By part (i) of the local-to-global cylinder observation, there is an isometry $J_h \colon (S^2 \times \mathbb{R}, \widetilde{h}) \to (S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$. Set $\gamma := J_h \circ \tau \circ J_h^{-1} \in \mathrm{Isom}(S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$. Because γ is conjugate to τ, the action of $\langle\gamma\rangle$ is free and γ has infinite order.

The Ricci endomorphism of the round cylinder has eigendistributions TS^2 and $T\mathbb{R}$, with the distinct eigenvalues 1 and 0. Hence every isometry preserves both distributions. Writing an isometry as $\Psi = (\Psi_1, \Psi_2)$, this gives $\partial_t\Psi_1 = 0$ and $d_x\Psi_2 = 0$. Since both factors are connected, it follows that $\Psi(x, t) = (A(x), b(t))$. Bijectivity of Ψ makes A and b diffeomorphisms, while $\Psi^*g_{\mathrm{cyl}} = g_{\mathrm{cyl}}$ gives $A^*g_{st} = g_{st}$ and $b^*dt^2 = dt^2$. Consequently,

$$\mathrm{Isom}(S^2 \times \mathbb{R}, g_{\mathrm{cyl}}) \cong \mathrm{Isom}(S^2, g_{st}) \times \mathrm{Isom}(\mathbb{R}, dt^2) \cong \mathrm{O}(3) \times (\mathbb{R} \rtimes \{\pm 1\}).$$

Thus $\gamma(x,t) = (Ax, \delta t + L)$ for some $A \in \mathrm{O}(3)$, $\delta \in \{\pm 1\}$, and $L \in \mathbb{R}$.

We claim that $\delta = 1$. If $\delta = -1$, then $\gamma^2(x,t) = (A^2x, t)$. Since $A^2 \in \mathrm{SO}(3)$, there exists $x_0 \in S^2$ such that $A^2x_0 = x_0$. Hence $\gamma^2(x_0,t) = (x_0,t)$ for all $t \in \mathbb{R}$. Since γ has infinite order, $\gamma^2 \neq \mathrm{id}$, contradicting the freeness of the $\langle\gamma\rangle$-action. Thus $\delta = 1$.

We next claim that $A \in \mathrm{SO}(3)$. The manifold $S^2 \times S^1$ is oriented, and τ preserves the lifted orientation on $S^2 \times \mathbb{R}$. Since orientation preservation is invariant under conjugation, the conjugate $\gamma = J_h \tau J_h^{-1}$ also preserves the product orientation. The translation $t \mapsto t+L$ preserves the orientation of $\mathbb{R}$; hence A preserves the orientation of S^2. Therefore $A \in \mathrm{SO}(3)$.

We have $L \neq 0$. Indeed, if $L = 0$, then $\gamma(x,t) = (Ax,t)$. Since $A \in \mathrm{SO}(3)$, there exists $x_0 \in S^2$ such that $Ax_0 = x_0$. Hence $\gamma(x_0,t) = (x_0,t)$ for all $t \in \mathbb{R}$, again contradicting freeness. Thus $L \neq 0$.

Finally, after postcomposing J_h with the reflection $(x,t) \mapsto (x,-t)$ if necessary, we may assume $L > 0$. After this normalization, the parameters do not depend on the choices. Indeed, if $J_h' = \Psi \circ J_h$, where $\Psi(x,t) = (Bx, \varepsilon t + c)$, then $(\Psi\gamma\Psi^{-1})(x,t) = (BAB^{-1}x, t + \varepsilon L)$. After restoring the convention that the translation length is positive, L is unchanged and A is replaced by an $\mathrm{O}(3)$-conjugate. If the deck generator τ is replaced by τ^{-1}, then the holonomy element becomes $\gamma^{-1}(x,t) = (A^{-1}x, t - L)$; conjugating by $(x,t) \mapsto (x,-t)$ to restore the normalization gives $(A^{-1}x, t+L)$. Thus neither operation changes the positive translation length L or the rotation angle $\theta \in [0,\pi]$ of A. Hence h determines the well-defined parameters (L,θ), and its holonomy admits the normalized representative

$$\gamma_{L,A}(x,t) := (Ax, t+L), \qquad A \in \mathrm{SO}(3), \quad L > 0.$$

We now construct the continuous map $Q\colon (0,\infty) \times [0,\pi] \to \mathcal{M}^{CC}(S^2 \times S^1)$. For $(L,\theta) \in (0,\infty) \times [0,\pi]$, let $\gamma_{L,\theta}(x,t) := (R_\theta x, t+L)$, where $R_\theta \in \mathrm{SO}(3)$ denotes the standard rotation by angle θ about the x_3-axis. Let $(S^2 \times \mathbb{R})/\langle\gamma_{L,\theta}\rangle$ carry the quotient metric $\bar{g}_{L,\theta}$.

To identify this quotient with $S^2 \times S^1$, define

$$\widetilde{F}_{L,\theta}\colon S^2 \times \mathbb{R} \longrightarrow S^2 \times \mathbb{R}, \qquad \widetilde{F}_{L,\theta}(x,s) = (R_{s\theta}x, Ls).$$

This is a diffeomorphism, with inverse $\widetilde{F}_{L,\theta}^{-1}(y,t) = \big(R_{-(t/L)\theta}y, t/L\big)$. Moreover, $\widetilde{F}_{L,\theta} \circ \tau = \gamma_{L,\theta} \circ \widetilde{F}_{L,\theta}$. Hence $\widetilde{F}_{L,\theta}$ descends to a diffeomorphism $F_{L,\theta}\colon S^2 \times S^1 \to (S^2 \times \mathbb{R})/\langle\gamma_{L,\theta}\rangle$. Define $h_{L,\theta} := F_{L,\theta}^* \bar{g}_{L,\theta} \in \mathrm{Met}^{CC}(S^2 \times S^1)$.

On the fixed universal cover one has $\varpi^* h_{L,\theta} = \widetilde{F}_{L,\theta}^* g_{\mathrm{cyl}}$. Let Z be the Killing field on S^2 generating $a \mapsto R_a$. For $(V,a), (W,b) \in T_xS^2 \oplus \mathbb{R}\partial_s$, direct differentiation gives

$$(\varpi^* h_{L,\theta})\big((V,a),(W,b)\big) = g_{st}(V + a\theta Z_x, W + b\theta Z_x) + L^2 ab.$$

This tensor is τ-invariant and depends smoothly on (L,θ), including at $\theta = 0$ and $\theta = \pi$. Hence the map $(L,\theta) \longmapsto h_{L,\theta}$ is continuous in the C^∞-topology. Thus there is a continuous map

$$Q\colon (0,\infty) \times [0,\pi] \longrightarrow \mathcal{M}^{CC}(S^2 \times S^1), \qquad Q(L,\theta) := [h_{L,\theta}].$$

We now construct the map $\overline{P}$. Let $h \in \mathrm{Met}^{CC}(S^2 \times S^1)$, and let $\widetilde{h} := \varpi^* h$. Define

$$f_h \colon S^2 \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad f_h(p) := d_{\widetilde{h}}(p, \tau p).$$

Since τ is an isometry of $(S^2 \times \mathbb{R}, \widetilde{h})$, one has $f_h(\tau p) = f_h(p)$, so f_h descends to a continuous function on the compact quotient $S^2 \times S^1$, still denoted f_h.

Set

$$L(h) := \min_{S^2 \times S^1} f_h, \qquad D(h) := \max_{S^2 \times S^1} f_h, \qquad \Theta(h) := \sqrt{D(h)^2 - L(h)^2}.$$

Since τ acts freely, $f_h(p) > 0$ for every p. Since f_h is continuous on the compact quotient, it attains a positive minimum. Hence $L(h) > 0$.

We now compute these invariants from the normalized holonomy. Let J_h be the isometry above satisfying $J_h \tau J_h^{-1} = \gamma_{L,A}$, where $A \in \mathrm{SO}(3)$ has rotation angle $\theta \in [0, \pi]$ and $L > 0$. Because isometries preserve the distance function, the displacement $f_h(p)$ transforms as

$$f_h(p) = d_{g_{\mathrm{cyl}}}(J_h(p), J_h(\tau p)) = d_{g_{\mathrm{cyl}}}((x,t), \gamma_{L,A}(x,t)),$$

where $(x,t) := J_h(p)$. Thus we obtain

$$f_h(p)^2 = d_{g_{st}}(x, Ax)^2 + d_{\mathbb{R}}(t, t+L)^2 = d_{g_{st}}(x, Ax)^2 + L^2.$$

Let v be a unit vector on the rotation axis of A, chosen arbitrarily if $\theta = 0$, and put $a := \langle x, v \rangle$. Then $\langle x, Ax \rangle = a^2 + (1 - a^2)\cos\theta$, so $d_{g_{st}}(x, Ax)$ ranges over $[0, \theta]$. If $\theta > 0$, its minimum is attained on the rotation axis and its maximum on the orthogonal equator; if $\theta = 0$, it vanishes everywhere. It follows that $L(h)^2 = L^2$ and $D(h)^2 = \theta^2 + L^2$. Therefore $L(h) = L$, $D(h) = \sqrt{L^2 + \theta^2}$, and $\Theta(h) = \theta$.

Thus

$$P \colon \mathrm{Met}^{CC}(S^2 \times S^1) \longrightarrow (0, \infty) \times [0, \pi], \qquad P(h) := \big(L(h), \Theta(h)\big)$$

is well defined.

We next prove that P is continuous. Because the C^∞-topology on $\mathrm{Met}^{CC}(S^2 \times S^1)$ is metrizable, it suffices to prove sequential continuity. Let $h_i \to h$ in the C^∞-topology. In particular, $h_i \to h$ in C^0. Since h is positive definite on the compact manifold $S^2 \times S^1$, after discarding finitely many terms and renumbering, we may choose $\varepsilon_i \in (0,1)$, $\varepsilon_i \to 0$, such that

$$(1 - \varepsilon_i) h \le h_i \le (1 + \varepsilon_i) h$$

as bilinear forms on $T(S^2 \times S^1)$. Pulling back by ϖ, we obtain

$$(1 - \varepsilon_i) \widetilde{h} \le \widetilde{h}_i \le (1 + \varepsilon_i) \widetilde{h}$$

on $T(S^2 \times \mathbb{R})$, where $\widetilde{h}_i := \varpi^* h_i$.

For every piecewise C^1 curve c in $S^2 \times \mathbb{R}$,

$$\sqrt{1 - \varepsilon_i}\, \ell_{\widetilde{h}}(c) \le \ell_{\widetilde{h}_i}(c) \le \sqrt{1 + \varepsilon_i}\, \ell_{\widetilde{h}}(c).$$

Taking infima over curves joining fixed endpoints yields

$$\sqrt{1-\varepsilon_i}\, d_{\widetilde{h}} \le d_{\widetilde{h}_i} \le \sqrt{1+\varepsilon_i}\, d_{\widetilde{h}}.$$

Hence

$$\sqrt{1-\varepsilon_i}\, f_h \le f_{h_i} \le \sqrt{1+\varepsilon_i}\, f_h$$

pointwise on $S^2 \times S^1$. If $\eta_i := \max\{\sqrt{1+\varepsilon_i}-1,\ 1-\sqrt{1-\varepsilon_i}\}$, then $\eta_i \to 0$ and $\|f_{h_i} - f_h\|_{C^0(S^2\times S^1)} \le \eta_i\, D(h) \to 0$. Therefore $L(h_i) \to L(h)$, $D(h_i) \to D(h)$, and $\Theta(h_i) \to \Theta(h)$. Thus P is continuous.

We next establish that P is invariant under pullback by $\mathrm{Diff}(S^2 \times S^1)$. Let $\psi \in \mathrm{Diff}(S^2 \times S^1)$, and let $\widetilde{\psi}\colon S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ be a lift. Conjugation by $\widetilde{\psi}$ induces an automorphism of $\mathrm{Deck}(\varpi) \cong \mathbb{Z}$. Hence there exists $\varsigma \in \{\pm 1\}$ such that $\widetilde{\psi} \circ \tau \circ \widetilde{\psi}^{-1} = \tau^{\varsigma}$. By naturality of pullback, $\varpi^*(\psi^* h) = \widetilde{\psi}^*(\varpi^* h) = \widetilde{\psi}^* \widetilde{h}$. Therefore, for every $p \in S^2 \times \mathbb{R}$,

$$\begin{aligned} f_{\psi^* h}(p) &= d_{\widetilde{\psi}^* \widetilde{h}}(p, \tau p) \\ &= d_{\widetilde{h}}(\widetilde{\psi} p, \widetilde{\psi} \tau p) \\ &= d_{\widetilde{h}}(\widetilde{\psi} p, \tau^{\varsigma} \widetilde{\psi} p). \end{aligned}$$

If $\varsigma = 1$, the last expression equals $f_h(\widetilde{\psi} p)$. If $\varsigma = -1$, the symmetry of the distance function and the fact that τ is an isometry of $\widetilde{h}$ give $d_{\widetilde{h}}(\widetilde{\psi} p, \tau^{-1} \widetilde{\psi} p) = d_{\widetilde{h}}(\tau \widetilde{\psi} p, \widetilde{\psi} p) = f_h(\widetilde{\psi} p)$. Thus $f_{\psi^* h} = f_h \circ \widetilde{\psi}$ in both cases. The two functions have the same range, and hence $L(\psi^* h) = L(h)$, $D(\psi^* h) = D(h)$, and $\Theta(\psi^* h) = \Theta(h)$. It follows that P is $\mathrm{Diff}(S^2 \times S^1)$-invariant. By the universal property of the quotient, P descends to a unique continuous map $\overline{P}\colon \mathcal{M}^{CC}(S^2 \times S^1) \to (0,\infty) \times [0,\pi]$.

We conclude by showing that Q and $\overline{P}$ are inverse homeomorphisms. For any $(L,\theta) \in (0,\infty)\times[0,\pi]$, the construction of the model metric $h_{L,\theta}$ provides an isometry $\widetilde{F}_{L,\theta}\colon (S^2 \times \mathbb{R}, \varpi^* h_{L,\theta}) \to (S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$ that conjugates τ to $\gamma_{L,\theta}(x,t) = (R_\theta x, t + L)$. For $p \in S^2 \times \mathbb{R}$, write $(x,t) := \widetilde{F}_{L,\theta}(p)$. Then $f_{h_{L,\theta}}(p) = \sqrt{d_{g_{st}}(x, R_\theta x)^2 + L^2}$. By the displacement computation above, $d_{g_{st}}(x, R_\theta x)$ ranges over $[0,\theta]$; in particular, this statement also covers $\theta = 0$. It follows that $L(h_{L,\theta}) = L$ and $\Theta(h_{L,\theta}) = \theta$. Thus, $\overline{P}(Q(L,\theta)) = \overline{P}([h_{L,\theta}]) = (L,\theta)$. Conversely, let $h \in \mathrm{Met}^{CC}(S^2 \times S^1)$ have invariants $P(h) = (L,\theta)$. By the holonomy normalization established above, there is an isometry $J_h\colon (S^2 \times \mathbb{R}, \varpi^* h) \to (S^2 \times \mathbb{R}, g_{\mathrm{cyl}})$ such that $J_h \tau J_h^{-1} = \gamma_{L,A}$, where $A \in \mathrm{SO}(3)$ has rotation angle θ. Choose $C \in \mathrm{SO}(3)$ such that $CAC^{-1} = R_\theta$, and set $K_C(x,t) := (Cx, t)$. Then $(K_C \circ J_h)\tau(K_C \circ J_h)^{-1} = K_C \gamma_{L,A} K_C^{-1} = \gamma_{L,\theta}$. Hence $K_C \circ J_h$ descends to an isometry

$$\overline{K}\colon (S^2 \times S^1, h) \longrightarrow \big((S^2 \times \mathbb{R})/\langle \gamma_{L,\theta}\rangle, \bar{g}_{L,\theta}\big).$$

Composing with $F_{L,\theta}^{-1}$ gives an isometry $F_{L,\theta}^{-1} \circ \overline{K}\colon (S^2 \times S^1, h) \to (S^2 \times S^1, h_{L,\theta})$. Consequently, $Q(\overline{P}([h])) = Q(L,\theta) = [h_{L,\theta}] = [h]$.

The identities $\overline{P} \circ Q = \mathrm{id}$ and $Q \circ \overline{P} = \mathrm{id}$, together with the continuity of $\overline{P}$ and Q, show that $\overline{P}$ and Q are inverse homeomorphisms. Thus $\mathcal{M}^{CC}(S^2 \times S^1) \cong (0,\infty) \times [0,\pi]$. □

We now consider a closed, oriented smooth manifold M^n, $n \ge 3$, that is homeomorphic to a spherical space form but not diffeomorphic to S^n. We do not assume that the given smooth structure is the quotient smooth structure; in particular, M need not carry a round metric. If M admits an LCF metric, then Kuiper's theorem [Kui49], followed by the Cartan fixed-point argument used below, shows that M is diffeomorphic to a quotient S^n/Γ, where $\Gamma \subset \mathrm{O}(n+1)$ is finite and acts freely on S^n. This implication is used only under the standing assumption $X(M) \neq \varnothing$.

Proposition 4.4. *Assume $n \geq 3$. Let M^n be a closed, oriented smooth manifold homeomorphic to a spherical space form but not diffeomorphic to S^n. Define*

$$\mathrm{Met}^{CC}(M) := \{h \in \mathrm{Met}(M) \mid h \textit{ is locally isometric to } (S^n, g_{st})\}.$$

If $X(M)$ is nonempty, then $\mathrm{Met}^{CC}(M)$ is nonempty, there is a $\mathrm{Diff}(M)$-equivariant strong deformation retraction of $X(M)$ onto $\mathrm{Met}^{CC}(M)$, and $\mathrm{Met}^{CC}(M)$ is a single $\mathrm{Diff}(M)$-orbit. Consequently, the moduli space $\mathcal{M}(M) := X(M)/\mathrm{Diff}(M)$ is either empty or contractible.

Proof strategy. Assuming $X(M) \neq \varnothing$, the strategy of the proof is to construct a canonical, $\mathrm{Diff}(M)$-equivariant strong deformation retraction from $X(M)$ onto the subspace of round metrics $\mathrm{Met}^{CC}(M)$. Marques' result [Mar12, Corollary 3.2] gives a round representative in every conformal class in $X(M)$, and, because M is not diffeomorphic to S^n, Obata's theorem [Oba71] makes this representative unique after normalizing its scalar curvature to $n(n-1)$. This defines a projection $r\colon X(M) \to \mathrm{Met}^{CC}(M)$. We establish the continuity of r by applying the Banach-space implicit function theorem to the Yamabe-type equation for the conformal factor; the crucial invertibility of the linearized conformal Laplacian follows directly from the Lichnerowicz–Obata spectral gap ($\lambda_1 > n$). A simple convex interpolation of these continuous conformal factors yields the required retraction. Finally, de Rham rigidity implies that $\mathrm{Met}^{CC}(M)$ is a single $\mathrm{Diff}(M)$-orbit, so the equivariant homotopy descends to the quotient and contracts $\mathcal{M}(M)$ to a point.

Proof. If $X(M) = \varnothing$, there is nothing to prove. Assume $X(M) \neq \varnothing$, and choose $g_* \in X(M)$. Since $\pi_1(M)$ is finite, the universal cover is a finite-sheeted cover of M and hence compact. The lift of g_* makes it a closed, simply connected LCF manifold. Kuiper's theorem [Kui49] identifies the universal cover conformally and diffeomorphically with S^n. Under this identification, the deck group acts freely by conformal transformations of S^n. This group is finite and, after a conformal conjugation, is contained in $\mathrm{O}(n+1)$; for example, this follows by applying the Cartan fixed-point theorem to its isometric action on $\mathbb{H}^{n+1}$. Thus M is diffeomorphic to S^n/Γ for a finite subgroup $\Gamma \subset \mathrm{O}(n+1)$ acting freely on S^n. Since M is not diffeomorphic to S^n, the group Γ is nontrivial.

Moreover, n is odd. Indeed, if n were even, then $2 = \chi(S^n) = |\Gamma|\,\chi(S^n/\Gamma)$ would force $|\Gamma| = 2$. The only free orthogonal involution of S^n is the antipodal map, whose quotient $\mathbb{RP}^n$ is nonorientable when n is even, contradicting the orientability of M.

For any $g \in X(M)$, Marques' result [Mar12, Corollary 3.2], applied through the above diffeomorphism, shows that the conformal class $[g]$ contains a metric of constant sectional curvature. This curvature is positive because the path in Marques' corollary consists of metrics with positive scalar curvature. After multiplying this metric by a positive constant, we obtain a metric $g_0 \in [g]$ of constant sectional curvature $+1$. Since M is not diffeomorphic to S^n, (M, g_0) is not conformally equivalent to the round sphere. Obata's theorem [Oba71] therefore implies that any metric of constant scalar curvature in $[g]$ is a constant multiple of g_0. Normalizing the scalar curvature to $n(n-1)$ therefore gives a unique round representative $h \in [g] \cap \mathrm{Met}^{CC}(M)$. We define $r\colon X(M) \to \mathrm{Met}^{CC}(M)$, $r(g) := h$. This uniqueness immediately gives $r(\psi \cdot g) = \psi \cdot r(g)$ for $\psi \in \mathrm{Diff}(M)$.

For each $\eta \in X(M)$, let $v_\eta > 0$ be the unique smooth function such that $r(\eta) = v_\eta^{\frac{4}{n-2}}\eta$. We show that r is continuous. Fix $g \in X(M)$, and write $h := r(g) = u^{\frac{4}{n-2}}g$, where $u := v_g$. Set

$$a_n := \frac{4(n-1)}{n-2}, \qquad q := \frac{n+2}{n-2}, \qquad L_{g'} := -a_n\Delta_{g'} + \mathrm{Sc}_{g'}.$$

By the conformal covariance of the conformal Laplacian,

$$L_{g'}(vw) = v^q L_{\widehat{g'}}w, \qquad \widehat{g'} = v^{\frac{4}{n-2}}g';$$

in particular, $L_{g'}(v) = \mathrm{Sc}_{\widehat{g'}}\, v^q$. Since h is round, $\mathrm{Sc}_h = n(n-1)$, and therefore $L_g(u) = n(n-1)u^q$.

Fix $k \geq 0$ and $0 < \alpha < 1$. Let

$$\mathcal{G}^{k+2,\alpha} := \big\{g' \in C^{k+2,\alpha}(\mathrm{Sym}^2(T^*M)) \mid g' \text{ is positive definite}\big\},$$

and

$$\mathcal{P}^{k+2,\alpha} := \big\{v \in C^{k+2,\alpha}(M) \mid v > 0\big\}.$$

Since pointwise positive definiteness and pointwise positivity are open conditions, $\mathcal{G}^{k+2,\alpha}$ and $\mathcal{P}^{k+2,\alpha}$ are open subsets of Banach spaces. Define $\mathcal{Y}\colon \mathcal{G}^{k+2,\alpha} \times \mathcal{P}^{k+2,\alpha} \to C^{k,\alpha}(M)$ by $\mathcal{Y}(g', v) := L_{g'}v - n(n-1)v^q$. The inversion map $g' \mapsto (g')^{-1}$ is C^∞ on $\mathcal{G}^{k+2,\alpha}$, differentiation is bounded between the relevant Hölder spaces, and multiplication and tensor contraction are continuous multilinear, hence smooth, operations on these spaces. In local coordinates, $L_{g'}v$ is a finite sum of contractions of $(g')^{-1}$, derivatives of g' of order at most two, and derivatives of v of order at most two. It follows, by localization, that $(g', v) \longmapsto L_{g'}v$ is a C^∞ map from $\mathcal{G}^{k+2,\alpha} \times C^{k+2,\alpha}(M)$ to $C^{k,\alpha}(M)$. Moreover, for every $v_0 \in \mathcal{P}^{k+2,\alpha}$, compactness of M gives a neighborhood of v_0 on which all functions are bounded above and bounded away from zero. For $j \geq 1$, the derivatives of the power map on this neighborhood are

$$D^j(v \mapsto v^q)_v(\varphi_1, \ldots, \varphi_j) = q(q-1)\cdots(q-j+1)v^{q-j}\varphi_1\cdots\varphi_j.$$

The Hölder algebra estimates therefore show that $v \mapsto v^q$ is C^∞ from $\mathcal{P}^{k+2,\alpha}$ to $C^{k+2,\alpha}(M)$, and hence also to $C^{k,\alpha}(M)$. Thus $\mathcal{Y}$ is C^∞ (in particular, C^1), and $\mathcal{Y}(g, u) = 0$.

The derivative of $\mathcal{Y}$ in the v-variable at (g, u) is

$$D_v\mathcal{Y}_{(g,u)}(\varphi) = \frac{d}{dt}\bigg|_{t=0} \mathcal{Y}(g, u + t\varphi) = L_g\varphi - q\,n(n-1)u^{q-1}\varphi.$$

Since $u > 0$, every $\varphi \in C^{k+2,\alpha}(M)$ can be written uniquely as $\varphi = uw$ with $w = u^{-1}\varphi \in C^{k+2,\alpha}(M)$.

Since $L_g(uw) = u^q L_h(w)$ and $L_h = -a_n\Delta_h + n(n-1)$, we obtain

$$\begin{aligned} D_v\mathcal{Y}_{(g,u)}(uw) &= L_g(uw) - qn(n-1)u^q w \\ &= u^q\big(L_h(w) - qn(n-1)w\big) \\ &= u^q\big(-a_n\Delta_h w + (1-q)n(n-1)w\big). \end{aligned}$$

Since

$$q = \frac{n+2}{n-2}, \qquad a_n = \frac{4(n-1)}{n-2}, \qquad (1-q)n(n-1) = -na_n,$$

we have $D_v\mathcal{Y}_{(g,u)}(uw) = -a_n\, u^q(\Delta_h + n)w$. Equivalently,

$$D_v\mathcal{Y}_{(g,u)} = M_{u^q} \circ A_h \circ M_{u^{-1}}, \qquad A_h := -a_n(\Delta_h + n),$$

where M_f denotes multiplication by f. We claim that $D_v\mathcal{Y}_{(g,u)} : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ is an isomorphism. Let Γ_h denote the deck group of the round universal covering $(S^n, g_{st}) \to (M,h)$. This group is nontrivial, because otherwise M would be diffeomorphic to S^n. Since h has constant sectional curvature 1, one has $\mathrm{Ric}_h = (n-1)h$. The Lichnerowicz estimate [Lic58] therefore gives $\lambda_1(-\Delta_h) \geq n$. If equality held, a first eigenfunction would lift to a Γ_h-invariant first spherical harmonic on S^n, hence to a nonzero linear function $x \mapsto \langle v, x\rangle$ with $\zeta v = v$ for every $\zeta \in \Gamma_h$. Thus every element of Γ_h would fix $v/|v| \in S^n$. Since $\Gamma_h \neq \{1\}$, choosing $\zeta \in \Gamma_h \setminus \{1\}$ contradicts the freeness of the Γ_h-action. Therefore $\lambda_1(-\Delta_h) > n$. Since $\lambda_0(-\Delta_h) = 0$ and every nonzero eigenvalue is at least $\lambda_1(-\Delta_h) > n$, the number n is not in the spectrum of $-\Delta_h$. Hence $\ker A_h = \{0\}$.

Since $n \notin \mathrm{spec}(-\Delta_h)$, the spectral theorem and the identification of the domain of $-\Delta_h$ with $W^{2,2}(M)$ show that $A_h = a_n(-\Delta_h - n)\colon W^{2,2}(M) \to L^2(M)$ is an isomorphism. Hence, for every $\rho \in C^{k,\alpha}(M)$, there is a unique $w \in W^{2,2}(M)$ satisfying $A_h w = \rho$. Equivalently, $\Delta_h w = -nw - a_n^{-1}\rho$. Starting from $w \in W^{2,2}(M)$, repeated application of global L^p regularity [LP87, Theorem 2.5(a)], together with the Sobolev embedding theorem, gives $w \in W^{2,p}(M)$ for every $p < \infty$. Indeed, one iterates the Sobolev embedding and the equation until $w \in W^{2,p_0}(M)$ for some $p_0 > n/2$; the Sobolev exponent strictly increases at each subcritical step. At a critical step $p = n/2$, one uses $W^{2,n/2}(M) \hookrightarrow L^r(M)$ for every finite r and chooses $r > n/2$; the equation and global L^r regularity then give $w \in W^{2,r}(M)$. Thus in all cases w is bounded once this threshold is crossed. The right-hand side then belongs to $L^p(M)$ for every $p < \infty$, and another application of global L^p regularity gives the displayed conclusion. Choose p sufficiently large and then choose β with $\alpha < \beta < 1 - \frac{n}{p}$. The Sobolev embedding gives $w \in C^{1,\beta}(M)$, so the right-hand side belongs to $C^{0,\alpha}(M)$. Global Hölder regularity [LP87, Theorem 2.5(b)] then gives $w \in C^{2,\alpha}(M)$. Iterating the same regularity theorem in the equation yields $w \in C^{k+2,\alpha}(M)$.

Since A_h is a second-order linear differential operator with smooth coefficients on the closed manifold M, the map $A_h\colon C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ is bounded. It is injective because its kernel is already trivial in $W^{2,2}(M)$, and it is surjective by the preceding argument. The bounded inverse theorem therefore implies that $A_h^{-1}\colon C^{k,\alpha}(M) \to C^{k+2,\alpha}(M)$ is bounded. Since multiplication by each of the positive smooth functions u and u^q is an isomorphism on Hölder spaces, $D_v\mathcal{Y}_{(g,u)}$ is an isomorphism as well.

By the Banach-space implicit function theorem, there exist an open neighborhood $\mathcal{U}_g^{k+2,\alpha} \subset \mathcal{G}^{k+2,\alpha}$ of g, an open neighborhood $\mathcal{W}_u^{k+2,\alpha} \subset \mathcal{P}^{k+2,\alpha}$ of u, and a C^1 map $\sigma_{k,\alpha}\colon \mathcal{U}_g^{k+2,\alpha} \to \mathcal{W}_u^{k+2,\alpha}$ such that $\sigma_{k,\alpha}(g) = u$ and the zero set of $\mathcal{Y}$ in $\mathcal{U}_g^{k+2,\alpha} \times \mathcal{W}_u^{k+2,\alpha}$ is precisely the graph of $\sigma_{k,\alpha}$. In particular, $\mathcal{Y}\big(g', \sigma_{k,\alpha}(g')\big) = 0$ for $g' \in \mathcal{U}_g^{k+2,\alpha}$.

Now let $g' \in \mathcal{U}_g^{k+2,\alpha} \cap X(M)$, and set $u_{g'} := \sigma_{k,\alpha}(g')$. Then $u_{g'} \in C^{k+2,\alpha}(M)$ satisfies $L_{g'}(u_{g'}) = n(n-1)u_{g'}^q$. Rewriting this equation as

$$-a_n \Delta_{g'} u_{g'} = n(n-1)u_{g'}^q - \mathrm{Sc}_{g'} u_{g'}$$

and using that g' is smooth and $u_{g'} > 0$, we see that the right-hand side lies in $C^{k+2,\alpha}(M)$. Hence the global Schauder estimate [LP87, Theorem 2.5(b)], applied with k replaced by $k+2$, gives $u_{g'} \in C^{k+4,\alpha}(M)$. Iterating this estimate gives $u_{g'} \in C^\infty(M)$.

Define $\widehat{h}(g') := u_{g'}^{\frac{4}{n-2}} g'$. Then $\widehat{h}(g')$ is smooth and has scalar curvature $n(n-1)$. By the existence and uniqueness results established earlier, $r(g') \in [g'] \cap \mathrm{Met}^{CC}(M)$ is the unique round representative of $[g']$. In particular, $r(g')$ is Einstein and has scalar curvature $n(n-1)$. Since $\widehat{h}(g') \in [g'] = [r(g')]$ also has constant scalar curvature, Obata's theorem implies that $\widehat{h}(g') = c\, r(g')$ for some $c > 0$. Since both metrics have scalar curvature $n(n-1)$, we have $c = 1$. Hence $\widehat{h}(g') = r(g')$ for $g' \in \mathcal{U}_g^{k+2,\alpha} \cap X(M)$. Since both conformal factors are positive, this identity also gives $u_{g'} = v_{g'}$ on $\mathcal{U}_g^{k+2,\alpha} \cap X(M)$.

We now prove continuity in the Fréchet C^∞-topology. Let $g_i \in X(M)$ and suppose that $g_i \to g$ in $C^\infty(\mathrm{Sym}^2(T^*M))$. Fix $m \geq 0$ and $\alpha \in (0,1)$. For all sufficiently large i, one has $g_i \in \mathcal{U}_g^{m+2,\alpha}$. The identity just proved shows that the implicit-function branch agrees on $\mathcal{U}_g^{m+2,\alpha} \cap X(M)$ with the canonical conformal factor. Hence $v_{g_i} = \sigma_{m,\alpha}(g_i) \to \sigma_{m,\alpha}(g) = v_g$ in $C^{m+2,\alpha}(M)$. Since m is arbitrary, we have $v_{g_i} \to v_g$ in $C^\infty(M)$. Consequently, $r(g_i) = v_{g_i}^{\frac{4}{n-2}} g_i \to v_g^{\frac{4}{n-2}} g = r(g)$ in $C^\infty(\mathrm{Sym}^2(T^*M))$. Thus both $g \mapsto v_g$ and r are sequentially continuous. Since the relevant C^∞ topologies are metrizable, both maps are continuous. In particular, $r\colon X(M) \to \mathrm{Met}^{CC}(M)$ is continuous.

For $g \in X(M)$, set

$$H(g,t) := \big((1-t) + t\, v_g\big)^{\frac{4}{n-2}} g, \qquad t \in [0,1].$$

Since affine interpolation, pointwise positive powers, and tensor multiplication are continuous operations in the C^∞ Fréchet topology, the map $(g,t) \mapsto H(g,t)$ is jointly continuous. Since $L_g(1) = \mathrm{Sc}_g > 0$ and $L_g(v_g) = v_g^{\frac{n+2}{n-2}} \mathrm{Sc}_{r(g)} > 0$, linearity of L_g gives $L_g\big((1-t) + t\, v_g\big) > 0$. Hence $\mathrm{Sc}_{H(g,t)} > 0$. Moreover, $H(g,t)$ is conformal to g, so it is locally conformally flat. Therefore $H(g,t) \in X(M)$ for every t. By construction, $H(g,0) = g$ and $H(g,1) = r(g) \in \mathrm{Met}^{CC}(M)$. Moreover, if $g \in \mathrm{Met}^{CC}(M)$, the uniqueness of the round representative yields $r(g) = g$, whence the conformal factor v_g is identically 1. It follows that $H(g,t) = g$, $t \in [0,1]$). Because H continuously deforms $X(M)$ into $\mathrm{Met}^{CC}(M)$ while keeping $\mathrm{Met}^{CC}(M)$ pointwise fixed at all times, H is a strong deformation retraction of $X(M)$ onto $\mathrm{Met}^{CC}(M)$.

Since $r(\psi \cdot g) = \psi \cdot r(g)$, it follows that $v_{\psi \cdot g} = v_g \circ \psi^{-1}$, and hence $H(\psi \cdot g, t) = \psi \cdot H(g,t)$ for $\psi \in \mathrm{Diff}(M)$. Let $q_M\colon X(M) \to \mathcal{M}(M)$ be the quotient map. Since

$\mathrm{Diff}(M)$ acts on $X(M)$ by homeomorphisms, q_M is open: for every open $U \subset X(M)$, $q_M^{-1}(q_M(U)) = \bigcup_{\psi \in \mathrm{Diff}(M)} \psi \cdot U$ is open. Hence $q_M \times \mathrm{id}_{[0,1]}$ is an open continuous surjection, and therefore a quotient map. Consequently, H descends to a well-defined continuous homotopy $\overline{H} \colon \mathcal{M}(M) \times [0,1] \to \mathcal{M}(M)$ from the identity map to the map induced by r.

It remains to observe that the time-one map $\overline{H}(\cdot, 1)$ is constant. Any two metrics $h_1, h_2 \in \mathrm{Met}^{CC}(M)$ are round metrics on the same smooth manifold M, which has already been shown to be diffeomorphic to a spherical space form. Since n is odd, de Rham's rigidity theorem [dR50, pp. 66–67] implies that diffeomorphic spherical space forms are isometric. Hence $\mathrm{Met}^{CC}(M)$ is a single $\mathrm{Diff}(M)$-orbit, and $\mathrm{Met}^{CC}(M)/\mathrm{Diff}(M)$ is a singleton. The homotopy $\overline{H}$ therefore contracts $\mathcal{M}(M)$ to a point. □

Remark 4.5. The difficulty in extending Proposition 4.4 to the case $M = S^n$ is not the existence of a path to a round metric within each conformal class: this already follows from [Mar12, Corollary 3.2]. Rather, the Obata normalization used above does not provide a canonical choice of a round representative.

Indeed, when the deck group of a round representative is nontrivial, the proof above uses the non-spherical case of Obata's rigidity theorem together with the strict Lichnerowicz–Obata spectral gap to single out a unique normalized round metric in each conformal class. This makes the retraction map $r \colon X(M) \to \mathrm{Met}^{CC}(M)$ canonical and allows one to prove its continuity. In the sphere case $M = S^n$, this mechanism breaks down: the round conformal class is exceptional, since it contains the entire $\mathrm{Conf}(S^n, g_{st})$-orbit of g_{st}. Thus, even after fixing the normalization $\mathrm{Sc}_h = n(n-1)$, there is no unique round representative in a given conformal class.

Consequently, the above argument does not directly extend to S^n.

Proof of Theorem D. Part (i) follows from Propositions 4.1 and 4.2.

For part (ii), recall that $X(M)$ is the space of LCF metrics with positive scalar curvature on M. If $X(M) = \varnothing$, then $\mathcal{M}(M) = \varnothing$. If $X(M) \neq \varnothing$, Proposition 4.4 gives a $\mathrm{Diff}(M)$-equivariant strong deformation retraction of $X(M)$ onto $\mathrm{Met}^{CC}(M)$, which is a single $\mathrm{Diff}(M)$-orbit. The induced homotopy therefore contracts $\mathcal{M}(M)$. Hence $\mathcal{M}(M)$ is either empty or contractible. □

5 Euclidean Rigidity for LCF Manifolds

This section establishes Euclidean rigidity for complete, open, simply connected LCF manifolds with nonnegative scalar curvature. In dimension three, any such contractible manifold is conformally diffeomorphic to Euclidean space: the Ma–Qing dimension bound and Čech–Alexander duality force the limit set to be a single point. In higher dimensions, $\lfloor (n-2)/2 \rfloor$-connectivity at infinity together with the small loops condition on the limit set implies, through shape-theoretic arguments, that the manifold is homeomorphic to $\mathbb{R}^n$. We also obtain conformal Euclidean rigidity under several analytic hypotheses and a smooth rigidity theorem in dimension four under bounded geometry.

5.1 Contractible LCF 3-manifolds

Since Schoen and Yau proved that every complete, noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$ [SY82, Theorem 3, p. 217], it is natural to ask which complete noncompact Riemannian 3-manifolds with PSC are diffeomorphic to $\mathbb{R}^3$. Consider the smooth metric

$$g_0 := g_E + \Big(\sum_{i=1}^{3} x_i \, dx_i\Big)^2$$

on $\mathbb{R}^3$. In spherical coordinates,

$$g_0 = (1 + r^2)\, dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$

Since $g_0 \geq g_E$, every d_{g_0}-Cauchy sequence is d_{g_E}-Cauchy and hence converges in the Euclidean topology. The local uniform equivalence of the two smooth metrics then gives convergence in d_{g_0}. Thus g_0 is complete. A warped-product computation after the change of variables $dt = \sqrt{1 + r^2}\, dr$ gives $\mathrm{Sc}_{g_0} = 2(r^2 + 3)/(r^2 + 1)^2 > 0$. In addition, $(S^2 \times \mathbb{R}, g_{st} \oplus dt^2)$ also has PSC. The round cylinder is LCF, since, with $r = e^\tau$,

$$g_E = dr^2 + r^2 g_{st} = r^2\big(d\tau^2 + g_{st}\big).$$

The metric g_0 is also LCF. Indeed, with t as above, on $r > 0$,

$$g_0 = dt^2 + r(t)^2 g_{st} = r(t)^2\big(ds^2 + g_{st}\big), \qquad ds = \frac{dt}{r(t)}.$$

Thus g_0 is conformal to the round cylinder away from the origin. Since local conformal flatness is conformally invariant, g_0 is LCF on $\mathbb{R}^3 \setminus \{0\}$. In dimension three, local conformal flatness is equivalent to the vanishing of the Cotton tensor. Hence the Cotton tensor of g_0 vanishes on $\mathbb{R}^3 \setminus \{0\}$, and, by continuity, also at the origin. Therefore g_0 is LCF on all of $\mathbb{R}^3$.

These two examples show that additional topological or geometric conditions are required to settle the question completely. Gromov and Lawson [GL83, Corollary 10.9] proved that every complete noncompact 3-manifold with uniformly positive scalar curvature and finitely generated fundamental group is simply connected at infinity. Hence, if such a manifold is contractible, it is homeomorphic to $\mathbb{R}^3$. Chang, Weinberger, and Yu later gave an index-theoretic proof [CWY10, Theorem 1].

This leads to the following conjecture, which has circulated in the field for decades: every complete, connected, open, contractible 3-manifold with nonnegative scalar curvature is diffeomorphic to $\mathbb{R}^3$. The conjecture has been verified using inverse mean curvature flow under the assumption of bounded geometry (i.e., $|\mathrm{Rm}| \leq C$ and the injectivity radius is bounded below by C^{-1}); see [Car26, p. 2]. Here we replace the condition of bounded geometry with that of local conformal flatness and establish the following result.

Theorem 5.1. *Let M^3 be a connected, open, contractible 3-manifold admitting a complete, locally conformally flat metric g with nonnegative scalar curvature. Then (M^3, g) is conformally diffeomorphic to $(\mathbb{R}^3, g_E)$.*

The proof is as follows. By Theorem 2.19, M embeds conformally as a contractible domain $\Omega \subset S^3$, and $\partial\Omega$ has zero Newtonian 2-capacity. Čech–Alexander duality first shows that Ω is dense in S^3, so that $S^3 \setminus \Omega = \partial\Omega$, and a second application shows that $\partial\Omega$ is connected. The Ma–Qing estimate gives $\dim_{\mathcal{H}}(\partial\Omega) \leq \frac{1}{2}$. Since every nonsingleton connected metric space has Hausdorff dimension at least 1, the boundary consists of a single point. Hence $\Omega \cong \mathbb{R}^3$, and the developing map followed by stereographic projection is the required conformal diffeomorphism.

Improving Schoen–Yau's estimate [SY88, Theorem 2.7], Ma and Qing proved the following special case of their theorem [MQ25, Theorem 1.3].

Theorem 5.2 (Ma–Qing)**.** *Let $(M^n, \bar{g})$, $n \geq 3$, be a complete Riemannian manifold, and let $S \subset M$ be compact. Let $D \subset M$ be a bounded open neighborhood of S. Suppose that $g = u^{\frac{4}{n-2}}\bar{g}$ is a conformal metric on $D \setminus S$ that is geodesically complete near S. If the scalar curvature of g is nonnegative, then*

$$\dim_{\mathcal{H}}^{\bar{g}}(S) \leq \frac{n-2}{2}.$$

Here the superscript in $\dim_{\mathcal{H}}^{\bar{g}}$ records the background metric; when the background is g_{st}, we suppress it.

In the applications below, we verify the following sufficient condition for geodesic completeness near S: every locally rectifiable curve in $D \setminus S$ that converges in the background topology to a point of S has infinite length with respect to g.

We shall also use the following capacity–dimension implication. Let $E \subsetneq S^n$ be compact with zero Newtonian 2-capacity, and choose a stereographic projection $\sigma\colon S^n \setminus \{p\} \to \mathbb{R}^n$ with $p \notin E$. On a relatively compact neighborhood of E in this chart, the round and Euclidean metrics and volume forms are uniformly comparable. Consequently, the corresponding local Sobolev 2-capacities have the same null sets, so $\sigma(E)$ has zero Euclidean Newtonian 2-capacity. The latter agrees, up to normalization, with the Riesz $(1,2)$-capacity. Since $\sigma(E)$ is bounded, [AH95, Proposition 5.1.4(b)] gives $C_{1,2}(\sigma(E)) = 0$. For every $0 < \varepsilon < 2$, [AH95, Theorem 5.1.13], with $(\alpha, p) = (1, 2)$ and $h(r) = r^{n-2+\varepsilon}$, applies because

$$\int_0^1 \frac{h(r)}{r^{n-2}}\,\frac{dr}{r} = \int_0^1 r^{\varepsilon-1}\,dr < \infty.$$

It gives $\Lambda_h(\sigma(E)) = 0$, where Λ_h denotes the associated ball-gauge Hausdorff measure. For this power gauge, vanishing of Λ_h is equivalent to $\mathcal{H}^{n-2+\varepsilon}(\sigma(E)) = 0$. It follows that $\dim_{\mathcal{H}} \sigma(E) \leq n-2$. Finally, σ is bi-Lipschitz on a neighborhood of E, and hence $\dim_{\mathcal{H}} E \leq n-2$.

Proof of Theorem 5.1. Since (M^3, g) is a simply connected, complete, LCF manifold with nonnegative scalar curvature, its developing map is defined on M itself, and $\Phi\colon M \to S^3$ is conformal. By Theorem 2.19, Φ is injective and the boundary $\partial\Phi(M)$ has zero Newtonian 2-capacity. Because Φ is an injective local diffeomorphism, it is a conformal diffeomorphism from M onto the open domain $\Omega := \Phi(M) \subset S^3$. In particular, Ω is contractible.

We now show that $\partial\Omega = S^3 \setminus \Omega$. Set $K := S^3 \setminus \Omega$. Since M is noncompact and Φ is a diffeomorphism onto Ω, we have $\Omega \neq S^3$, and hence $K \neq \varnothing$. Since

$\partial\Omega$ has zero Newtonian 2-capacity, the capacity–dimension implication above gives $\dim_{\mathcal{H}}(\partial\Omega) \le 1$. By the Szpilrajn inequality [HW41, Chapter VII], the covering dimension of $\partial\Omega$ is at most 1, so $\check{H}^2(\partial\Omega;\mathbb{Z}) = 0$. By Čech–Alexander duality, $\widetilde{H}_0(S^3 \setminus \partial\Omega;\mathbb{Z}) \cong \widetilde{\check{H}}^2(\partial\Omega;\mathbb{Z}) = 0$. Therefore $S^3 \setminus \partial\Omega$ is connected. On the other hand, $S^3 \setminus \partial\Omega = \Omega \sqcup \operatorname{int}(K)$, a disjoint union of open sets. Since $\Omega \neq \varnothing$, connectedness forces $\operatorname{int}(K) = \varnothing$. Thus Ω is dense in S^3, and hence $\partial\Omega = \overline{\Omega} \setminus \Omega = S^3 \setminus \Omega = K$.

Since $\partial\Omega = S^3\setminus\Omega$ and Ω is contractible, Čech–Alexander duality yields $\widetilde{\check{H}}^0(\partial\Omega;\mathbb{Z}) \cong \widetilde{H}_2(\Omega;\mathbb{Z}) = 0$. Because $\partial\Omega$ is nonempty and compact, this implies that $\partial\Omega$ is connected.

Set $S := S^3\setminus\Omega = \partial\Omega$. Since S is closed in the compact manifold S^3, it is compact. Choose $x_0 \in \Omega$ and $\varepsilon > 0$ such that $\overline{B_{g_{st}}(x_0,\varepsilon)} \subset \Omega$, and set $D := S^3 \setminus \overline{B_{g_{st}}(x_0,\varepsilon)}$. Then D is a bounded open neighborhood of S in S^3.

Let $\tilde{g} := (\Phi^{-1})^* g$ be the metric transported from M to Ω. Since $\Phi\colon M \to \Omega$ is a conformal diffeomorphism, the restriction of $\tilde{g}$ to $D \setminus S$ has the form $\tilde{g} = u^4 g_{st}$ for some smooth positive function u, and $\mathrm{Sc}_{\tilde{g}} = \mathrm{Sc}_g \circ \Phi^{-1} \ge 0$.

It remains to verify that $\tilde{g}$ is geodesically complete near S. Since $\Phi\colon (M,g) \to (\Omega,\tilde{g})$ is an isometry and (M,g) is complete, the metric space $(\Omega, d_{\tilde{g}})$ is complete.

Suppose, for contradiction, that there exists a locally rectifiable curve $\gamma\colon [0,1) \to D \setminus S \subset \Omega$ such that $\gamma(t) \to p \in S$ as $t \uparrow 1$ and $L_{\tilde{g}}(\gamma) < \infty$. Then for $0 \le s < t < 1$, $d_{\tilde{g}}(\gamma(s),\gamma(t)) \le L_{\tilde{g}}\big(\gamma|_{[s,t]}\big)$. Since the total length of γ is finite, the right-hand side tends to 0 as $s,t \uparrow 1$. Thus $\gamma(t)$ is a Cauchy curve in $(\Omega, d_{\tilde{g}})$. By completeness, there exists $q \in \Omega$ such that $\gamma(t) \to q$ with respect to $d_{\tilde{g}}$.

But the distance topology of a smooth Riemannian metric agrees with the manifold topology. Thus $\gamma(t) \to q$ in Ω, and continuity of the inclusion $\Omega \hookrightarrow S^3$ gives $\gamma(t) \to q$ in S^3. Since S^3 is Hausdorff and $\gamma(t) \to p \in S = S^3 \setminus \Omega$ in the background topology, uniqueness of limits would give $p = q$, a contradiction. Therefore every locally rectifiable curve in $D \setminus S$ converging in the background topology to a point of S has infinite $\tilde{g}$-length. Hence $\tilde{g}$ is geodesically complete near S.

We may therefore apply Theorem 5.2 with background manifold (S^3, g_{st}), singular set S, and the bounded open neighborhood D. It follows that $\dim_{\mathcal{H}}(S) \le \frac{3-2}{2} = \frac{1}{2}$. Since $S = \partial\Omega$, it follows that $\dim_{\mathcal{H}}(\partial\Omega) \le \frac{1}{2}$.

Hence the topological dimension of $\partial\Omega$ is zero; in particular, $\partial\Omega$ is totally disconnected. On the other hand, $\partial\Omega$ is connected, which forces $\partial\Omega$ to consist of a single point. Consequently, Ω is diffeomorphic to $\mathbb{R}^3$. Because stereographic projection provides a conformal diffeomorphism from the punctured sphere to Euclidean space, we conclude that (M,g) is conformally diffeomorphic to $(\mathbb{R}^3, g_E)$. □

Remark 5.3. The nonnegativity assumption on the scalar curvature in Theorem 5.1 cannot be removed. Indeed, let $\Omega \subset S^3$ be a round ball. By the Loewner–Nirenberg theorem [LN74], Ω admits a complete metric g, conformal to g_{st}, with $\mathrm{Sc}_g = -1$. Although Ω is contractible, it is not conformally diffeomorphic to $S^3 \setminus \{q\} \cong \mathbb{R}^3$. Otherwise, by Liouville theorem 2.19, such a conformal diffeomorphism would extend to a Möbius transformation of S^3, which would map the round sphere $\partial\Omega$ to the single point q, a contradiction. More generally, let $F^d \subset S^n$ be a closed smooth submanifold. Then there exists a smooth positive function u on $S^n \setminus F$ such that $g = u^{\frac{4}{n-2}} g_{st}$ is complete and satisfies $\mathrm{Sc}_g = -1$ if and only if $d > \frac{n-2}{2}$; see [AM88].

This argument does not extend directly to higher dimensions $n \geq 4$, since Theorem 5.2 only yields the bound $\dim_{\mathcal{H}}(\partial\Omega) \leq \frac{n-2}{2}$, which is no longer sufficient to force $\partial\Omega$ to be totally disconnected. However, under additional assumptions, the same strategy remains valid in higher dimensions, with Theorem 5.2 replaced by results implying that $\dim_{\mathcal{H}}(\partial\Omega) < 1$.

Corollary 5.4. *Let M^n $(n \geq 4)$ be a connected, open, simply connected n-manifold such that $\widetilde{H}_{n-1}(M;\mathbb{Z}) = 0$. Assume that M admits a complete locally conformally flat metric g with $\mathrm{Sc}_g \geq 0$. Assume in addition that at least one of the following holds:*

(i) $\mathrm{Ric}_g \geq 0$ *outside a compact subset of M;*

(ii)

$$\mathrm{Ric}_g^- \in L^1(M,g) \cap L^\infty(M,g), \qquad \mathrm{Sc}_g \in L^\infty(M,g), \qquad |\nabla_g \mathrm{Sc}_g| \in L^\infty(M,g),$$

where

$$\mathrm{Ric}_g^-(x) := \max\big\{0, -\lambda_{\min}(g^{-1}\mathrm{Ric}_g)(x)\big\};$$

(iii) The Ricci tensor satisfies the critical integrability condition $|\mathrm{Ric}_g| \in L^{n/2}(M,g)$*;*

(iv) There exists a point $p \in M$ such that

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g\big(B_p(r)\big)}{r^n(\log r)^{n-1}} = 0;$$

(v) There exists $p_0 \in (n-2, n)$ such that

$$A_g^{(p_0)} := \frac{1}{n-2}\left((p_0-2)\,\mathrm{Ric}_g \;+\; \frac{n-p_0}{2(n-1)}\,\mathrm{Sc}_g\, g\right) \geq 0.$$

Then (M,g) is conformally diffeomorphic to $(\mathbb{R}^n, g_E)$.

Proof. As in the proof of Theorem 5.1, since (M^n,g) is simply connected and LCF with $\mathrm{Sc}_g \geq 0$, Theorem 2.19 yields an injective conformal map $\Phi\colon M \to S^n$. Moreover, the boundary $\partial\Phi(M)$ has vanishing Newtonian 2-capacity. Thus Φ is a conformal diffeomorphism from M onto an open domain $\Omega := \Phi(M) \subset S^n$. Since $\partial\Omega$ has zero Newtonian 2-capacity, the capacity–dimension implication above gives $\dim_{\mathcal{H}}(\partial\Omega) \leq n-2$. In particular, the covering dimension of $\partial\Omega$ is at most $n-2$, so $\widetilde{\check{H}}^{n-1}(\partial\Omega;\mathbb{Z}) = 0$.

By Čech–Alexander duality, $\widetilde{H}_0(S^n \setminus \partial\Omega;\mathbb{Z}) \cong \widetilde{\check{H}}^{n-1}(\partial\Omega;\mathbb{Z}) = 0$. Therefore $S^n \setminus \partial\Omega$ is connected. Set $K := S^n \setminus \Omega$. Since M is noncompact whereas S^n is compact, we have $K \neq \varnothing$. Thus, $S^n \setminus \partial\Omega = \Omega \sqcup \mathrm{int}(K)$, a disjoint union of open sets. Since $\Omega \neq \varnothing$, connectedness forces $\mathrm{int}(K) = \varnothing$. Thus Ω is dense in S^n, and hence $\partial\Omega = \overline{\Omega} \setminus \Omega = S^n \setminus \Omega = K$.

Now $\Omega \cong M$, so our topological hypothesis gives $\widetilde{H}_{n-1}(\Omega;\mathbb{Z}) = 0$. By Čech–Alexander duality, $\widetilde{\check{H}}^0(\partial\Omega;\mathbb{Z}) \cong \widetilde{H}_{n-1}(\Omega;\mathbb{Z}) = 0$. Hence $\partial\Omega$ is connected.

Next define the pushforward metric $\widetilde{g} := (\Phi^{-1})^*g$ on Ω. Then $\widetilde{g}$ is conformal to the standard round metric on S^n, and $\Phi\colon (M,g) \to (\Omega,\widetilde{g})$ is an isometry. Therefore

$(\Omega, \widetilde{g})$ is complete. All five additional hypotheses are preserved by this isometry; under condition (iv), the distinguished point becomes $\widetilde{p} := \Phi(p)$.

Hence, if any one of conditions (i), (ii), or (iii) holds, the finite-point conformal compactification theorem applies: for conditions (i) and (ii), this is due to Ma–Qing [MQ22, Corollary 1.1], while for condition (iii), it follows from Chen–Li [CL22, Theorem 1.4]. In all three cases, one concludes that $\partial\Omega = S^n \setminus \Omega$ is a finite set. Since $\partial\Omega = S^n \setminus \Omega$ is nonempty and connected, it follows that $\partial\Omega$ consists of a single point, say $\partial\Omega = \{q\}$.

If condition (iv) holds, then there exists a point $\widetilde{p} \in \Omega$ such that

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_{\widetilde{g}}\big(B_{\widetilde{p}}(r)\big)}{r^n(\log r)^{n-1}} = 0.$$

Carron and Herzlich [CH02, Proposition 2.2] then imply that $\dim_{\mathcal{H}}(\partial\Omega) = 0$.

If condition (v) is satisfied, we apply Liu–Ma–Qing–Zhong's theorem [LMQZ25, Theorem 5.1.1]. We verify its hypotheses as follows. The set $\partial\Omega$ is closed in S^n, and the metric $\widetilde{g}$ on $S^n \setminus \partial\Omega = \Omega$ is conformal to g_{st}. Moreover, $\widetilde{g}$ is complete, hence geodesically complete near $\partial\Omega$ as in the proof of Theorem 5.1. Condition (v) gives $p_0 \in (n-2, n)$ such that $A_{\widetilde{g}}^{(p_0)} \geq 0$. Since $n \geq 4$, we have $(n-2, n) \subset [2, n)$. Therefore, Liu–Ma–Qing–Zhong's theorem yields $\dim_{\mathcal{H}}(\partial\Omega) \leq (n - p_0)/2 < 1$.

Thus conditions (iv) and (v) each imply that $\partial\Omega$ consists of a single point, since a nonempty connected metric space of Hausdorff dimension < 1 is a singleton. Consequently, under any one of the five additional conditions, there exists $q \in S^n$ such that $\partial\Omega = \{q\}$, and hence $\Omega = S^n \setminus \{q\}$. By stereographic projection from q, Ω is conformally diffeomorphic to $(\mathbb{R}^n, g_E)$. Composing with Φ, we conclude that (M, g) is conformally diffeomorphic to $(\mathbb{R}^n, g_E)$. □

5.2 LCF 4-manifolds

In dimension four, the author proved in [Den25, Theorem 5.3] that if (M^4, g) is closed, LCF, and scalar-flat, with $\pi_2(M) \neq 0$, then its Riemannian universal cover $(\widetilde{M}, \widetilde{g})$ is, up to homothety, isometric to $\big(\mathbb{H}^2 \times S^2, g_{\mathbb{H}} \oplus g_{st}\big)$, where $g_{\mathbb{H}}$ denotes the hyperbolic metric of curvature -1. In the present subsection, we use bounded geometry to prove a Euclidean rigidity statement for complete LCF 4-manifolds. The argument uses Huang's Ricci-flow classification theorem and his analysis of infinite connected sums [Hua17, Theorem 1.1 and §§ 2–3].

Theorem 5.5 (Euclidean rigidity for LCF 4-manifolds)**.** *Let (M^4, g) be a complete, smooth, connected, open, locally conformally flat Riemannian 4-manifold. Assume that (M, g) has bounded geometry, in the sense that $|\mathrm{Rm}_g| \leq C$ and $\mathrm{inj}_g(x) \geq C^{-1}$ for all $x \in M$ and some constant $C \geq 1$, and has uniformly positive scalar curvature, in the sense that $\mathrm{Sc}_g \geq \sigma > 0$ for some constant σ. If $\pi_1(M) = 0$ and $H_3(M; \mathbb{R}) = 0$, then M is diffeomorphic to the standard $\mathbb{R}^4$.*

Proof. Since $\pi_1(M) = 0$, the manifold M is orientable. The curvature bound gives, after changing constants, a two-sided bound $|\mathrm{sec}_g| \leq C_0$, while $\mathrm{inj}_g \geq C^{-1} > 0$. Thus (M, g) has bounded geometry in the sense required by Huang's theorem.

Since (M^4, g) is LCF, its Weyl tensor vanishes. In an orthonormal frame, with R_{ijij} denoting the sectional curvature of the plane spanned by e_i and e_j, the curvature tensor satisfies

$$R_{ijkl} = \frac{1}{2}\big(\mathrm{Ric}_{ik}\delta_{jl} + \mathrm{Ric}_{jl}\delta_{ik} - \mathrm{Ric}_{il}\delta_{jk} - \mathrm{Ric}_{jk}\delta_{il}\big) - \frac{\mathrm{Sc}_g}{6}\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big).$$

In particular, $R_{1234} = 0$. Consequently, for every point $x \in M$ and every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\} \subset T_xM$,

$$\begin{aligned} &R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\quad = \sum_{i=1}^{4} \mathrm{Ric}_{ii} - \frac{2}{3}\mathrm{Sc}_g = \mathrm{Sc}_g - \frac{2}{3}\mathrm{Sc}_g = \frac{1}{3}\mathrm{Sc}_g \geq \frac{\sigma}{3} > 0. \end{aligned}$$

Hence (M, g) has uniformly positive isotropic curvature.

Moreover, M contains no essential incompressible space form. Indeed, every such space form is, in particular, an embedded $N^3 \cong S^3/\Gamma \subset M$ with Γ finite and nontrivial whose fundamental group injects into that of the ambient manifold, so that $\Gamma \cong \pi_1(N) \hookrightarrow \pi_1(M) = 0$, which is impossible.

Huang's classification theorem [Hua17, Theorem 1.1 and § 1] applies. Thus M is diffeomorphic to an infinite connected sum of S^4, $\mathbb{RP}^4$, $S^3 \times S^1$, and $S^3 \widetilde{\times} S^1$ along a countably infinite, connected, locally finite graph G, with loops and multiple edges allowed. Here the four vertex manifolds carry their standard smooth structures. Huang states in the proof of his Theorem 1.2 that the triviality of $\pi_1(M)$ forces G to be a tree and every vertex piece to be the standard S^4 [Hua17, § 3]. We give the details. Working in the equivalent boundary-gluing model obtained from Huang's quotient model by inserting collars, for each vertex v, let Y_v be the corresponding vertex manifold X_v with the interior of one 4-ball removed for each incident half-edge (so a loop contributes two balls). Since local finiteness makes this a finite family and the boundary spheres are simply connected, Seifert–van Kampen gives $\pi_1(Y_v) \cong \pi_1(X_v)$. Choose a nested exhaustion of G by finite connected subgraphs. Local finiteness ensures that the corresponding partial sums are compact and that their interiors exhaust M. Applying Seifert–van Kampen to these partial sums and using that fundamental groups commute with this directed union gives

$$\pi_1(M) \cong \Big(*_{v\in V(G)}\pi_1(Y_v)\Big) * \pi_1(G) \cong \Big(*_{v\in V(G)}\pi_1(X_v)\Big) * \pi_1(G).$$

Since a free product is trivial only when every factor is trivial, every vertex group is trivial, and $\pi_1(G)$ is trivial. Among Huang's four pieces, only S^4 has trivial fundamental group. Thus every vertex piece is the standard S^4, and G is a tree.

In the proof of Huang's Theorem 1.2, the one-endedness of G follows from the assumption that the manifold is homeomorphic to $\mathbb{R}^4$. That assumption is unavailable here; instead, we use $H_3(M; \mathbb{R}) = 0$.

Since G is an infinite locally finite tree, König's lemma implies that G contains a ray and hence has at least one end. We claim that G has exactly one end. Suppose otherwise. Then G contains a bi-infinite geodesic. Let e be an edge on this geodesic. Then $G \setminus e$ has two infinite components. In Huang's quotient model for the infinite connected sum, the edge e is represented by a smooth neck; its central sphere $S_e \cong S^3$

separates M into two connected noncompact open sets M_- and M_+. Choose a smooth function $\chi\colon M \to [0,1]$ which is identically 0 on M_- outside this neck, identically 1 on M_+ outside the same neck, and depends only on the neck coordinate inside the neck. Although χ is not compactly supported, its differential is supported in a compact subcylinder of the neck. Hence $\alpha = d\chi$ is a closed compactly supported 1-form. Choose rays in the two infinite components of $G \setminus e$, beginning at the endpoints of e, and concatenate the corresponding rays in M_- and M_+ across the neck. The local finiteness of G and the compactness of the vertex blocks imply that the resulting block decomposition of M is locally finite. Consequently, the inverse image of a compact subset of M meets only a bounded subarc of the concatenated double ray. After smoothing inside the necks, we therefore obtain a proper smooth curve $\gamma\colon \mathbb{R} \to M$ crossing the neck once and joining ends represented by rays in M_- and M_+, respectively. Therefore $\int_\gamma \alpha = 1$. Thus $[\alpha] \neq 0$ in $H^1_c(M;\mathbb{R})$: if $\alpha = d\varphi$ with $\varphi \in C^\infty_c(M)$, then $\varphi(\gamma(t)) \to 0$ as $t \to \pm\infty$, and hence $\int_\gamma d\varphi = 0$, a contradiction. By the compactly supported de Rham theorem and Poincaré duality for the oriented manifold M, $0 \neq [\alpha] \in H^1_c(M;\mathbb{R}) \cong H_3(M;\mathbb{R})$, contradicting $H_3(M;\mathbb{R}) = 0$. Hence G is a one-ended infinite tree.

It remains to identify the resulting infinite connected sum. We have shown that G is a locally finite one-ended tree and that every vertex piece is the standard S^4. Choose a proper ray in G. Every branch off this ray is finite; otherwise, König's lemma would produce a second end of G. The final reduction in Huang's proof of Theorem 1.2 then applies: after absorbing these finite branches, the connected sum becomes an infinite connected sum of standard S^4's along the ray $[0,\infty)$ [Hua17, § 3]. The standard $\mathbb{R}^4$ is such a ray sum, obtained from the decomposition of $\mathbb{R}^4$ into the unit 4-ball and the closed annuli between successive concentric 3-spheres. Huang's well-definedness theorem for infinite connected sums, together with the fact that the standard S^4 admits an orientation-reversing diffeomorphism, shows that the diffeomorphism type is independent of the gluing embeddings and orientation choices [Hua17, Theorem 2.2, Remark 1, and § 3]. Hence M is diffeomorphic to the standard $\mathbb{R}^4$. □

Remark 5.6. For $n \geq 7$, a closed Riemannian n-manifold equipped with an LCF metric of PSC does not necessarily have positive isotropic curvature. Indeed, for a Riemannian n-manifold, positive isotropic curvature is equivalent to positive $(n-4)$-curvature [Lab00, Proposition, p. 1470]. A closed LCF n-manifold with positive isotropic curvature implies $b_p = 0$ for all $2 \leq p \leq n-2$ [Lab00, Corollary I, p. 1471]. Thus, if the manifold is connected and oriented and also satisfies $H_1(M;\mathbb{Z}) = 0$, Poincaré duality implies that it is a rational homology sphere. On the other hand, let (N^3, g_{hyp}) be a closed, connected, hyperbolic integer homology 3-spheres. Then the product manifold $(N^3 \times S^m, g_{\mathrm{hyp}} \oplus g_{st})$ is LCF, and $\mathrm{Sc}_{g_{\mathrm{hyp}} \oplus g_{st}} = -6 + m(m-1)$. Thus its scalar curvature is negative when $m = 2$, zero when $m = 3$, and positive when $m \geq 4$. Furthermore, $H_1(N^3 \times S^m;\mathbb{Z}) = 0$ and $\pi_1(N^3 \times S^m)$ is infinite. However, $N^3 \times S^m$ is not a rational homology sphere. This example demonstrates that an LCF metric with PSC need not itself have positive isotropic curvature.

5.3 Applying shape theory to limit sets

In this subsection, we use topological conditions on the conformal boundary Λ to characterize Ω, thereby addressing Yau's Problem 36 [Yau93, Problem 36]. Throughout this subsection, dim denotes covering dimension.

Since the limit set Λ of a developing map may be a fractal, we use shape theory instead of classical homotopy theory to study it. An important shape invariant, introduced by Borsuk, is *movability.* A compact metrizable space X, embedded in the Hilbert cube Q, is said to be *movable* if for every neighborhood U of X there exists a neighborhood $U' \subset U$ of X such that, for any neighborhood $U'' \subset U$ of X, there exists a homotopy $H\colon U' \times I \to U$ satisfying $H(x, 0) = x$ and $H(x, 1) \in U''$ for all $x \in U'$ [Gev21, Definition 4]. This definition is independent of the embedding of X into Q.

Inspired by the turning (diameter) condition in the definition of homotopically $(1, c)$-uniform domains due to Alestalo and Väisälä [AV96], we introduce the following weaker spherical loop-filling condition.

Definition 5.7. *Let $\Omega \subset (S^n, g_{st})$ be a domain, and let $c \geq 1$. We say that Ω has the* diameter-controlled loop-filling property with constant c *if every continuous map $f\colon S^1 \to \Omega$ extends to a continuous map $F\colon B^2 \to \Omega$ such that*

$$\operatorname{diam}_{g_{st}}\big(F(B^2)\big) \leq c \operatorname{diam}_{g_{st}}\big(f(S^1)\big).$$

Recall that a *topological n-cell* is a space homeomorphic to the standard closed unit cube I^n. A compact subset X of an n-manifold N is *cellular in N* if there exist n-cells $B_1, B_2, \dots$ in N such that

$$B_{i+1} \subset \operatorname{Int}_N(B_i), \qquad X = \bigcap_{i=1}^{\infty} B_i.$$

A compact metric space X is called *cell-like* if there exist a manifold N and an embedding $\varphi\colon X \to N$ such that $\varphi(X)$ is cellular in N. For any nonempty finite-dimensional compact metric space X, Lacher [Lac69, Theorem 1.1] shows that being cell-like is equivalent to having the shape of a point, and also equivalent to the statement that every embedding of X into any absolute neighborhood retract (ANR) has *Property UV^∞*. Here a compact subset X of an ANR space Y is said to have Property UV^∞ if, for every neighborhood $U \subset Y$ of X, there exists a neighborhood $V \subset U$ of X such that the inclusion $V \hookrightarrow U$ is null-homotopic in U.

Following Venema [Ven76], we make the following definitions. Let X be a compact subset of an n-manifold N^n equipped with a compatible metric. We say that X satisfies the *cellularity criterion* if, for every neighborhood U of X, there exists a neighborhood $V \subset U$ of X such that every loop in $V \setminus X$ is null-homotopic in $U \setminus X$. We say that X satisfies the *small loops condition* (SLC) if, for every neighborhood U of X, there exist a neighborhood $V \subset U$ of X and a number $\varepsilon > 0$ such that every loop in $V \setminus X$ of diameter $< \varepsilon$ is null-homotopic in $U \setminus X$. Finally, X satisfies the *inessential loops condition* (ILC) if, for every neighborhood U of X, there exists a neighborhood $V \subset U$ of X such that every loop in $V \setminus X$ that is null-homotopic in V is also null-homotopic in $U \setminus X$.

Although SLC is stated using a compatible metric, it is independent of this choice. Indeed, let d_1 and d_2 be compatible metrics and suppose that SLC holds for d_1. Given a neighborhood U of X, choose, as in the definition, a neighborhood $V_0 \subset U$ and $\varepsilon_1 > 0$ for d_1. Since X is compact and the ambient manifold is locally compact, there is a neighborhood V of X such that $\overline{V}$ is compact and $\overline{V} \subset V_0$. Uniform continuity of the identity map $(\overline{V}, d_2) \to (\overline{V}, d_1)$ gives $\varepsilon_2 > 0$ such that every subset of $\overline{V}$ of d_2-diameter less than ε_2 has d_1-diameter less than ε_1. Hence every loop in $V \setminus X$ of d_2-diameter less than ε_2 is null-homotopic in $U \setminus X$. Thus SLC for d_1 implies SLC for d_2; interchanging the metrics proves equivalence.

Consequently, SLC is invariant under homeomorphism. Indeed, after pulling back a compatible metric by the homeomorphism, small loops and null-homotopies in the complement correspond exactly. The preceding metric independence then gives the claim for arbitrary compatible metrics.

These loop conditions are closely related. For a proper compact subset $X \subset S^n$, choose $p \in S^n \setminus X$ and use stereographic projection to regard X as a compact subset of $\mathbb{R}^n$; all three loop conditions, as well as covering dimension, are preserved. Venema's results [Ven76, p. 444] then show that, if $\dim X \leq n-2$, ILC is equivalent to the small loops condition. Moreover, if X has the shape of a point, then ILC is equivalent to the cellularity criterion.

Proposition 5.8. *Let $n = 4$ or 5, and let (M^n, g) be a connected, simply connected, complete, open, locally conformally flat manifold with nonnegative scalar curvature. Assume*

$$\widetilde{H}_{n-1}(M;\mathbb{Z}) = \widetilde{H}_{n-2}(M;\mathbb{Z}) = 0.$$

Let $\Phi\colon M \to S^n$ be the developing map, set $\Omega := \Phi(M)$, and let $\Lambda := S^n \setminus \Omega$. If Λ is movable and Ω has the diameter-controlled loop-filling property with some constant $c \geq 1$, then Λ is cellular in S^n. Consequently, M is homeomorphic to $\mathbb{R}^n$.

The strategy of the proof is to combine the geometric and homological assumptions with movability to show that Λ has the shape of a point. Using the diameter-controlled loop-filling property, we then verify the cellularity criterion. The conclusion follows from the 4- and 5-dimensional cellularity theorems.

Proof. By Theorem 2.19, the developing map is injective and therefore identifies M conformally and diffeomorphically with the domain $\Omega \subset S^n$, and $\partial\Omega$ has zero Newtonian 2-capacity. Since M is noncompact, $\Omega \neq S^n$, and $\Lambda := S^n \setminus \Omega$ is a nonempty compact subset of S^n.

We first show that $\Lambda = \partial\Omega$. The capacity–dimension implication above gives $\dim_{\mathcal{H}}(\partial\Omega) \leq n-2$. Hence the covering dimension of $\partial\Omega$ is at most $n-2$, and therefore $\check{\widetilde{H}}^{n-1}(\partial\Omega;\mathbb{Z}) = 0$. By Čech–Alexander duality, $\widetilde{H}_0(S^n \setminus \partial\Omega;\mathbb{Z}) = 0$. Thus $S^n \setminus \partial\Omega$ is connected. But $S^n \setminus \partial\Omega = \Omega \sqcup \operatorname{int}(\Lambda)$. Since $\Omega \neq \varnothing$, it follows that $\operatorname{int}(\Lambda) = \varnothing$, and hence $\Lambda = \partial\Omega$.

Choose $x_0 \in \Omega$ and $\varepsilon > 0$ such that $\overline{B_{g_{st}}(x_0,\varepsilon)} \subset \Omega$, and set $D := S^n \setminus \overline{B_{g_{st}}(x_0,\varepsilon)}$. Then D is a bounded open neighborhood of Λ.

Transfer g to Ω by setting $h := (\Phi^{-1})^* g$. Then h is complete and, on $D \setminus \Lambda$, $h = u^{\frac{4}{n-2}} g_{st}$ for some smooth positive function u. Moreover, $\mathrm{Sc}_h = \mathrm{Sc}_g \circ \Phi^{-1} \geq 0$.

As in the proof of Theorem 5.1, completeness implies that h is geodesically complete near Λ. Applying Theorem 5.2 with background manifold (S^n, g_{st}), singular set Λ, and the bounded open neighborhood D, we obtain $\dim_{\mathcal{H}}(\Lambda) \le \frac{n-2}{2}$. Since covering dimension does not exceed Hausdorff dimension and is integer-valued, $n = 4$ or 5 gives $\dim \Lambda \le 1$.

For $j = 0, 1$, Čech–Alexander duality and $\Omega \cong M$ give

$$\widetilde{\check{H}}^j(\Lambda; \mathbb{Z}) \cong \widetilde{H}_{n-j-1}(\Omega; \mathbb{Z}) \cong \widetilde{H}_{n-j-1}(M; \mathbb{Z}) = 0.$$

In particular, Λ is connected. First consider the case $\dim \Lambda = 0$. A connected compact 0-dimensional metric space is a point. In particular, Λ is cellular in S^n, and $\Omega = S^n \setminus \Lambda \cong S^n \setminus \{\mathrm{pt}\} \cong \mathbb{R}^n$.

We may therefore assume that $\dim \Lambda = 1$. Since $\widetilde{\check{H}}^1(\Lambda; \mathbb{Z}) = 0$, Lemma 5.9 gives $\check{H}_1(\Lambda; \mathbb{Z}) = 0$.

Since Λ is a connected one-dimensional compact metric space, it is a curve. In the terminology used by Borsuk–Dydak, the first Betti number is the rank of the first integral homology group in the sense of Vietoris or Čech [BD80, p. 163]. Thus Lemma 5.9 shows that the first Betti number of Λ is zero. The consequence of Trybulec's theorem recorded in [BD80, Theorem 8.2 and the ensuing paragraph, p. 180] states that the shape of a movable curve is determined by this number. The interval $[0, 1]$ is a movable curve with first Betti number 0, and it has the shape of a point. Therefore $\mathrm{Sh}(\Lambda) = \mathrm{Sh}([0, 1]) = \mathrm{Sh}(*)$. Thus Λ has the shape of a point. Since Λ is a nonempty finite-dimensional compact metric space, Lacher's theorem [Lac69, Theorem 1.1] implies that Λ is cell-like and that its embedding in S^n has Property UV^∞.

We next prove that Λ satisfies the small loops condition SLC. Let U be an arbitrary open neighborhood of Λ in S^n. Since $\Lambda \subsetneq S^n$, choose proper open neighborhoods U_0 and V of Λ such that $\overline{V} \subset U_0$ and $\overline{U_0} \subset U$. Set $\eta := d_{g_{st}}\big(\overline{V}, S^n \setminus U_0\big) > 0$ and $\varepsilon := \frac{\eta}{c}$. Let $\gamma\colon S^1 \to V \setminus \Lambda$ be any loop satisfying $\mathrm{diam}_{g_{st}}\big(\gamma(S^1)\big) < \varepsilon$. Since $V \setminus \Lambda \subset \Omega$, the loop γ lies in Ω. By the diameter-controlled loop-filling property, γ extends to a map $G\colon B^2 \to \Omega$ such that

$$\mathrm{diam}_{g_{st}}\big(G(B^2)\big) \le c\, \mathrm{diam}_{g_{st}}\big(\gamma(S^1)\big) < \eta.$$

Because $G(B^2)$ contains $\gamma(S^1) \subset \overline{V}$, every point of $G(B^2)$ lies within distance $< \eta$ of $\overline{V}$. By the definition of η, this implies $G(B^2) \cap (S^n \setminus U_0) = \varnothing$. Hence $G(B^2) \subset U_0 \setminus \Lambda \subset U \setminus \Lambda$. Thus every loop of sufficiently small diameter in $V \setminus \Lambda$ bounds a disk in $U \setminus \Lambda$. This is precisely the small loops condition.

Because $\dim \Lambda \le 1$ and $n \in \{4, 5\}$, we have $\dim \Lambda \le n - 2$. By Venema [Ven76, p. 444], for a compact metric space $X \subset N^n$ with $\dim X \le n - 2$, the small loops condition is equivalent to the inessential loops condition. Therefore Λ satisfies ILC.

Because Λ has the shape of a point and satisfies ILC, Venema's equivalence of loop conditions implies that Λ satisfies the cellularity criterion [Ven76, p. 444].

We now apply the appropriate cellularity theorem by dimension.

If $n = 4$, then Repovš's 4-dimensional cellularity criterion [Rep87, Theorem, p. 564] applies: a compact metric space with Property UV^∞ in the interior of a topological 4-manifold is cellular if and only if it satisfies the cellularity criterion.

Since Λ has Property UV^∞ and satisfies the cellularity criterion, it follows that Λ is cellular in S^4.

If $n = 5$, then S^5 is a PL 5-manifold. The cell-like form of McMillan's cellularity criterion [DV09, Theorem 3.2.3, p. 107], originating in [McM64, Theorem 1, p. 327], states that a compact cell-like subset in the interior of a PL n-manifold, $n \geq 5$, is cellular if and only if it satisfies the cellularity criterion. Therefore Λ is cellular in S^5.

Thus, in both cases, $\Lambda \subset S^n$ is cellular. We now apply Morton Brown's cellular quotient theorem, in the form proved by Uspenskij [Usp05, Theorem 1.3]. If $K \subset S^n$ is cellular, then the quotient space S^n/K is homeomorphic to S^n. Applying this to $K = \Lambda$, we obtain $S^n/\Lambda \cong S^n$. The quotient map $\mathfrak{q}\colon S^n \to S^n/\Lambda$ collapses exactly Λ to a single point. Since $S^n \setminus \Lambda$ is saturated and open, the quotient-map criterion shows that its restriction is a homeomorphism $S^n \setminus \Lambda \cong (S^n/\Lambda) \setminus \{\mathfrak{q}(\Lambda)\}$. Since $S^n/\Lambda \cong S^n$, this gives $S^n \setminus \Lambda \cong S^n \setminus \{\mathrm{pt}\} \cong \mathbb{R}^n$. Since $\Omega = S^n \setminus \Lambda$, we have $M \cong \Omega \cong \mathbb{R}^n$. $\square$

We record the following elementary relation between Čech cohomology and Čech–Vietoris homology in covering dimension at most one.

Lemma 5.9. *Let X be a compact metrizable space with $\dim X \leq 1$. Suppose that $\check{H}^1(X;\mathbb{Z}) = 0$. Then $\check{H}_1(X;\mathbb{Z}) = 0$, where $\check{H}_*$ denotes Čech–Vietoris homology.*

Proof. Let $\mathcal{C}$ be the set of all finite open covers of X of multiplicity at most 2, ordered by refinement. Since $\dim X \leq 1$, every finite open cover of X has a finite open refinement in $\mathcal{C}$, so $\mathcal{C}$ is cofinal among all finite open covers. It is also directed: if $\mathcal{U}, \mathcal{V} \in \mathcal{C}$, then the finite open cover $\{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \neq \varnothing\}$ has a refinement in $\mathcal{C}$. Hence, by cofinality,

$$\check{H}_1(X;\mathbb{Z}) \cong \varprojlim_{\mathcal{U}\in\mathcal{C}} H_1(N(\mathcal{U});\mathbb{Z}), \qquad \check{H}^1(X;\mathbb{Z}) \cong \varinjlim_{\mathcal{U}\in\mathcal{C}} H^1(N(\mathcal{U});\mathbb{Z}).$$

If $\mathcal{V} \succeq \mathcal{U}$, choose a refinement projection $p_{\mathcal{U}\mathcal{V}}\colon N(\mathcal{V}) \to N(\mathcal{U})$. Different choices are contiguous, hence induce the same maps on homology and cohomology; similarly, if $\mathcal{W} \succeq \mathcal{V} \succeq \mathcal{U}$, then $p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{V}\mathcal{W}}$ and $p_{\mathcal{U}\mathcal{W}}$ are contiguous. Thus the induced maps form the inverse and direct systems above.

For each $\mathcal{U} \in \mathcal{C}$, since $\mathcal{U}$ has multiplicity at most 2, its nerve $N(\mathcal{U})$ is a finite simplicial complex of dimension at most 1, hence a finite graph. In particular, $H_1(N(\mathcal{U});\mathbb{Z})$ is finitely generated free abelian, and $H_0(N(\mathcal{U});\mathbb{Z})$ is free abelian. Therefore the universal coefficient theorem gives a natural isomorphism $H^1(N(\mathcal{U});\mathbb{Z}) \cong \mathrm{Hom}(H_1(N(\mathcal{U});\mathbb{Z}),\mathbb{Z})$, under which $p^*_{\mathcal{U}\mathcal{V}}$ corresponds to precomposition with $(p_{\mathcal{U}\mathcal{V}})_*$. Equivalently, $\langle p^*_{\mathcal{U}\mathcal{V}}\alpha, y\rangle = \langle \alpha, (p_{\mathcal{U}\mathcal{V}})_* y\rangle$.

We prove the contrapositive. Assume $\check{H}_1(X;\mathbb{Z}) \neq 0$. Then there exists a nonzero compatible family $x = (x_\mathcal{U})_{\mathcal{U}\in\mathcal{C}} \in \varprojlim_{\mathcal{U}\in\mathcal{C}} H_1(N(\mathcal{U});\mathbb{Z})$. Choose $\mathcal{U}_0 \in \mathcal{C}$ with $x_{\mathcal{U}_0} \neq 0$. Since $H_1(N(\mathcal{U}_0);\mathbb{Z})$ is free abelian, there exists $\phi \in H^1(N(\mathcal{U}_0);\mathbb{Z})$ such that $\langle \phi, x_{\mathcal{U}_0}\rangle \neq 0$.

We claim that the image of ϕ in $\varinjlim_{\mathcal{U}\in\mathcal{C}} H^1(N(\mathcal{U});\mathbb{Z}) \cong \check{H}^1(X;\mathbb{Z})$ is nonzero. Otherwise, since the index category is directed, an element of this filtered colimit is zero if and only if it becomes zero after passing to some refinement; thus there exists $\mathcal{V} \succeq \mathcal{U}_0$ such that $p^*_{\mathcal{U}_0\mathcal{V}}(\phi) = 0$. Pairing with $x_{\mathcal{V}}$ and using compatibility of x, we get

$$0 = \langle p^*_{\mathcal{U}_0\mathcal{V}}(\phi), x_{\mathcal{V}} \rangle = \langle \phi, (p_{\mathcal{U}_0\mathcal{V}})_*(x_{\mathcal{V}}) \rangle = \langle \phi, x_{\mathcal{U}_0} \rangle,$$

a contradiction. Thus $\check{H}_1(X;\mathbb{Z}) \neq 0$ implies $\check{H}^1(X;\mathbb{Z}) \neq 0$. The contrapositive proves the lemma. □

To replace the additional analytic or geometric hypotheses in Corollary 5.4 by topological conditions, and to extend Proposition 5.8 to higher dimensions, we recall several notions from shape theory, beginning with the following definition; see [Gev21, Section 7].

Definition 5.10 (Homotopy pro-groups). *Let (X, x_0) be a pointed compact metric space embedded in an absolute neighborhood retract P. Let $V_1 \supset V_2 \supset V_3 \supset \cdots \supset X$ be a nested cofinal sequence of open neighborhoods of X in P. For $q \geq 1$, the inclusions $V_j \hookrightarrow V_i$, $j \geq i$, induce homomorphisms $p_{ij} \colon \pi_q(V_j, x_0) \to \pi_q(V_i, x_0)$. The q-th homotopy pro-group of (X, x_0), denoted* pro-$\pi_q(X, x_0)$, *is the inverse system $\{\pi_q(V_i, x_0), p_{ij}\}_{j \geq i}$, viewed as an object of the pro-category* pro-**Grp**.

Since open subsets of an ANR are ANRs, the V_i form an associated ANR-neighborhood system for X. The resulting pro-isomorphism class is independent of the chosen cofinal neighborhood system. Indeed, if $\{U_i\}$ and $\{U'_a\}$ are two nested cofinal systems in the same ambient ANR, then cofinality gives, after passing to subsequences, alternating inclusions

$$U_{i_1} \supset U'_{a_1} \supset U_{i_2} \supset U'_{a_2} \supset \cdots \supset X,$$

which induce the usual pro-isomorphism between the two inverse systems.

Independence of the ambient ANR is part of the standard well-definedness of shape invariants [Gev21, Sections 2.2 and 7]. Any two associated ANR-neighborhood systems for X, even arising from different ANR embeddings of X, are pro-homotopy equivalent. Consequently, their homotopy pro-groups are pro-isomorphic. Thus pro-$\pi_q(X, x_0)$ is an invariant of the pointed shape of (X, x_0).

We write pro-$\pi_q(X, x_0) = 0$ if this inverse system is pro-isomorphic to the trivial system. For a nested sequential representative, this is equivalent to saying that for every i there exists $j \geq i$ such that the bonding homomorphism $\pi_q(V_j, x_0) \to \pi_q(V_i, x_0)$ is the zero homomorphism.

For a contractible open n-manifold ($n \geq 3$), simple connectivity at infinity is the classical end condition appearing in the recognition of Euclidean space. In the present setting, the open manifold need not be contractible. Thus we require a higher-dimensional vanishing condition on the homotopy groups at infinity.

Definition 5.11 (d-connected at infinity). *Let $d \geq 1$ be an integer, and let M be a noncompact, path-connected manifold. We say that M is d-*connected at infinity *if M is one-ended and, for every compact set $C \subset M$, there exists a compact set $K \subset M$, with $C \subset K$, such that every map $S^q \to M \setminus K$ extends to a map $B^{q+1} \to M \setminus C$ for every $1 \leq q \leq d$.*

With these definitions, we obtain the following Euclidean rigidity theorem when $n \geq 4$.

Theorem 5.12. *Let (M^n, g), $n \geq 4$, be a complete, open, simply connected, locally conformally flat n-manifold with nonnegative scalar curvature. Let $\Phi\colon M \to S^n$ be the developing map, set $\Omega := \Phi(M)$, and let $\Lambda := S^n \setminus \Omega$. Assume that M is $\lfloor \frac{n-2}{2} \rfloor$-connected at infinity and that Λ satisfies the small loops condition in S^n. Then M is homeomorphic to $\mathbb{R}^n$.*

The strategy of the proof closely follows that of Proposition 5.8. First, it utilizes $\lfloor \frac{n-2}{2} \rfloor$-connectedness at infinity and the dimension bounds to establish that the neighborhood system around the compact set Λ exhibits vanishing homotopy pro-groups. This vanishing permits the use of the shape-theoretic Whitehead theorem to deduce that Λ is cell-like. Finally, by verifying the cellularity criterion, we guarantee that collapsing Λ to a point yields a space homeomorphic to the ambient sphere S^n; consequently, its complement Ω is forced to be homeomorphic to $\mathbb{R}^n$.

Proof. Theorem 2.19 implies that the developing map identifies M conformally and diffeomorphically with the domain $\Omega \subset S^n$, and $\partial\Omega$ has zero Newtonian 2-capacity. Since M is noncompact, $\Omega \neq S^n$, so $\Lambda := S^n \setminus \Omega$ is a nonempty compact subset of S^n.

We first show that $\Lambda = \partial\Omega$. The capacity–dimension implication above gives $\dim_{\mathcal{H}}(\partial\Omega) \leq n-2$, so the covering dimension of $\partial\Omega$ is at most $n-2$, and hence $\widetilde{\check{H}}^{n-1}(\partial\Omega; \mathbb{Z}) = 0$. By Čech–Alexander duality, $\widetilde{H}_0(S^n \setminus \partial\Omega; \mathbb{Z}) = 0$. Thus $S^n \setminus \partial\Omega$ is connected. Since $S^n \setminus \partial\Omega = \Omega \sqcup \operatorname{int}(\Lambda)$ and $\Omega \neq \varnothing$, we obtain $\operatorname{int}(\Lambda) = \varnothing$. Therefore $\Lambda = \partial\Omega$.

Choose $x_0 \in \Omega$ and $\varepsilon > 0$ such that $\overline{B_{g_{st}}(x_0, \varepsilon)} \subset \Omega$, and set $D := S^n \setminus \overline{B_{g_{st}}(x_0, \varepsilon)}$. Then D is a bounded open neighborhood of Λ.

The metric $h := (\Phi^{-1})^* g$ on Ω is complete and, on $D \setminus \Lambda$, has the form $h = u^{\frac{4}{n-2}} g_{st}$ for some smooth positive function u. Moreover, $\mathrm{Sc}_h = \mathrm{Sc}_g \circ \Phi^{-1} \geq 0$. As in the proof of Theorem 5.1, h is geodesically complete near Λ. Applying Theorem 5.2 with background manifold (S^n, g_{st}), singular set Λ, and the bounded open neighborhood D gives $\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2}{2}$. Since topological dimension is bounded above by Hausdorff dimension, we obtain $\dim(\Lambda) \leq \lfloor \frac{n-2}{2} \rfloor =: d$.

Since $\Phi\colon M \to \Omega$ is a diffeomorphism, the assumed d-connectedness at infinity of M transfers to Ω. In particular, Ω is noncompact, path-connected, and one-ended. We claim that Λ is connected. For a space A, let $\mathcal{C}(A)$ denote its set of connected components. Since $\dim \Lambda \leq d = \lfloor (n-2)/2 \rfloor$ and $n \geq 4$, we have $d \leq n-3$. Let $W \subset S^n$ be connected and open. If $W = S^n$, then $W \setminus \Lambda = \Omega$ is connected. Suppose that $W \neq S^n$. Choose $a \in S^n \setminus W$ and identify $S^n \setminus \{a\}$ with $\mathbb{R}^n$. Then W is a region in $\mathbb{R}^n$, and $\dim(\Lambda \cap W) \leq \dim \Lambda \leq d \leq n-3$. Corollary 1 to Hurewicz–Wallman's non-separation theorem [HW41, p. 48, Corollary 1 to Theorem IV.4] therefore gives that $W \setminus \Lambda = W \setminus (\Lambda \cap W)$ is connected.

Let U be an open neighborhood of Λ. Because U is locally connected, every component W of U is open. Moreover, $W \setminus \Lambda \neq \varnothing$: otherwise the nonempty open subset W of S^n would be contained in Λ, contrary to $\dim \Lambda < n$. The preceding non-separation argument shows that $W \setminus \Lambda$ is connected. It follows that the inclusion $U \setminus \Lambda \hookrightarrow U$ induces a natural bijection

$$\mathcal{C}(U \setminus \Lambda) \longrightarrow \mathcal{C}(U). \qquad (*)$$

These bijections commute with the bonding maps arising from inclusions of neighborhoods.

Let $\mathcal{N}_0$ be the set, ordered by reverse inclusion, of open neighborhoods U of Λ for which every component of U meets Λ. This family is cofinal: for any open neighborhood U of Λ, the union U' of the components of U that meet Λ belongs to $\mathcal{N}_0$ and satisfies $U' \subset U$. Applying the same construction to $U_1 \cap U_2$, where $U_1, U_2 \in \mathcal{N}_0$, shows that $\mathcal{N}_0$ is directed. If $U \in \mathcal{N}_0$ and W is a component of U, then the closure in S^n of $W \setminus \Lambda$ meets $W \cap \Lambda$, because $\operatorname{int}_{S^n} \Lambda = \varnothing$ and hence $W \setminus \Lambda$ is dense in W. Thus $W \setminus \Lambda$ is not relatively compact in Ω; in the terminology of ends, it is an unbounded component of the corresponding neighborhood of infinity. Thus every component of $U \setminus \Lambda$, for $U \in \mathcal{N}_0$, is unbounded in Ω. Guilbault's description of the Freudenthal end set [Gui16, Section 3.3.1] uses an efficient exhaustion $\{K_i\}$ of Ω by compacta and identifies an end with a coherent choice of a component of $\Omega \setminus K_i$. Since $\{K_i\}$ is cofinal among the compact subsets of Ω and restriction to a cofinal subsystem does not change the inverse limit, that description, together with $(*)$, gives natural bijections

$$\begin{aligned} \mathcal{E}(\Omega) &\cong \varprojlim_{K \subset \Omega \text{ compact}} \mathcal{C}(\Omega \setminus K) \\ &\cong \varprojlim_{U \in \mathcal{N}_0} \mathcal{C}(U \setminus \Lambda) \cong \varprojlim_{U \in \mathcal{N}_0} \mathcal{C}(U). \end{aligned} \tag{$**$}$$

Here the second bijection uses the correspondence $U = S^n \setminus K$ between neighborhoods of Λ and compact subsets K of Ω, followed by cofinality of $\mathcal{N}_0$.

For completeness, we identify the last inverse limit in $(**)$. For $x \in \Lambda$, let $C_U(x)$ denote the component of U containing x. The coherent family $(C_U(x))_{U \in \mathcal{N}_0}$ depends only on the connected component of x in Λ. If $x, y \in \Lambda$ lie in distinct components, then, because components and quasi-components agree in a compact Hausdorff space, a clopen partition of Λ separates x from y. Disjoint open neighborhoods in S^n of the two compact parts of that partition, followed by passage to $\mathcal{N}_0$, give a neighborhood U for which $C_U(x) \neq C_U(y)$. Thus the induced map $\mathcal{C}(\Lambda) \to \varprojlim_{U \in \mathcal{N}_0} \mathcal{C}(U)$ is injective.

To prove surjectivity, let $(C_U)_{U \in \mathcal{N}_0}$ be a coherent element of the inverse limit. Each $A_U := C_U \cap \Lambda$ is a nonempty clopen, hence compact, subset of Λ. The family $\{A_U\}_{U \in \mathcal{N}_0}$ has the finite intersection property: given $U_1, \ldots, U_r$, choose $V \in \mathcal{N}_0$ with $V \subset \bigcap_{a=1}^r U_a$; coherence gives $A_V \subset \bigcap_{a=1}^r A_{U_a}$. Compactness of Λ therefore gives a point $x \in \bigcap_U A_U$, and then $C_U = C_U(x)$ for every U. Hence the map is surjective. Combining this identification with $(**)$, the ends of Ω correspond bijectively to the connected components of Λ. Since Ω is one-ended, Λ is connected.

Let $m := \dim \Lambda$. By the dimension bound above, $m \leq d = \left\lfloor \frac{n-2}{2} \right\rfloor$. If $m = 0$, then Λ is a connected compact zero-dimensional space, hence a point. Therefore $M \cong \Omega = S^n \setminus \{\mathrm{pt}\} \cong \mathbb{R}^n$.

Thus, for the rest of the proof, assume $m \geq 1$. We use the preceding observation that SLC is independent of the compatible metric and invariant under homeomorphism. Choose $p \in S^n \setminus \Lambda$ and identify $S^n \setminus \{p\} \cong \mathbb{R}^n$ by stereographic projection. Let $X \subset \mathbb{R}^n$ be the image of Λ. Since $p \notin \Lambda$, the set X is compact and homeomorphic to Λ. Therefore $\dim X = \dim \Lambda$, and X satisfies SLC in $\mathbb{R}^n$.

By Hollingsworth–Rushing [HR75, Lemma 1, p. 105], if $X \subset \mathbb{R}^n$ is compact, satisfies SLC, and $\dim X = k \le n-3$, then X has arbitrarily small neighborhoods W such that

$$\pi_i(W, W \setminus X, x) = 0 \qquad (1 \le i \le n-k-1,\ x \in W \setminus X).$$

Applying this with $k = m$ and using the dimensional hypothesis together with $n \ge 4$, we have $m \le d = \left\lfloor \frac{n-2}{2} \right\rfloor \le n-3$, and hence we obtain arbitrarily small neighborhoods W of X satisfying

$$\pi_i(W, W \setminus X, x) = 0 \qquad (1 \le i \le n-m-1,\ x \in W \setminus X).$$

Pull these neighborhoods back to $S^n \setminus \{p\}$. We now choose a nested cofinal sequence of open neighborhoods of Λ, $V_1 \supset V_2 \supset V_3 \supset \cdots \supset \Lambda$, as follows. Let $V_0 = S^n \setminus \{p\}$. Inductively choose a Hollingsworth–Rushing neighborhood W_i of Λ, small enough that $W_i \subset V_{i-1}$ and small enough to lie in a prescribed neighborhood basis of Λ. Let V_i be the connected component of W_i containing Λ. This component is well defined because Λ is connected, and it is open because W_i is an open subset of a manifold.

Set $N_i := V_i \setminus \Lambda$. The relative vanishing passes to the component pair. Fix $x \in N_i = V_i \setminus \Lambda$ and $1 \le j \le d+1$. Let $[f] \in \pi_j(V_i, N_i, x)$ be represented by a based relative map $f\colon (B^j, S^{j-1}, *) \to (V_i, N_i, x)$. Composing with the inclusion of pairs $(V_i, N_i) \hookrightarrow (W_i, W_i \setminus \Lambda)$, we regard f as a class in $\pi_j(W_i, W_i \setminus \Lambda, x)$. By the Hollingsworth–Rushing vanishing for W_i, this class is trivial. Hence there is a based relative null-homotopy $H\colon B^j \times I \to W_i$ such that

$$\begin{aligned} H(\,\cdot\,, 0) &= f, & H(*, t) &= x \qquad (t \in I),\\ H(S^{j-1} \times I) &\subset W_i \setminus \Lambda, & H(B^j \times \{1\}) &\subset W_i \setminus \Lambda. \end{aligned}$$

Since $B^j \times I$ is connected and $H(B^j \times I)$ meets V_i, for instance at the basepoint x, the image $H(B^j \times I)$ is a connected subset of W_i meeting the component V_i. Therefore $H(B^j \times I) \subset V_i$. Consequently $H(S^{j-1} \times I) \subset V_i \setminus \Lambda = N_i$ and $H(B^j \times \{1\}) \subset V_i \setminus \Lambda = N_i$. Thus the original class $[f]$ is already trivial in $\pi_j(V_i, N_i, x)$. Hence this relative homotopy object is trivial for $1 \le j \le d+1$; for $j = 1$, this means that the relative homotopy pointed set is a singleton. With this convention, we write

$$\pi_j(V_i, N_i, x) = 0 \qquad (1 \le j \le d+1,\ x \in N_i).$$

Here we use $d + 1 \le n - m - 1$, which follows from $m \le d = \lfloor (n-2)/2 \rfloor$.

Each V_i is a connected open subset of S^n. Since $\dim(\Lambda \cap V_i) \le d \le n-3$, the same non-separation theorem used above shows that $N_i = V_i \setminus \Lambda$ is connected. It is an open subset of a manifold and hence locally path connected; consequently, N_i is path connected.

The sequence (N_i) is cofinal among neighborhoods of infinity in Ω. Indeed, if $C \subset \Omega$ is compact, then $S^n \setminus C$ is a neighborhood of Λ in S^n. Hence, for all sufficiently large i, $V_i \subset S^n \setminus C$, and therefore $N_i = V_i \setminus \Lambda \subset \Omega \setminus C$.

We now show directly that the compact-set definition of d-connectedness at infinity gives the based pro-triviality needed below. Choose a proper ray $r\colon [0, \infty) \to \Omega$;

such rays exist in every connected open manifold [Gui16, Section 3.3.3]. Since each N_i is a neighborhood of infinity, choose $0 < t_1 < t_2 < \cdots$, with $t_i \to \infty$, such that $r([t_i, \infty)) \subset N_i$, and set $x_i := r(t_i) \in N_i$. The path segment $r|_{[t_i,t_j]}$ from x_i to x_j lies in N_i whenever $j \geq i$, and gives the usual change-of-basepoint path for the based end tower.

Fix i. Since N_i is a neighborhood of infinity, $C_i := \Omega \setminus N_i = S^n \setminus V_i$ is compact. By d-connectedness at infinity, there exists a compact set $K_i \subset \Omega$, $C_i \subset K_i$, such that every map $S^q \to \Omega \setminus K_i$ extends to a map $B^{q+1} \to \Omega \setminus C_i = N_i$ for every $1 \leq q \leq d$.

Since the sequence (N_i) is cofinal among neighborhoods of infinity, choose $j \geq i$ such that $N_j \subset \Omega \setminus K_i$. We claim that, for every $1 \leq q \leq d$, the bonding homomorphism $\pi_q(N_j, x_j) \to \pi_q(N_i, x_i)$ is the zero homomorphism.

Indeed, let $[f] \in \pi_q(N_j, x_j)$ be represented by a based map $f\colon S^q \to N_j$. Since $N_j \subset \Omega \setminus K_i$, the compact-set hypothesis gives an extension $F\colon B^{q+1} \to N_i$ of f. Hence the image of $[f]$ in $\pi_q(N_i, x_j)$ is trivial. The bonding map to $\pi_q(N_i, x_i)$ is obtained by inclusion followed by the standard change-of-basepoint isomorphism along the ray inside N_i, and change of basepoint sends the zero element to the zero element. Therefore $\pi_q(N_j, x_j) \to \pi_q(N_i, x_i)$ is zero.

Thus, for every i, there exists $j \geq i$ such that the bonding map to the i-th term is zero. Consequently, for every $1 \leq q \leq d$, the tower $\{\pi_q(N_i, x_i)\}_{i \geq 1}$ is pro-trivial.

Next consider the based long exact homotopy sequence of the pair (V_i, N_i, x_i). Since

$$\pi_j(V_i, N_i, x_i) = 0 \qquad (1 \leq j \leq d+1),$$

exactness gives, for each $1 \leq q \leq d$, an isomorphism induced by inclusion,

$$\pi_q(N_i, x_i) \xrightarrow{\cong} \pi_q(V_i, x_i).$$

These isomorphisms are compatible with the based bonding maps. Indeed, if $j \geq i$, then the inclusion of pairs $(V_j, N_j) \hookrightarrow (V_i, N_i)$ induces a morphism of the corresponding long exact homotopy sequences. The bonding maps are obtained from these inclusions together with change of basepoint along the ray segment from x_i to x_j. Since this segment lies in $N_i \subset V_i$, the change-of-basepoint maps for the N-tower and the V-tower commute with the maps induced by $N_\ell \hookrightarrow V_\ell$. Thus the levelwise isomorphisms above define an isomorphism of pro-systems

$$\{\pi_q(N_i, x_i)\} \cong \{\pi_q(V_i, x_i)\} \qquad (1 \leq q \leq d).$$

Since the first pro-system is pro-trivial, it follows that, for every $1 \leq q \leq d$, the tower $\{\pi_q(V_i, x_i)\}_{i \geq 1}$ is pro-trivial.

Choose a point $* \in \Lambda$. Since each V_i is path connected, choose a path β_i in V_i from $*$ to x_i. For $j \geq i$, let $\alpha_{ij} = r|_{[t_i,t_j]}$, regarded as a path in $N_i \subset V_i$ from x_i to x_j. After changing basepoints by β_i and β_j, the fixed-basepoint bonding homomorphism differs from the ray-based bonding homomorphism by the automorphism determined by the loop $\beta_i * \alpha_{ij} * \overline{\beta_j}$ in V_i: this is an inner automorphism for $q = 1$ and the standard $\pi_1(V_i, *)$-action for $q \geq 2$. In either case the identity element is fixed. Hence whenever the ray-based bonding homomorphism is zero, so is the corresponding fixed-basepoint bonding homomorphism. Therefore, for every $1 \le q \le d$, the tower $\{\pi_q(V_i, *)\}_{i \ge 1}$ is pro-trivial.

The nested ANR-neighborhood system $(V_i, *)$ is an associated pointed ANR-neighborhood system for $(\Lambda, *)$. Hence the inverse system $\{\pi_q(V_i, *), (V_j \hookrightarrow V_i)_\#\}_{j \ge i}$ represents $\text{pro-}\pi_q(\Lambda, *)$. Since this representative system is pro-trivial, we obtain

$$\text{pro-}\pi_q(\Lambda, *) = 0 \qquad (1 \le q \le d).$$

Since $m = \dim \Lambda \le d$, we have

$$\text{pro-}\pi_q(\Lambda, *) = 0 \qquad (1 \le q \le m).$$

We now apply Morita's finite-dimensional Whitehead theorem [Mor75, Theorem 1.2, p. 394]. In the form needed here, it says that if $f\colon (X, x_0) \to (Y, y_0)$ is a shape morphism of pointed connected finite-dimensional topological spaces, and if the induced morphism of homotopy pro-groups

$$\text{pro-}\pi_k(f)\colon\ \text{pro-}\pi_k(X, x_0) \longrightarrow \text{pro-}\pi_k(Y, y_0)$$

is an isomorphism for every $1 \le k \le r$, where $r := \max\{\dim X, \dim Y\}$, then f is a shape equivalence.

The ordinary constant map $c\colon (\Lambda, *) \to (\{*\}, *)$ induces a pointed shape morphism. Indeed, every continuous map between pointed compact metric spaces induces a morphism in the pointed shape category. Concretely, if $\{V_i\}$ is an associated pointed ANR-neighborhood system for $(\Lambda, *)$, then the constant maps $V_i \to \{*\}$ are compatible with the bonding maps and represent the induced shape morphism.

We apply Morita's theorem to this morphism. The spaces $(\Lambda, *)$ and $(\{*\}, *)$ are pointed, connected, and finite-dimensional. Moreover, $\max\{\dim \Lambda, \dim\{*\}\} = m$. For every $1 \le k \le m$, we have already shown that $\text{pro-}\pi_k(\Lambda, *) = 0$, and clearly $\text{pro-}\pi_k(\{*\}, *) = 0$. Therefore

$$\text{pro-}\pi_k(c)\colon\ \text{pro-}\pi_k(\Lambda, *) \longrightarrow \text{pro-}\pi_k(\{*\}, *)$$

is an isomorphism for every $1 \le k \le m$. By Morita's theorem, c is a shape equivalence. Hence $\mathrm{Sh}(\Lambda) = \mathrm{Sh}(*)$.

The set Λ is a nonempty finite-dimensional compact metric space. Lacher's theorem [Lac69, Theorem 1.1] now implies both that Λ is cell-like and that its embedding $\Lambda \hookrightarrow S^n$ has Property UV^∞. Explicitly, for every neighborhood U of Λ in S^n, there is a neighborhood $V \subset U$ of Λ such that the inclusion $V \hookrightarrow U$ is null-homotopic.

It remains to prove cellularity. Since $n \ge 4$, we have $d = \left\lfloor \frac{n-2}{2} \right\rfloor \ge 1$. Thus the preceding pro-triviality includes the pro-fundamental group at infinity.

Let U be any neighborhood of Λ in S^n. Choose i such that $V_i \subset U$. From the pro-triviality of the tower $\{\pi_1(N_i, x_i)\}$, choose $j \ge i$ such that the bonding homomorphism $\pi_1(N_j, x_j) \to \pi_1(N_i, x_i)$ is trivial. Put $V := V_j$. Let $\gamma\colon S^1 \to V \setminus \Lambda = N_j$ be any loop. Since N_j is path connected, join the basepoint of γ to x_j by a path in N_j. This conjugates γ to a loop based at x_j. Its image in $\pi_1(N_i, x_i)$ is trivial, so the conjugate loop, and hence γ, is null-homotopic in $N_i \subset U \setminus \Lambda$.

Therefore, for every neighborhood U of Λ, there exists a smaller neighborhood $V \subset U$ such that every loop in $V \setminus \Lambda$ is null-homotopic in $U \setminus \Lambda$. This is the cellularity criterion.

The remainder is exactly as in the proof of Proposition 5.8. Namely, the cellularity criterion obtained above, together with Property UV^∞ and the cell-like conclusion, implies that Λ is cellular in S^n: for $n = 4$ this follows from Repovš's theorem [Rep87, Theorem, p. 564], and for $n \geq 5$ from the cell-like form of McMillan's cellularity criterion [DV09, Theorem 3.2.3, p. 107], originating in [McM64, Theorem 1, p. 327].

Finally, collapse Λ to a point: $\mathfrak{q}: S^n \to S^n/\Lambda$. Since Λ is cellular in S^n, Uspenskij's form of Brown's cellular quotient theorem [Usp05, Theorem 1.3] shows that the quotient space S^n/Λ is homeomorphic to S^n. The complement $S^n \setminus \Lambda$ is saturated and open for $\mathfrak{q}$, so the quotient-map criterion shows that $\mathfrak{q}$ restricts to a homeomorphism $S^n \setminus \Lambda \cong (S^n/\Lambda) \setminus \{\mathfrak{q}(\Lambda)\}$. Therefore

$$M \cong \Omega = S^n \setminus \Lambda \cong (S^n/\Lambda) \setminus \{\mathfrak{q}(\Lambda)\} \cong S^n \setminus \{\mathrm{pt}\} \cong \mathbb{R}^n.$$

□

Proof of Theorem E. For $n = 3$, a connected open 3-manifold M^3 with $\pi_1(M) = 0$ and $H_2(M;\mathbb{Z}) = 0$ is contractible. Indeed, $\pi_1(M) = 0$ implies that M is orientable and that $H_1(M;\mathbb{Z}) = 0$. Poincaré duality for the connected noncompact oriented 3-manifold M gives $H_3(M;\mathbb{Z}) \cong H^0_c(M;\mathbb{Z}) = 0$. Moreover, a smooth 3-manifold has the homotopy type of a 3-dimensional CW complex, so $H_i(M;\mathbb{Z}) = 0$ for all $i > 3$. Together with the assumption $H_2(M;\mathbb{Z}) = 0$, this gives $\widetilde{H}_i(M;\mathbb{Z}) = 0$ for all $i \geq 0$. If some $\pi_k(M)$ were nonzero, choose the least such $k \geq 2$. The Hurewicz theorem would give $\pi_k(M) \cong H_k(M;\mathbb{Z}) = 0$, a contradiction. Thus $\pi_k(M) = 0$ for all $k \geq 1$. Since M has the homotopy type of a CW complex, Whitehead's theorem implies that M is contractible.

Theorem 5.1 for $n = 3$ and Theorem 5.12 for $n \geq 4$ complete the proof. □

6 Non-Euclidean contractible LCF n-manifolds with PSC for $n \geq 4$

The aim of this section is to show that, for $n \geq 4$, the conclusion of Theorem E(ii) need not hold if both of its additional topological hypotheses are omitted, and that the disjunction of conditions (i)–(v) in Corollary 5.4 cannot be omitted, even at the level of homeomorphism type. These two sharpness conclusions are recorded explicitly in Corollary 6.20 below.

The construction below does not separate the two hypotheses in Theorem E(ii). Indeed, since $n \geq 4$, $\lfloor (n-2)/2 \rfloor \geq 1$, so $\lfloor (n-2)/2 \rfloor$-connectedness at infinity implies simple connectivity at infinity, whereas the manifold constructed below is not simply connected at infinity. Thus it fails the hypothesis in Theorem E(ii) of being $\lfloor (n-2)/2 \rfloor$-connected at infinity. We do not determine whether its limit continuum K satisfies the small loops condition. Consequently, the examples show only that the two hypotheses in Theorem E(ii) cannot be omitted simultaneously; they do not establish that either hypothesis can be omitted while the other is retained.

Theorem 6.1. *For every integer $n \geq 4$, there exists a nondegenerate continuum $K \subset S^n$ with $\dim_{\mathcal{H}} K = 1$ such that, with $M := S^n \setminus K$, the following hold.*

1. *M is a contractible smooth open manifold that is not simply connected at infinity.*
2. *There exists a smooth function $\phi : M \to (0, \infty)$ with $m_0 := \inf_M \phi > 0$ such that, for every $t \in (-m_0, \infty)$,*

$$g^{(t)} := (\phi + t)^{4/(n-2)} g_{st}|_M$$

is smooth, complete, and locally conformally flat, with

$$\mathrm{Sc}_{g^{(t)}} = n(n-1)t(\phi + t)^{-(n+2)/(n-2)}.$$

Hence $\mathrm{Sc}_{g^{(t)}}$ is negative, zero, or positive according as $t < 0$, $t = 0$, or $t > 0$, and

$$\lim_{t \to 0} \int_M |\mathrm{Sc}_{g^{(t)}}|^{n/2} \, dV_{g^{(t)}} = 0.$$

For $t > 0$, the rescaled metric

$$\bar{g}^{(t)} := t^{-4/(n-2)} g^{(t)} = \left(1 + \frac{\phi}{t}\right)^{4/(n-2)} g_{st}|_M$$

satisfies

$$0 < \mathrm{Sc}_{\bar{g}^{(t)}} < n(n-1),$$

and its scalar curvature decays uniformly to zero at K in the sense that

$$\lim_{\delta \downarrow 0} \sup_{\substack{x \in M \\ d_{g_{st}}(x,K) < \delta}} \mathrm{Sc}_{\bar{g}^{(t)}}(x) = 0.$$

The construction of the metric is inspired by Karakhanyan's existence theorem [Kar24, Theorem 1.1], which states that $S^n \setminus K$ admits a complete scalar-flat metric conformal to the round metric if and only if the corresponding spherical Bessel capacity vanishes; see [Kar24, Section 2.2]. We first fix the notation for this section and then outline the proof. A *continuum* is a nonempty compact connected metric space; it is *nondegenerate* if it contains more than one point. All manifolds and metrics are smooth unless stated otherwise. For $\beta > 0$, $1 < p < \infty$, and $E \subset \mathbb{R}^d$, the Bessel capacity of E is

$$\mathcal{C}^{\mathbb{R}^d}_{\beta,p}(E) := \inf \left\{ \|f\|^p_{L^p(\mathbb{R}^d)} : f \in L^p(\mathbb{R}^d),\ f \geq 0,\ G_\beta * f \geq 1 \text{ everywhere on } E \right\},$$

where $G_\beta := \mathcal{F}^{-1}((1 + |\xi|^2)^{-\beta/2})$ is the Bessel kernel; see [AH95, §1.2.4]. The displayed infimum is the specialization of [AH95, Definition 2.3.3] to the kernel G_β. Fix $n \geq 4$ and set

$$\alpha := 1 + \frac{2}{n}, \qquad q := \frac{n}{2}, \qquad s := n - \alpha q = \frac{n-2}{2}, \qquad \theta := \frac{1}{q-1} = \frac{2}{n-2}. \tag{11}$$

Then $q \geq 2$, $\quad \alpha q = \frac{n+2}{2} < n$. The estimates below repeatedly use

$$n - 2 - s = s, \qquad s\theta = (n - 2 - s)\theta = 1, \qquad (n-2)\theta = 2. \tag{12}$$

Throughout this section we use the unnormalized Hausdorff convention

$$\mathcal{H}^t_\rho(A) := \inf\left\{\sum_i (\operatorname{diam} E_i)^t : A \subset \bigcup_i E_i,\ \operatorname{diam} E_i \le \rho\right\}, \qquad \mathcal{H}^t(A) := \lim_{\rho\downarrow 0} \mathcal{H}^t_\rho(A),$$

where the infimum is over countable covers. Thus no dimensional normalizing constant is included in $\mathcal{H}^t$. Fix $p_\sigma \in S^n$ and a stereographic chart $\sigma : S^n \setminus \{p_\sigma\} \to \mathbb{R}^n$ with pole p_σ, and choose a closed round ball $B \Subset S^n \setminus \{p_\sigma\}$. All compact sets in the tower construction lie in $\operatorname{int} B$. For $E \subset B$, write $\operatorname{cap}(E) := \mathcal{C}^{\mathbb{R}^n}_{\alpha,q}\big(\sigma(E)\big)$, the Bessel capacity of $\sigma(E)$ computed in this fixed chart. For compact $E \subset B$, the numerical value of $\operatorname{cap}(E)$ may depend on the chart, but its vanishing does not. Indeed, if $\widetilde{\sigma}$ is any other stereographic chart defined on a neighborhood of E, then the transition diffeomorphism $T := \widetilde{\sigma} \circ \sigma^{-1}$ is smooth on a neighborhood of the compact set $\sigma(E)$. If $E = \varnothing$, the assertion below is immediate, so assume that $E \neq \varnothing$. Choose $\delta_0 > 0$ so that the closed δ_0-neighborhood of $\sigma(E)$ is contained in the domain of T, and set $M_0 := \sup\big\{\|DT(z)\| : \operatorname{dist}(z, \sigma(E)) \le \delta_0\big\} < \infty$. If $x, y \in \sigma(E)$ and $|x-y| < \delta_0$, then the segment joining x to y remains in this neighborhood, so the mean-value estimate applies. If $|x-y| \ge \delta_0$, then $|T(x)-T(y)| \le \operatorname{diam} \widetilde{\sigma}(E) \le \delta_0^{-1} \operatorname{diam} \widetilde{\sigma}(E)\, |x-y|$. Consequently,

$$|T(x) - T(y)| \le \max\big\{M_0, \delta_0^{-1} \operatorname{diam} \widetilde{\sigma}(E)\big\}\, |x-y|,$$

and hence $T|_{\sigma(E)}$ is Lipschitz. The same argument applied to T^{-1} shows that $T^{-1}|_{\widetilde{\sigma}(E)}$ is Lipschitz. By the Lipschitz-image theorem for Bessel capacity [AH95, Theorem 5.2.1], if $A \subset \mathbb{R}^n$, $\Phi : A \to \mathbb{R}^n$ is Lipschitz, $\beta > 0$, and $1 < p < n/\beta$ (equivalently, $0 < \beta p < n$), then

$$\mathcal{C}^{\mathbb{R}^n}_{\beta,p}\big(\Phi(A)\big) \le C\mathcal{C}^{\mathbb{R}^n}_{\beta,p}(A)$$

for some $C = C(n, \beta, p, \operatorname{Lip}(\Phi)) < \infty$. Since $\alpha > 0$ and $1 < q < \frac{n}{\alpha}$ (equivalently, $0 < \alpha q = \frac{n+2}{2} < n$), the theorem applies with $(\beta, p) = (\alpha, q)$. The theorem is formulated for a Lipschitz map whose domain is the arbitrary subset A; no extension to all of $\mathbb{R}^n$ is required. Applying the estimate first to $T|_{\sigma(E)}$ and then to $T^{-1}|_{\widetilde{\sigma}(E)}$ proves

$$\mathcal{C}^{\mathbb{R}^n}_{\alpha,q}\big(\sigma(E)\big) = 0 \quad\Longleftrightarrow\quad \mathcal{C}^{\mathbb{R}^n}_{\alpha,q}\big(\widetilde{\sigma}(E)\big) = 0.$$

This is precisely the spherical capacity-zero condition in [Kar24, Section 2.2 and the proof of Theorem 1.1]: after choosing a stereographic pole in $S^n \setminus E$, that condition is expressed by the vanishing of the Euclidean Bessel capacity of the stereographic image. The displayed equivalence shows that it is independent of the allowed pole. Thus $\operatorname{cap}(E) = 0$ is precisely the fixed-chart expression of the spherical capacity-zero condition used above. For noncompact subsets of B, the notation cap always refers to this fixed chart; below we use only its monotonicity and make no chart-independence assertion for such sets.

The proof is divided into five steps. Steps 1–2 are topological and construct K and the complement $M := S^n \setminus K$, while Steps 3–5 are analytic and construct the metrics. Once K is known to be nonempty and compact, the analytic part uses only the connectedness of M, the inclusion $K \subset \operatorname{int} B$, and the assumption $\operatorname{cap}(K) = 0$.

Step 1 (seed and tower). Let $F_2 = \langle a, b \rangle$. The endomorphism $\omega : F_2 \to F_2$ determined by $\omega(a) = [a, b]$ and $\omega(b) = [a, b^{-1}]$ is injective and induces the zero map on $H_1(F_2; \mathbb{Z})$ (Lemma 6.2). We build nested closed regular neighborhoods $V_{j+1} \Subset \operatorname{int} V_j \subset \operatorname{int} B$ of embedded rank-two roses with a common wedge point x, marked by isomorphisms $\mathfrak{m}_j : F_2 \to \pi_1(V_j, x)$, such that

$$(i_j)_* \circ \mathfrak{m}_{j+1} = \mathfrak{m}_j \circ \omega, \qquad \operatorname{cap}(V_j) < 2^{-j},$$

where $i_j : V_{j+1} \hookrightarrow V_j$; we also choose ball covers of V_j of small $(1 + \frac{1}{j})$-content (Lemma 6.5 and Proposition 6.6). The marking constrains only the homotopy class of each successor rose, whereas the capacity bound constrains only the open set into which its regular neighborhood is squeezed. Thus the two conditions can be imposed independently. The continuum $K := \bigcap_j V_j$ satisfies $\operatorname{cap}(K) = 0$ and $\dim_{\mathcal{H}} K = 1$ (Proposition 6.7).

Step 2 (topology of the complement). Since $H_1(\omega) = 0$, the inclusion $i_j : V_{j+1} \hookrightarrow V_j$ induces the zero homomorphism $(i_j)^* : H^1(V_j; \mathbb{Z}) \to H^1(V_{j+1}; \mathbb{Z})$ for every j. Thus Čech continuity makes K Čech-acyclic (Proposition 6.10), while natural Alexander duality for the finite stages, followed by passage to the direct limit, makes M acyclic (Proposition 6.12). General position for 2-disks against the one-dimensional spines $(2 + 1 - n < 0)$ makes the stages $C_j := S^n \setminus \operatorname{int} V_j$ simply connected, so M is contractible (Proposition 6.14). Injectivity of ω, on the other hand, yields loops on ∂V_k, arbitrarily far out in M, that stay essential in V_1; so M is not simply connected at infinity and $M \ncong \mathbb{R}^n$ (Proposition 6.15). The nondegeneracy of K is part of Proposition 6.7.

Step 3 (the measure). Put $F := \sigma(K)$. By the Hedberg–Wolff inequality, $\operatorname{cap}(K) = 0$ forces every probability measure on F to have infinite Wolff self-energy. This averaged statement does not pass pointwise to weak-$*$ limits. Applying Sion's minimax theorem to smoothed truncated potentials, whose dependence on the measure is concave because $\theta \le 1$, and then taking a convex series, we obtain a probability measure μ with $\operatorname{supp} \mu = F$ and $\mathcal{W}^\mu \equiv +\infty$ on F (Theorem 6.16).

Step 4 (the scalar-flat metric). The Newtonian potential $u(x) := \int_F |x - y|^{2-n} \, d\mu(y)$ is harmonic off F and, because $s\theta = 1$, identically $+\infty$ on F. With U_{rd} the round conformal factor in the chart, $\phi := u/U_{\mathrm{rd}}$ extends smoothly and positively across the pole p_σ by Kelvin inversion. The probability normalization makes the inverted potential bounded there, and $\phi(p_\sigma) = 2^{-(n-2)/2}$. Hence $h := \phi^{4/(n-2)} g_{st}|_M$ is smooth and scalar-flat, and a dyadic annulus estimate, via $(n-2)\theta = 2$, converts $\mathcal{W}^\mu \equiv +\infty$ into infinite length of every divergent curve: h is complete (Lemma 6.18).

Step 5 (the shifts). Since μ is a probability measure with bounded support, $m_0 := \inf_M \phi > 0$. For $t \in (-m_0, \infty)$ the metric $g^{(t)} := (\phi + t)^{4/(n-2)} g_{st}|_M$ is uniformly comparable to h, hence complete, and linearity of the conformal Laplacian gives

$$\mathrm{Sc}_{g^{(t)}} = n(n-1)\, t\, (\phi + t)^{-(n+2)/(n-2)},$$

so the sign of the scalar curvature is the sign of t, and $\int_M |\mathrm{Sc}_{g^{(t)}}|^{n/2} \, dV_{g^{(t)}} \to 0$ as $t \to 0$; for $t > 0$ the rescaling $t^{-4/(n-2)} g^{(t)}$ has scalar curvature in $(0, n(n-1))$ tending to 0 at K (Lemma 6.19). One harmonic function thus yields complete locally conformally flat metrics of all three signs.

The restriction $n \geq 4$ enters through three thresholds, each equivalent to it: polarity of finite graphs, $s = \frac{n-2}{2} \geq 1$ (Lemma 6.3); general position, $2 + 1 - n < 0$ (Lemma 6.13); and Sion concavity, $\theta = \frac{2}{n-2} \leq 1$ (Theorem 6.16).

6.1 The thin marked tower

This subsection constructs the nested marked tower and its limit continuum K, which provide the topological and capacity-theoretic input for the metric construction carried out later in the section. The construction is organized as follows: Lemma 6.2 supplies the group-theoretic input, Lemma 6.3 provides the analytic input for the tower construction, Lemma 6.4 supplies the level-uniform regular-neighborhood construction, Lemma 6.5 combines these inputs, Proposition 6.6 iterates the construction, and Proposition 6.7 records the properties of the limiting continuum needed below.

Let $\Delta^{n+1} \subset \mathbb{R}^{n+1}$ be a regular Euclidean simplex centered at the origin. Radial projection $x \mapsto x/|x|$ transports the boundary complex $\partial\Delta^{n+1}$ to a smooth triangulation of S^n; fix the compatible PL structure induced by this triangulation. Whenever a finite PL graph is considered, we pass to a common adapted subdivision containing it as a subcomplex. A *rank-two rose* is a CW graph $R = R_a \vee R_b$, where R_a and R_b are oriented petal circles with a specified common wedge vertex and labels a, b. Whenever R is treated as a PL graph, each petal is triangulated by at least three simplicial edges; regular neighborhoods are taken in the sense of [Bry02, §3]. Write $F_2 = \langle a, b\rangle$ and $[x, y] = xyx^{-1}y^{-1}$.

Lemma 6.2 (Algebraic seed). *The endomorphism $\omega : F_2 \to F_2$, $\omega(a) = [a, b]$, $\omega(b) = [a, b^{-1}]$, is injective and induces the zero map on $H_1(F_2; \mathbb{Z})$.*

Proof. Put $w_a := aba^{-1}b^{-1}$ and $w_b := ab^{-1}a^{-1}b$. The products $w_a w_b$ and $w_b w_a$ are freely reduced with second letters b and b^{-1}, respectively, so $w_a w_b \neq w_b w_a$. Thus w_a and w_b do not commute. Since every cyclic group is abelian, $G := \langle w_a, w_b\rangle$ is noncyclic. By the Nielsen–Schreier theorem, G is free. Since it is noncyclic and generated by two elements, it has rank two. Let $\omega_G : F_2 \twoheadrightarrow G$ be the corestriction of ω, so that $\omega_G(a) = w_a$ and $\omega_G(b) = w_b$, and choose an isomorphism $\psi : G \xrightarrow{\sim} F_2$. The composite $\psi \circ \omega_G$ is a surjective endomorphism of the Hopfian group F_2 and is therefore injective. Hence ω_G is injective. If $\iota_G : G \hookrightarrow F_2$ denotes the inclusion, then $\omega = \iota_G \circ \omega_G$, so ω is injective. Finally, $w_a = [a, b]$ and $w_b = [a, b^{-1}]$ lie in $[F_2, F_2]$, so the map induced by ω on $H_1(F_2; \mathbb{Z})$ is zero. □

The next lemma is the entire analytic input of the construction. Part (a) controls Hausdorff content and gives $\dim_{\mathcal{H}} K \leq 1$ in the limit; part (b) says that both capacity conclusions hold: the graph has zero capacity, and its open neighborhoods can be chosen with arbitrarily small capacity. This is the outer regularity that lets smallness be imposed on the codimension-zero stages of the tower, not merely on their spines.

Lemma 6.3 (Thin neighborhoods of finite graphs). *Let Γ_0 be a finite embedded PL graph and let W be an open set such that $\Gamma_0 \subset W \subset \operatorname{int} B$. Let $\delta, \eta > 0$ and $\tau > 1$. Then:*

(a) *there is a finite family* $\{U_\ell\}$ *of open spherical balls with*

$$\Gamma_0 \subset U := \bigcup_\ell U_\ell \Subset W, \qquad \operatorname{diam} U_\ell < \delta, \qquad \sum_\ell (\operatorname{diam} U_\ell)^\tau < \eta;$$

(b) *if* $n \geq 4$, *then* $\operatorname{cap}(\Gamma_0) = 0$. *Moreover, for every family satisfying part* (a), *with* $U = \bigcup_\ell U_\ell$, *and every* $\varepsilon > 0$, *there is an open set* O *such that* $\Gamma_0 \subset O \Subset U$ *and* $\operatorname{cap}(O) < \varepsilon$.

Proof. (a) Let Γ_0 consist of closed edges $e_1, \dots, e_m$ and isolated vertices $v_1, \dots, v_{m'}$. Since Γ_0 is compact and W is open, choose $\rho_W > 0$ such that the closed ρ_W-neighborhood of Γ_0 is contained in W. Let L_i be the length of e_i in the round metric. For $r > 0$, parametrize e_i by arclength and divide it into at most $\lfloor L_i/r \rfloor + 1$ subarcs of length less than r. A spherical ball of radius r centered at an endpoint of such a subarc contains that subarc, because round distance is bounded above by arclength. Cover each edge in this way and each isolated vertex by one open ball of radius r. Denote the resulting family by $\{U_\ell\}$ and put $U := \bigcup_\ell U_\ell$.

Each ball has diameter at most $2r$, and

$$\sum_\ell (\operatorname{diam} U_\ell)^\tau \leq \sum_{i=1}^m \left(\frac{L_i}{r} + 1\right)(2r)^\tau + m'(2r)^\tau = 2^\tau \Big(\sum_{i=1}^m L_i\Big) r^{\tau-1} + 2^\tau (m + m') r^\tau.$$

Since $\tau > 1$, both exponents $\tau - 1$ and τ are positive, so the right-hand side tends to zero as $r \to 0$. Choose $r < \rho_W$ sufficiently small that $2r < \delta$ and the displayed sum is less than η. Every closed ball in the family is then contained in W; since the family is finite, $\overline{U} \subset W$. Thus $U \Subset W$.

(b) The argument proceeds in four steps. The hypothesis $n \geq 4$ is used at the junction of Steps 1 and 2, where the critical dimension $s = \frac{n-2}{2}$ from (11) must satisfy $s \geq 1$. All spherical neighborhoods constructed below are contained in $W \subset \operatorname{int} B$.

Step 1 (finite critical measure for the spine). The restriction $\sigma|_B$ is smooth on the compact set $B \subset S^n \setminus \{p_\sigma\}$ and is therefore Lipschitz. Since Γ_0 is a finite one-dimensional PL graph, $\sigma(\Gamma_0)$ is a finite union of rectifiable curves, and hence $\mathcal{H}^1(\sigma(\Gamma_0)) < \infty$. We also claim that $\mathcal{H}^t(\sigma(\Gamma_0)) = 0$ for every $t > 1$. For $0 < \rho \leq 1$, choose a countable cover $\{E_i\}$ of $\sigma(\Gamma_0)$ such that $\operatorname{diam} E_i \leq \rho$ and $\sum_i \operatorname{diam} E_i \leq \mathcal{H}^1_\rho(\sigma(\Gamma_0)) + 1$. The same cover gives

$$\begin{aligned}
\mathcal{H}^t_\rho(\sigma(\Gamma_0)) &\leq \sum_i (\operatorname{diam} E_i)^t \\
&\leq \rho^{t-1} \sum_i \operatorname{diam} E_i \\
&\leq \rho^{t-1} \big(\mathcal{H}^1(\sigma(\Gamma_0)) + 1\big).
\end{aligned}$$

Letting $\rho \downarrow 0$ proves the claim. Since $s = (n-2)/2$, we obtain $\mathcal{H}^s(\sigma(\Gamma_0)) < \infty$: when $n = 4$, one has $s = 1$, and this follows from the finite $\mathcal{H}^1$-measure; when $n \geq 5$, one has $s > 1$, and the preceding claim gives the stronger identity $\mathcal{H}^s(\sigma(\Gamma_0)) = 0$. This is precisely where the dimensional hypothesis $n \geq 4$, which ensures $s \geq 1$, is used.

Step 2 (polarity of the spine: checking the hypotheses of Adams–Hedberg). We apply the finite-critical-measure theorem [AH95, Theorem 5.1.9]: if $p > 1$ and $0 < \beta p < n$, and if $\Lambda_h(E) < \infty$ for the ball-gauge Hausdorff measure with gauge $h(r) = r^{\,n-\beta p}$, then $\mathcal{C}^{\mathbb{R}^n}_{\beta,p}(E) = 0$. Apply the theorem with $(\beta, p) = (\alpha, q) = (1 + 2/n, n/2)$. Then $p = q = n/2 \geq 2 > 1$, while $\beta p = \alpha q = ((n + 2)/n)(n/2) = (n + 2)/2 < n$ because $n > 2$. Thus the parameters lie in the admissible range, and the corresponding critical gauge exponent is $n - \beta p = s$.

The cited theorem uses the ball-gauge measure Λ_h for $h(r) = r^s$, whereas Step 1 bounds the unnormalized Hausdorff measure $\mathcal{H}^s$ fixed above. For $A \subset \mathbb{R}^n$ and $\rho > 0$, let $\Lambda_h^{(\rho)}(A)$ denote the infimum in the definition of $\Lambda_h(A)$ over ball covers of A whose radii are at most ρ. Let $\{E_i\}_{i\geq 1}$ be a countable cover of A by nonempty sets with $d_i := \operatorname{diam} E_i \leq \rho/2$ and $\sum_i d_i^s < \infty$, and choose $x_i \in E_i$. Given $\zeta > 0$, if $d_i > 0$, choose $d_i < r_i \leq \rho$ so close to d_i that $r_i^s < d_i^s + 2^{-i}\zeta$; if $d_i = 0$, choose instead $0 < r_i \leq \rho$ with $r_i^s < 2^{-i}\zeta$. Then the balls $\{B(x_i, r_i)\}_{i\geq 1}$ cover A and $\sum_{i\geq 1} h(r_i) < \sum_{i\geq 1} d_i^s + \zeta$, because $\sum_{i\geq 1} 2^{-i} = 1$. Covers with infinite s-sum are irrelevant to the infimum. Taking infima over the covers $\{E_i\}_{i\geq 1}$, and then letting $\zeta \downarrow 0$, gives $\Lambda_h^{(\rho)}(A) \leq \mathcal{H}^s_{\rho/2}(A)$. Letting $\rho \downarrow 0$ yields $\Lambda_h(A) \leq \mathcal{H}^s(A)$. Thus Step 1 yields $\Lambda_h(\sigma(\Gamma_0)) \leq \mathcal{H}^s(\sigma(\Gamma_0)) < \infty$. The finite-critical-measure theorem now gives $\mathcal{C}^{\mathbb{R}^n}_{\alpha,q}(\sigma(\Gamma_0)) = 0, \qquad \text{or equivalently} \qquad \operatorname{cap}(\Gamma_0) = 0.$

Step 3 (admissible potentials of small norm). By the defining formula [AH95, Definition 2.3.3], the capacity of a set E is the infimum of $\|f\|^p_{L^p}$ over all nonnegative $f \in L^p(\mathbb{R}^n)$ such that $G_\beta * f \geq 1$ everywhere on E. The convolution is understood as an extended nonnegative Lebesgue integral; it is pointwise well defined and independent of the chosen nonnegative measurable representative of f. Since $\operatorname{cap}(\Gamma_0) = 0$ by Step 2, there is a nonnegative function $f_{\alpha,q} \in L^q(\mathbb{R}^n)$ such that

$$G_\alpha * f_{\alpha,q} \geq 1 \quad \text{on } \sigma(\Gamma_0), \qquad \|f_{\alpha,q}\|^q_{L^q} < 2^{-q}\varepsilon.$$

The scale factor 2^{-q} is reserved to offset the doubling operation in Step 4.

Step 4 (outer regularity: from the spine to an open set). The Bessel kernel G_α is nonnegative and lower semicontinuous. Thus Fatou's lemma, applied along every convergent sequence of base points, shows that the potential $G_\alpha * f_{\alpha,q}$ is lower semicontinuous on $\mathbb{R}^n$; cf. [AH95, Proposition 2.3.2]. Hence its strict superlevel set $\{G_\alpha * f_{\alpha,q} > \frac{1}{2}\}$ is open and contains $\sigma(\Gamma_0)$, where the potential is at least 1. Since $\sigma(\Gamma_0)$ is compact and $\sigma(U)$ is an open neighborhood of it, choose an open set U' such that $\sigma(\Gamma_0) \subset U' \Subset \sigma(U)$. The intersection $O' := U' \cap \{x : (G_\alpha * f_{\alpha,q})(x) > \frac{1}{2}\}$ is an open neighborhood of $\sigma(\Gamma_0)$. On O', the doubled function $2f_{\alpha,q}$ has potential > 1 and is therefore admissible. Hence, by the definition of capacity, $\mathcal{C}^{\mathbb{R}^n}_{\alpha,q}(O') \leq 2^q \|f_{\alpha,q}\|^q_{L^q} < \varepsilon$. Finally, $O := \sigma^{-1}(O')$ is open with $\Gamma_0 \subset O \Subset U$, because $O' \Subset \sigma(U)$ and σ is a homeomorphism onto its image; and since $O \subset U \subset B$, its capacity is computed in the fixed chart: $\operatorname{cap}(O) = \mathcal{C}^{\mathbb{R}^n}_{\alpha,q}(O') < \varepsilon$. □

Lemma 6.4 (Sublevel regular neighborhoods)**.** *Let T be a finite triangulation of the fixed PL sphere S^n, let L be a nonempty, proper, full subcomplex of T, and let $f_L : |T| \to [0, 1]$ be the simplicial map that takes value 0 at the vertices of L and value 1 at all other vertices. Then $f_L^{-1}(0) = |L|$. Define the simplicial complement of L in T by $C(L, T) := \{\Delta \in T : |\Delta| \cap |L| = \varnothing\}$. Then $C(L, T)$ is a nonempty, proper, full subcomplex of T, and its 0–1-level function is $f_{C(L,T)} = 1 - f_L$. Moreover, for every $c \in (0, 1)$, the set $N_c := f_L^{-1}([0, c])$ is a closed regular neighborhood of $|L|$*

in S^n and a compact PL n-manifold, with $\partial N_c = f_L^{-1}(c)$. Its topological interior in S^n is $\operatorname{int} N_c = f_L^{-1}([0,c))$. There is also a homeomorphism of pairs

$$\Phi_c : \big(N_c \setminus |L|, \partial N_c\big) \xrightarrow{\cong} \big(\partial N_c \times (0,1], \partial N_c \times \{1\}\big) \tag{13}$$

whose restriction to ∂N_c is $z \mapsto (z,1)$. Consequently, ∂N_c is a strong deformation retract of $N_c \setminus |L|$.

Proof. Let $y \in |T|$, and let Δ_y be its carrier. If $f_L(y) = 0$, then every vertex of Δ_y has f_L-value 0; fullness of L therefore gives $\Delta_y \in L$, and hence $y \in |L|$. The reverse inclusion is immediate, so $f_L^{-1}(0) = |L|$.

The collection $C(L,T)$ is a subcomplex: if $\Delta \in C(L,T)$ and τ is a face of Δ, then $|\tau| \subset |\Delta|$, and hence $|\tau| \cap |L| = \varnothing$. Its vertex set is

$$C(L,T)^{(0)} = T^{(0)} \setminus L^{(0)}. \tag{14}$$

Indeed, a vertex of L does not belong to $C(L,T)$; conversely, if $v \notin L^{(0)}$, then $f_L(v) = 1$, so $v \notin f_L^{-1}(0) = |L|$, and the 0-simplex $\{v\}$ belongs to $C(L,T)$. Because L is nonempty, it has a vertex, so $C(L,T) \neq T$. Because L is proper and full, some vertex of T does not belong to L: otherwise fullness would imply that every simplex of T belongs to L, contrary to $L \neq T$. Thus $C(L,T)$ is nonempty and proper.

To prove fullness, let $\Delta \in T$ have all its vertices in $C(L,T)^{(0)}$. Equation (14) gives $f_L = 1$ at every vertex of Δ, and affineness gives $f_L \equiv 1$ on $|\Delta|$. Since $|L| = f_L^{-1}(0)$, the simplex Δ is disjoint from $|L|$ and therefore belongs to $C(L,T)$. Finally, $f_{C(L,T)}$ and $1 - f_L$ agree at every vertex by (14); both are affine on each simplex of T, so uniqueness of the affine extension gives $f_{C(L,T)} = 1 - f_L$ on $|T|$.

Fix $c \in (0,1)$. Call a simplex $\Delta \in T$ *mixed* if its vertices occur at both f_L-levels 0 and 1. A mixed simplex belongs to neither L nor $C(L,T)$. Conversely, if $\Delta \notin L \cup C(L,T)$, fullness of L gives a vertex outside $L^{(0)}$, while fullness of $C(L,T)$ gives a vertex outside $C(L,T)^{(0)}$, hence in $L^{(0)}$ by (14). Thus the mixed simplices are precisely the simplices not belonging to $L \cup C(L,T)$. For every mixed simplex Δ, choose a derived vertex $\widehat{\Delta}_c \in \mathring{\Delta} \cap f_L^{-1}(c)$, which is possible by choosing positive barycentric coordinates whose total weight on the level-1 vertices is c. Let $\widehat{T}_c$ be the resulting derived subdivision of T modulo $L \cup C(L,T)$. Set

$$N(L,\widehat{T}_c) := \bigcup_{v \in L^{(0)}} \mathrm{St}(v,\widehat{T}_c), \qquad \dot{N}(L,\widehat{T}_c) := \big\{\Xi \in N(L,\widehat{T}_c) : |\Xi| \cap |L| = \varnothing\big\},$$

where St denotes the closed simplicial star.

Every simplex of $\widehat{T}_c$ that is not a simplex of $L \cup C(L,T)$ has the form

$$\Xi = v * \widehat{\Delta}_{0,c} * \cdots * \widehat{\Delta}_{r,c},$$

where $\Delta_0 \subsetneq \cdots \subsetneq \Delta_r$ are mixed simplices and v is either empty or a simplex of $L \cup C(L,T)$ contained in Δ_0. Consequently, the vertex levels of such a simplex are contained in either $\{0,c\}$ or $\{c,1\}$. A simplex of L has all its vertices at level 0. A simplex of $C(L,T)$ has all its vertices at level 1, since a vertex belonging to $L^{(0)}$ would give a point of its intersection with $|L|$. Thus the vertex levels of every simplex of $\widehat{T}_c$ are contained in either $\{0, c\}$ or $\{c, 1\}$; in particular, no simplex has both a level-0 and a level-1 vertex.

The same chain description shows that L is full in $\widehat{T}_c$. Indeed, suppose that every vertex of a simplex $\Xi \in \widehat{T}_c$ belongs to L. Every such vertex has level 0. A simplex of $C(L,T)$ has only level-1 vertices, whereas every simplex of $\widehat{T}_c \setminus \big(L \cup C(L,T)\big)$ contains at least one derived vertex $\widehat{\Delta}_{i,c}$ of level c. Hence Ξ is a simplex of L, which proves fullness in the subdivided complex. Thus L is the full subcomplex of $\widehat{T}_c$ induced by the level-0 vertices, and the complex $N(L, \widehat{T}_c)$ defined above is its simplicial neighborhood. Since L is full in T and $\widehat{T}_c$ is, by construction, a derived subdivision of T modulo $L \cup C(L,T)$, [Bry02, §3, pp. 227–228] shows that $N(L, \widehat{T}_c)$ is a derived neighborhood of L in T. Because every derived vertex $\widehat{\Delta}_c$ was chosen on $f_L^{-1}(c)$, this is precisely Bryant's c-neighborhood.

We claim that a simplex Ξ of $\widehat{T}_c$ belongs to $N(L, \widehat{T}_c)$ if and only if it has no level-1 vertex. Suppose first that Ξ has no level-1 vertex. If it has a level-0 vertex v, then $v \in L^{(0)}$ and $\Xi \in \mathrm{St}(v, \widehat{T}_c)$. If all the vertices of Ξ have level c, then Ξ is represented by a chain of mixed simplices as above with $v = \varnothing$. Choose $v \in L^{(0)} \cap \Delta_0$. The chain description of the derived subdivision shows that $v * \Xi$ is a simplex of $\widehat{T}_c$. Therefore $\Xi \in \mathrm{St}(v, \widehat{T}_c)$.

Conversely, suppose that $\Xi \in N(L, \widehat{T}_c)$. By the definition of the closed star, there exist $v \in L^{(0)}$ and a simplex $\Xi' \in \widehat{T}_c$ such that Ξ is a face of Ξ' and v is a vertex of Ξ'. The simplex Ξ' therefore has a level-0 vertex. By the preceding level dichotomy it has no level-1 vertex, and hence neither does its face Ξ. This proves the claim.

Every simplex of $\widehat{T}_c$ is contained in a simplex of T; hence the restriction of f_L to every simplex Ξ of $\widehat{T}_c$ is affine. If Ξ has no level-1 vertex, then $f_L \leq c$ on $|\Xi|$ and $\Xi \in N(L, \widehat{T}_c)$. If Ξ has a level-1 vertex, then the level dichotomy implies that it has no level-0 vertex. In that case, $f_L^{-1}([0,c]) \cap |\Xi|$ is the face spanned by the level-c vertices of Ξ, with the empty face allowed. The same is true of $|N(L, \widehat{T}_c)| \cap |\Xi|$, because the faces of Ξ belonging to $N(L, \widehat{T}_c)$ are, by the claim, precisely those having no level-1 vertex. It follows simplexwise that $f_L^{-1}([0,c]) = |N(L, \widehat{T}_c)|$. Together with the preceding identification, this equality identifies N_c with the realization of a derived neighborhood of L in T; hence N_c is a regular neighborhood of $|L|$ in S^n. Since $\widehat{T}_c$ is finite, $N(L, \widehat{T}_c)$ is a finite subcomplex, so N_c is compact and therefore closed in S^n. Moreover, since S^n is a PL manifold without boundary, [Bry02, Theorem 3.6] shows that N_c is a PL n-manifold and that $\partial N_c = |\dot{N}(L, \widehat{T}_c)|$.

By the preceding characterization of $N(L, \widehat{T}_c)$, its simplices disjoint from $|L|$ are precisely the simplices all of whose vertices have level c. Indeed, a simplex of the neighborhood that has a level-0 vertex meets $|L|$, whereas, if all the vertices of a simplex Ξ have level c, then $f_L \equiv c > 0$ on $|\Xi|$ and hence $|\Xi| \cap |L| = \varnothing$, because $|L| = f_L^{-1}(0)$. On every simplex Ξ of $\widehat{T}_c$, the intersection $f_L^{-1}(c) \cap |\Xi|$ is exactly the face spanned by its level-c vertices, again with the empty face allowed. Consequently, $\partial N_c = |\dot{N}(L, \widehat{T}_c)| = f_L^{-1}(c)$.

Since $[0, c)$ is open in $[0, 1]$ with its relative topology, $f_L^{-1}([0, c))$ is open in S^n and is contained in N_c. Conversely, let $z \in f_L^{-1}(c)$. Its carrier Δ in T is mixed. Normalizing separately the barycentric coordinates on the level-0 and level-1 vertices gives unique points z_i in the corresponding faces of Δ such that $z = (1-c)z_0 + cz_1$. Choose $c_m \in (c, 1)$ with $c_m \to c$ and put $z_m := (1-c_m)z_0 + c_m z_1$. Then z_m has the same carrier as z, $z_m \to z$, and $f_L(z_m) = c_m > c$. Thus no point of $f_L^{-1}(c)$ is interior to N_c, and consequently $\operatorname{int} N_c = f_L^{-1}([0, c))$.

It remains to prove the product assertion. Let $z \in N_c \setminus |L|$ and put $r := f_L(z) \in (0, c]$. The carrier Δ of z in T is mixed. Let Δ_0 and Δ_1 be the faces of Δ spanned by its level-0 and level-1 vertices, respectively. Normalizing separately the two groups of barycentric coordinates gives unique points $z_i \in |\Delta_i|$ such that $z = (1-r)z_0 + rz_1$. Define

$$q_c(z) := (1-c)z_0 + cz_1 \in f_L^{-1}(c) = \partial N_c, \qquad \Phi_c(z) := \big(q_c(z), r/c\big).$$

Conversely, if $w \in \partial N_c$ and $\vartheta \in (0, 1]$, write, by the same normalized-barycentric-coordinate construction, $w = (1-c)w_0 + cw_1$, and set $\Psi_c(w, \vartheta) := (1-\vartheta c)w_0 + \vartheta c w_1$. These definitions are independent of the choice of a mixed simplex containing the point: both are determined by the barycentric coordinates in its unique carrier, and the formulas restrict compatibly to every common face. On each mixed simplex the formulas are continuous. The sets

$$|\Delta| \cap f_L^{-1}((0, c]) \quad \text{and} \quad \big(|\Delta| \cap f_L^{-1}(c)\big) \times (0, 1],$$

as Δ ranges over the finitely many mixed simplices of T, are finite covers of the domain and codomain by relatively closed subsets, respectively. The pasting lemma therefore shows that Φ_c and Ψ_c are continuous. The displayed formulas show directly that $\Psi_c \circ \Phi_c = \mathrm{id}$ and $\Phi_c \circ \Psi_c = \mathrm{id}$, so Φ_c is the claimed homeomorphism. If $z \in \partial N_c$, then $r = c$ and $\Phi_c(z) = (z, 1)$. Finally,

$$R_u(z) := \Psi_c\big(q_c(z), (1-u)r/c + u\big), \qquad 0 \le u \le 1,$$

defines a strong deformation retraction $(R_u)_{0 \le u \le 1}$ of $N_c \setminus |L|$ onto ∂N_c. The construction uses only positive levels; no product structure across the generally degenerate spine $f_L^{-1}(0) = |L|$ is asserted.

Because $c \in (0, 1)$ was arbitrary, the conclusion holds for every c while T, L, and f_L remain fixed. □

The next lemma constructs one stage of the tower from the preceding stage. Its topological and analytic requirements are imposed successively. First, an embedded rose is chosen whose petals represent $\mathfrak{m}(\omega(a))$ and $\mathfrak{m}(\omega(b))$. A sufficiently small regular neighborhood of that rose is then chosen inside an open set of controlled capacity supplied by Lemma 6.3.

Lemma 6.5 (Thin marked successor)**.** *Assume $n \ge 4$. Let $H \subset \operatorname{int} B$ be a closed regular neighborhood of an embedded rank-two rose with wedge vertex x, assume $x \in \operatorname{int} H$, and let $\mathfrak{m} : F_2 \xrightarrow{\cong} \pi_1(H, x)$ be a marking. For every $\varepsilon > 0$, $\tau > 1$, $\delta > 0$, $\eta > 0$, there are an embedded rank-two PL rose $\Gamma' \subset \operatorname{int} H$ with wedge vertex x, a connected closed regular neighborhood $H' \Subset \operatorname{int} H$ of Γ' satisfying $\Gamma' \subset \operatorname{int} H'$ (in particular $x \in \operatorname{int} H'$), a marking $\mathfrak{m}' : F_2 \xrightarrow{\cong} \pi_1(H', x)$, and a finite family $\{U'_\ell\}$ of open balls with*

$$H' \subset \bigcup_\ell U'_\ell \Subset \operatorname{int} H, \qquad \operatorname{diam} U'_\ell < \delta, \qquad \sum_\ell (\operatorname{diam} U'_\ell)^\tau < \eta,$$

and, for the inclusion $i : H' \hookrightarrow H$,

$$i_* \circ \mathfrak{m}' = \mathfrak{m} \circ \omega, \qquad \operatorname{cap}(H') < \varepsilon.$$

The petals of Γ' may be oriented and labeled a and b so that $\mathfrak{m}'(a)$ and $\mathfrak{m}'(b)$ are represented by the corresponding oriented petal loops based at x. Moreover, the data may be chosen together with a triangulation $\mathcal{T}'$ of S^n in which Γ' is the polyhedron of a nonempty, proper, full subcomplex, the simplicial map $f' : |\mathcal{T}'| \to [0,1]$ that takes value 0 on the vertices of this subcomplex and value 1 on all other vertices, and a number $t' \in (0,1)$ such that

$$(f')^{-1}(0) = \Gamma', \qquad H' = (f')^{-1}([0,t']),$$

and, for every $c \in (0,1)$, $(f')^{-1}([0,c])$ is a closed regular neighborhood of Γ' with

$$\partial\big((f')^{-1}([0,c])\big) = (f')^{-1}(c).$$

In particular, i_* is injective.

The proof is divided into three steps. Step 1 uses relative embedding approximation to realize the prescribed classes by an embedded one-complex. Step 2 applies Lemma 6.3 to obtain a cover of arbitrarily small τ-content and a regular neighborhood of arbitrarily small capacity. Since this neighborhood deformation retracts onto its spine, it retains the topological data fixed in Step 1. Step 3 transfers the marking through this deformation retraction and uses the injectivity of ω from Lemma 6.2.

Proof. We work throughout in a fixed compatible PL structure on S^n, refining subdivisions as necessary so that the finite PL polyhedra under consideration are subcomplexes and the base point x is a vertex. Let $R = R_a \vee R_b$ denote the standard rank-two rose with wedge point $*$ and oriented petal circles labeled a and b, triangulated so that $*$ is a vertex and each petal contains at least three simplicial edges.

Step 1 (construction of an embedded PL rose realizing the algebraic data). Since H is a closed regular neighborhood of a compact polyhedron in the interior of a PL manifold, it is a compact PL manifold with boundary [Bry02, Theorem 3.6]. Consequently, ∂H has a collar in H by the PL collaring theorem [Bry02, Corollary 2.5]. Choose a collar $c_H \colon \partial H \times [0,1] \hookrightarrow H$ whose image is disjoint from the interior point x; this is possible after restricting and reparametrizing an initial collar to a sufficiently short subinterval. Put $H_0 := H \setminus c_H\big(\partial H \times [0,1)\big) \subset \operatorname{int} H$. Define a homotopy $(r_u)_{u\in[0,1]}$ on H by

$$r_u\big(c_H(y,t)\big) := c_H\big(y, (1-u)t + u\big) \quad (y \in \partial H,\ 0 \le t \le 1), \qquad r_u(z) := z \quad (z \in H_0).$$

The two formulas agree on $c_H(\partial H \times \{1\})$, so $(r_u)_{u\in[0,1]}$ is a strong deformation retraction of H onto H_0. The homotopy fixes x, and its restriction to $\operatorname{int} H$ is a strong deformation retraction onto the same set H_0, because $r_u(c_H(y,t)) \in c_H(\partial H \times (0,1])$ whenever $t > 0$. Consequently, the inclusion $\operatorname{int} H \hookrightarrow H$ is a based homotopy equivalence and induces an isomorphism $\pi_1(\operatorname{int} H, x) \xrightarrow{\cong} \pi_1(H,x)$. In particular, the classes $\mathfrak{m}(\omega(a))$ and $\mathfrak{m}(\omega(b))$ admit continuous based representatives γ_a, γ_b : $([0,1],\{0,1\}) \to (\operatorname{int} H, x)$ satisfying $[\gamma_a] = \mathfrak{m}(\omega(a))$ and $[\gamma_b] = \mathfrak{m}(\omega(b))$ in $\pi_1(H,x)$. Let $q_0 : [0,1] \to [0,1]/(0 \sim 1)$ be the quotient map, and, for $c \in \{a,b\}$, fix an orientation-preserving based homeomorphism $\vartheta_c : \big([0,1]/(0\sim 1), q_0(0)\big) \to (R_c, *)$. Since $\gamma_c(0) = \gamma_c(1) = x$, there is a unique continuous based loop $\overline{\gamma}_c : (R_c,*) \to (\operatorname{int} H, x)$ satisfying $\overline{\gamma}_c \circ \vartheta_c \circ q_0 = \gamma_c$. By the universal property of the wedge sum, there is a unique continuous based map $f_0 : (R,*) \to (\operatorname{int} H, x)$ whose restrictions to R_a and R_b are $\overline{\gamma}_a$ and $\overline{\gamma}_b$, respectively. Thus

$$[f_0|_{R_a}] = [\overline{\gamma}_a] = [\gamma_a] = \mathfrak{m}(\omega(a)), \qquad [f_0|_{R_b}] = [\overline{\gamma}_b] = [\gamma_b] = \mathfrak{m}(\omega(b))$$

in $\pi_1(H,x)$. The map f_0 may have self-intersections away from the wedge point x.

We record the precise relative approximation statement being used. Bryant's Corollary 4.4 [Bry02, p. 235] states that, if $X_0 \subset X$ is a subpolyhedron of a p-dimensional polyhedron, M_0 is a PL n-manifold with $2p+1 \le n$, and $f : X \to M_0$ is continuous with $f|_{X_0}$ a PL embedding, then, for every continuous control function $\varepsilon : X \to (0,\infty)$, the map f is ε-homotopic relative to X_0 to a PL embedding. Apply this statement with

$$X = R, \qquad X_0 = \{*\}, \qquad p = 1, \qquad M_0 = \operatorname{int} H, \qquad \varepsilon \equiv \varepsilon_0 > 0.$$

Here $\operatorname{int} H$ is an open PL n-manifold, $f_0|_{\{*\}}$ is a PL embedding, and $3 = 2p+1 \le n$. Consequently, there is a PL embedding $e : R \hookrightarrow \operatorname{int} H$ that is ε_0-homotopic to f_0 relative to $\{*\}$. Hence $e(*) = x$ and $e \simeq f_0$ relative to $*$ in $\operatorname{int} H$. No condition on $f_0^{-1}(f_0(*))$ occurs among the hypotheses of this corollary. The absence of such a hypothesis is not an oversight. Corollary 4.4 rests on [Bry02, Theorem 4.3]. The existence and PL character of e are supplied directly by Corollary 4.4; we use Theorem 4.3 only to explain why no additional hypothesis $f^{-1}(f(X_0)) = X_0$ is required. Put $p_{\mathrm{rel}} := \dim(X \setminus X_0)$, which is the dimension denoted by p in Theorem 4.3. In the present application, $p_{\mathrm{rel}} = \dim(R \setminus \{*\}) = 1 = p$, where $p = \dim R$ is the parameter in Corollary 4.4. Thus $p_{\mathrm{rel}} \le n$, and $f_0|_{\{*\}}$ is PL and nondegenerate on the one-vertex triangulation of $\{*\}$. Theorem 4.3 therefore gives an approximating map $\widetilde{f}$, relative to $X_0 = \{*\}$, satisfying

$$\dim\big(S(\widetilde{f}) \setminus X_0\big) \le 2p_{\mathrm{rel}} - n = 2 - n < 0, \qquad S(\widetilde{f}) := \overline{\{z \in X : \widetilde{f}^{-1}(\widetilde{f}(z)) \ne \{z\}\}}.$$

Here the closure is taken in X. Bryant denotes this approximating map by f'.[3] Because the right-hand side of the dimension estimate in the preceding display is negative, Bryant's dimension estimate asserts that $S(\widetilde{f}) \setminus X_0 = \varnothing$. If $\widetilde{f}(z_1) = \widetilde{f}(z_2)$ for two distinct points, then both z_1 and z_2 belong to $S(\widetilde{f})$ and hence to X_0, contradicting the injectivity of $\widetilde{f}|_{X_0} = f_0|_{X_0}$. Thus no point of $X \setminus X_0$ can be identified either with another such point or with a point of X_0; this is precisely why no hypothesis $f^{-1}(f(X_0)) = X_0$ is needed. In particular, possible interior returns of the original petal loops to x do not affect the application. No metric bound on e is needed; Step 2 constrains its neighborhood.

[3] In the two dimension estimates following the introduction of f', Bryant prints S_f; in context the intended singular set is $S_{f'}$.

Consequently, $\Gamma' := e(R)$ is an embedded PL rank-two rose with wedge vertex $x \in \operatorname{int} H$, and its oriented petal circles, regarded as based loops at x, satisfy

$$[e|_{R_a}] = [f_0|_{R_a}] = \mathfrak{m}(\omega(a)), \qquad [e|_{R_b}] = [f_0|_{R_b}] = \mathfrak{m}(\omega(b)) \qquad \text{in } \pi_1(H, x),$$

because a based homotopy in $\operatorname{int} H$ restricts to each petal circle as a based homotopy of loops, and the inclusion induces an isomorphism $\pi_1(\operatorname{int} H, x) \cong \pi_1(H, x)$.

Step 2 (construction of a thin regular neighborhood). Since Γ' is compact and lies in the open set $\operatorname{int} H$, the distance $d_1 := \operatorname{dist}_{g_{st}}\big(\Gamma', S^n \setminus \operatorname{int} H\big)$ is positive. The open set $W := \{z \in S^n : d_{g_{st}}(z, \Gamma') < d_1/2\}$ satisfies $\Gamma' \subset W \Subset \operatorname{int} H$.

Apply Lemma 6.3 to the finite graph $\Gamma' \subset W \subset \operatorname{int} B$. For the given parameters $\tau > 1$, $\delta > 0$, and $\eta > 0$, it produces a finite family of open balls $\{U'_\ell\}$ such that $\Gamma' \subset U' := \bigcup_\ell U'_\ell \Subset W$, $\operatorname{diam} U'_\ell < \delta$, and $\sum_\ell (\operatorname{diam} U'_\ell)^\tau < \eta$. The lemma also gives an open set O satisfying $\Gamma' \subset O \Subset U'$ and $\operatorname{cap}(O) < \varepsilon$.

We construct a closed regular neighborhood of Γ' inside O. Choose a triangulation T of S^n, in the fixed PL structure, such that Γ' is the underlying polyhedron of a full subcomplex L; fullness can be achieved by a derived subdivision. Since Γ' is a rank-two rose, L is nonempty; and since $\dim |L| = \dim \Gamma' = 1 < n = \dim S^n$, one has $L \neq T$. Thus L is a nonempty, proper, full subcomplex, so Lemma 6.4 applies. Let $f_L : |T| = S^n \to [0, 1]$ be the simplicial map that takes value 0 at the vertices of L and value 1 at all other vertices. By that lemma, $f_L^{-1}(0) = \Gamma'$, and, for every $c \in (0, 1)$, the set $f_L^{-1}([0, c])$ is a closed regular neighborhood of Γ' and a compact PL n-manifold, with $\partial\big(f_L^{-1}([0, c])\big) = f_L^{-1}(c)$. For $\varepsilon' \in (0, 1)$ put $N_{\varepsilon'} := f_L^{-1}([0, \varepsilon'])$.

The sets $N_{\varepsilon'}$ are nested compacta with $\bigcap_{0<\varepsilon'<1} N_{\varepsilon'} = f_L^{-1}(0) = \Gamma'$. Since $O \Subset U' \Subset W \Subset \operatorname{int} H \subset \operatorname{int} B \subsetneq S^n$, the set $S^n \setminus O$ is nonempty and compact. It is disjoint from $f_L^{-1}(0) = \Gamma'$, and hence $m_O := \min_{S^n \setminus O} f_L > 0$. Choose $0 < \varepsilon' < \min\{1, m_O\}$ and set $H' := N_{\varepsilon'}$. A point with $f_L \le \varepsilon' < m_O$ cannot lie in $S^n \setminus O$, so $\Gamma' \subset \operatorname{int} H' = f_L^{-1}([0, \varepsilon')) \subset H' \subset O$, where the equality follows from Lemma 6.4. In particular, $\Gamma' \subset \operatorname{int} H'$ and $x \in \operatorname{int} H'$, as required.

Retain the auxiliary data $\mathcal{T}' := T, \qquad f' := f_L, \qquad t' := \varepsilon'$. Then $(f')^{-1}(0) = \Gamma'$ and $H' = (f')^{-1}([0, t'])$.

Since $H' \subset O \Subset U'$, one has $H' \subset \bigcup_\ell U'_\ell$. The diameter and τ-content bounds are properties of the fixed family $\{U'_\ell\}$. Monotonicity of capacity with respect to inclusion gives $\operatorname{cap}(H') \le \operatorname{cap}(O) < \varepsilon$.

Step 3 (the marking identity and injectivity of the inclusion). Since a regular neighborhood collapses onto its spine [Bry02, Theorem 3.16], the resulting deformation retraction of H' onto Γ' fixes Γ' (and therefore the base point x) pointwise; in particular H' is connected, retracting onto the connected rose Γ'. Consequently, the inclusion $\iota : (\Gamma', x) \hookrightarrow (H', x)$ is a based homotopy equivalence and induces an isomorphism $\iota_* : \pi_1(\Gamma', x) \xrightarrow{\cong} \pi_1(H', x)$. The embedding $e : (R, *) \to (\Gamma', x)$ is a based PL homeomorphism, so e_* is an isomorphism. By the Seifert–van Kampen theorem, the fundamental group of the standard rose is free of rank two on its two oriented petal circles: $\pi_1(R, *) \cong \langle a, b \rangle \cong F_2$. Thus $\pi_1(\Gamma', x)$ is likewise free of rank two, generated by the images of the two oriented petal circles of Γ'.

Define $\mathfrak{m}' := \iota_* \circ e_* : F_2 \to \pi_1(H', x)$. Both factors are isomorphisms, so $\mathfrak{m}'$ is a marking. Explicitly, in $\pi_1(H', x)$, $\mathfrak{m}'(a) = [\iota \circ e|_{R_a}], \qquad \mathfrak{m}'(b) = [\iota \circ e|_{R_b}]$.

Let $i : H' \hookrightarrow H$ be the inclusion. For the generator a, $i_*\mathfrak{m}'(a) = (i \circ \iota \circ e)_*(a) = [e|_{R_a}]$ in $\pi_1(H, x)$, because $i \circ \iota \circ e$ regards $e|_{R_a}$ as a loop in H. By Step 1, $[e|_{R_a}] = \mathfrak{m}(\omega(a))$. The same identity holds for b. Thus the homomorphisms $i_* \circ \mathfrak{m}'$ and $\mathfrak{m} \circ \omega$ agree on the free generators of F_2 and are therefore equal: $i_* \circ \mathfrak{m}' = \mathfrak{m} \circ \omega$.

Rearranging this identity yields the factorization $i_* = \mathfrak{m} \circ \omega \circ (\mathfrak{m}')^{-1}$. Here $\mathfrak{m}$ and $(\mathfrak{m}')^{-1}$ are isomorphisms, while ω is injective by Lemma 6.2. Hence i_* is injective. $\square$

For the iteration, we use the rates $\varepsilon = \delta = \eta = 2^{-(j+1)}$ and $\tau = 1 + \frac{1}{j+1}$; these yield the estimates in Proposition 6.6(2)–(3) that are used in Proposition 6.7.

Proposition 6.6 (The tower). *There are closed regular neighborhoods $V_j \subset \operatorname{int} B$ of embedded rank-two roses Γ_j whose petals are oriented and labeled a and b, a common wedge point $x \in \bigcap_j \operatorname{int} V_j$, finite ball families $\mathcal{U}_j = \{U_{j,\ell}\}$, and markings $\mathfrak{m}_j : F_2 \xrightarrow{\cong} \pi_1(V_j, x)$ such that for all $j \geq 1$:*

1. *$V_{j+1} \Subset \operatorname{int} V_j$;*
2. *$\operatorname{cap}(V_j) < 2^{-j}$;*
3. *$V_j \subset \bigcup_\ell U_{j,\ell}$ with $\operatorname{diam} U_{j,\ell} < 2^{-j}$ and $\sum_\ell (\operatorname{diam} U_{j,\ell})^{1+1/j} < 2^{-j}$;*
4. *$(i_j)_* \circ \mathfrak{m}_{j+1} = \mathfrak{m}_j \circ \omega$ for $i_j : V_{j+1} \hookrightarrow V_j$;*
5. *$\mathfrak{m}_j(a)$ and $\mathfrak{m}_j(b)$ are represented by the corresponding oriented petal loops of Γ_j based at x;*
6. *there are a triangulation $\mathcal{T}_j$ of S^n in which Γ_j is the polyhedron of a nonempty, proper, full subcomplex, the simplicial map $f_j : |\mathcal{T}_j| \to [0,1]$ taking value 0 on the vertices of this subcomplex and value 1 on all other vertices, and a number $t_j \in (0,1)$ such that*

$$f_j^{-1}(0) = \Gamma_j, \qquad V_j = f_j^{-1}([0, t_j]),$$

and, for every $c \in (0,1)$, $f_j^{-1}([0,c])$ is a closed regular neighborhood of Γ_j with

$$\partial\big(f_j^{-1}([0,c])\big) = f_j^{-1}(c).$$

Consequently, for $k > j$ the inclusion $V_k \hookrightarrow V_j$ induces $\mathfrak{m}_j \circ \omega^{k-j} \circ \mathfrak{m}_k^{-1}$ on π_1; in particular it is injective.

The proof is inductive. At each stage, Lemma 6.5 constructs in the preceding neighborhood a marked rose whose oriented petals represent $\mathfrak{m}_j(\omega(a))$ and $\mathfrak{m}_j(\omega(b))$. Equivalently, its marking satisfies $(i_j)_* \circ \mathfrak{m}_{j+1} = \mathfrak{m}_j \circ \omega$. The lemma also provides capacity, diameter, and Hausdorff-content bounds for a sufficiently thin regular neighborhood. The exponentially decaying parameters yield the estimates used for the limit in Proposition 6.7.

Proof. Fix a compatible PL structure on S^n and pass to suitable subdivisions as necessary so that each finite PL polyhedron under consideration is a subcomplex and x is a vertex. Let $(R, *)$ denote the standard rank-two rose $R = R_a \vee R_b$, with oriented petal circles labeled a, b and with $*$ as their common vertex. Triangulate each petal by at least three simplicial edges. We identify $\pi_1(R,*)$, freely generated by the two oriented petal loops, with $F_2 = \langle a, b\rangle$.

Step 1 (the base stage). We prove the assertion for $j = 1$. Fix an open PL n-ball W_1 with $\overline{W_1} \subset \operatorname{int} B$ and a PL embedding $e_1 : (R,*) \hookrightarrow W_1$. For example, under a PL homeomorphism $W_1 \cong \mathbb{R}^n$, realize R as the union of the boundaries of two 2-simplices in $\mathbb{R}^2 \subset \mathbb{R}^n$ meeting at a single common vertex, and transport this compact 1-polyhedron back to W_1. Set $\Gamma_1 := e_1(R)$ and $x := e_1(*)$. Lemma 6.3, applied to (Γ_1, W_1) with $(\tau, \delta, \eta) = (2, 2^{-1}, 2^{-1})$, and with $\varepsilon = 2^{-1}$ in part (b), produces a finite family $\mathcal{U}_1 = \{U_{1,\ell}\}$ of open balls and an open set O_1 with

$$\Gamma_1 \subset O_1 \Subset U^{(1)} := \bigcup_\ell U_{1,\ell} \Subset W_1, \qquad \operatorname{diam} U_{1,\ell} < 2^{-1}, \qquad \sum_\ell (\operatorname{diam} U_{1,\ell})^2 < 2^{-1},$$

together with $\operatorname{cap}(O_1) < 2^{-1}$. Choose a triangulation of S^n in which Γ_1 is the polyhedron of a nonempty proper subcomplex, and pass to a derived subdivision. Denote the resulting triangulation by $\mathcal{T}_1$; the corresponding subcomplex with polyhedron Γ_1 is nonempty, proper, and full. Let $f_1 : |\mathcal{T}_1| \to [0,1]$ take value 0 on the vertices of that subcomplex and value 1 on all other vertices. By Lemma 6.4, $f_1^{-1}(0) = \Gamma_1$ and, for every $c \in (0,1)$, $f_1^{-1}([0,c])$ is a closed regular neighborhood of Γ_1 with $\partial\big(f_1^{-1}([0,c])\big) = f_1^{-1}(c)$. Since $m_{O_1} := \min_{S^n\setminus O_1} f_1 > 0$, choose $0 < t_1 < \min\{1, m_{O_1}\}$ and set $V_1 := f_1^{-1}([0,t_1])$. Then $\Gamma_1 \subset \operatorname{int} V_1 \subset V_1 \subset O_1$. Thus $\mathcal{T}_1$, f_1, and t_1 are the data required in property (6) at $j = 1$. Moreover, V_1 collapses onto Γ_1 by [Bry02, Theorem 3.16]; hence the inclusion $(\Gamma_1, x) \hookrightarrow (V_1, x)$ is a based homotopy equivalence. In particular, $V_1 \subset W_1 \subset \operatorname{int} B$, $x \in \operatorname{int} V_1$, and V_1 is connected. Property (2) at $j = 1$ follows from $V_1 \subset O_1$ by monotonicity of the capacity; property (3) at $j = 1$ holds because $V_1 \subset O_1 \subset U^{(1)}$, while the diameter and content bounds are properties of the fixed family $\mathcal{U}_1$. Define

$$\mathfrak{m}_1 := (\iota_1)_* \circ (e_1)_* : F_2 \longrightarrow \pi_1(V_1, x), \qquad \iota_1 : (\Gamma_1, x) \hookrightarrow (V_1, x);$$

here $(e_1)_*$ is an isomorphism because e_1 is a based PL homeomorphism onto Γ_1, and $(\iota_1)_*$ is an isomorphism, so $\mathfrak{m}_1$ is a marking. By its definition, $\mathfrak{m}_1(a)$ and $\mathfrak{m}_1(b)$ are represented by the correspondingly oriented a- and b-petals of Γ_1, which proves property (5) at $j = 1$. The triple $(V_1, x, \mathfrak{m}_1)$ therefore satisfies every hypothesis of Lemma 6.5: $V_1 \subset \operatorname{int} B$ is a closed regular neighborhood of the embedded rank-two rose Γ_1, the point x lies in $\operatorname{int} V_1$, and $\mathfrak{m}_1 : F_2 \xrightarrow{\cong} \pi_1(V_1, x)$ is a marking.

Step 2 (the inductive stage). Let $j \geq 1$ and suppose stage j has been constructed: an embedded PL rank-two rose Γ_j with wedge vertex x, a closed regular neighborhood $V_j \subset \operatorname{int} B$ of Γ_j with $x \in \operatorname{int} V_j$, and a marking $\mathfrak{m}_j : F_2 \xrightarrow{\cong} \pi_1(V_j, x)$. Properties (2) and (3) at index j are verified when the corresponding stage is constructed: in Step 1 for $j = 1$ and in the present step for $j \geq 2$. These data are exactly the hypotheses of Lemma 6.5 for $(H, x, \mathfrak{m}) = (V_j, x, \mathfrak{m}_j)$; apply it with the parameters $\varepsilon = \delta = \eta = 2^{-(j+1)}$ and $\tau = 1 + \frac{1}{j+1}$ (> 1), and name its output

$$\begin{aligned} &\Gamma_{j+1} := \Gamma', && V_{j+1} := H', && \mathfrak{m}_{j+1} := \mathfrak{m}', \\ &\mathcal{U}_{j+1} := \{U_{j+1,\ell}\} := \{U'_\ell\}, && \mathcal{T}_{j+1} := \mathcal{T}', && \\ &f_{j+1} := f', && t_{j+1} := t'. && \end{aligned}$$

The conclusions of the lemma then read: $\Gamma_{j+1} \subset \operatorname{int} V_j$ is an embedded rank-two PL rose with wedge vertex x; V_{j+1} is a closed regular neighborhood of Γ_{j+1} with $V_{j+1} \Subset \operatorname{int} V_j$, which is property (1);

$$\begin{aligned} V_{j+1} \subset \bigcup_{\ell} U_{j+1,\ell} \Subset \operatorname{int} V_j, \\ \operatorname{diam} U_{j+1,\ell} < 2^{-(j+1)}, \\ \sum_{\ell} (\operatorname{diam} U_{j+1,\ell})^{1+\frac{1}{j+1}} < 2^{-(j+1)}, \end{aligned}$$

which is property (3) at index $j+1$, with the required content exponent $1 + \frac{1}{j+1}$. Moreover, $\operatorname{cap}(V_{j+1}) < 2^{-(j+1)}$, which is property (2) at index $j+1$. Finally, for the inclusion $i_j : V_{j+1} \hookrightarrow V_j$, $(i_j)_* \circ \mathfrak{m}_{j+1} = \mathfrak{m}_j \circ \omega$, which is property (4) at index j (the lemma's final assertion, injectivity of $(i_j)_*$, is subsumed in Step 3 of the present proof). The petal assertion of Lemma 6.5 gives property (5) at index $j+1$, and its final auxiliary-data assertion gives property (6) at index $j+1$. To continue the recursion, we verify that stage $j+1$ again satisfies the hypotheses of Lemma 6.5: V_{j+1} is a closed regular neighborhood of the embedded rose Γ_{j+1}, and $V_{j+1} \subset V_j \subset \operatorname{int} B$; the marking $\mathfrak{m}_{j+1}$ is supplied by the lemma; and the lemma also gives $x \in \Gamma_{j+1} \subset \operatorname{int} V_{j+1}$. Hence $x \in \bigcap_{j \geq 1} \operatorname{int} V_j$. Each V_j is connected: this was shown for $j = 1$ in Step 1, and for $j \geq 2$ it follows because V_j collapses onto the connected graph Γ_j [Bry02, Theorem 3.16]. This completes the recursion.

Step 3 (composites and injectivity). Fix $j \geq 1$ and write $\iota_j^k : V_k \hookrightarrow V_j$ for the inclusion whenever $k > j$. We prove $(\iota_j^k)_* = \mathfrak{m}_j \circ \omega^{k-j} \circ \mathfrak{m}_k^{-1}$ by induction on $k - j$. For $k = j+1$ this is property (4), rearranged using the invertibility of $\mathfrak{m}_{j+1}$: $(i_j)_* = \mathfrak{m}_j \circ \omega \circ \mathfrak{m}_{j+1}^{-1}$. Assuming the identity for k, functoriality of π_1 applied to $\iota_j^{k+1} = \iota_j^k \circ i_k$ gives

$$(\iota_j^{k+1})_* = (\iota_j^k)_* \circ (i_k)_* = \left(\mathfrak{m}_j \circ \omega^{k-j} \circ \mathfrak{m}_k^{-1}\right) \circ \left(\mathfrak{m}_k \circ \omega \circ \mathfrak{m}_{k+1}^{-1}\right) = \mathfrak{m}_j \circ \omega^{k-j+1} \circ \mathfrak{m}_{k+1}^{-1}.$$

Since ω is injective (Lemma 6.2), so is every power ω^{k-j}, and pre- and post-composition with the isomorphisms $\mathfrak{m}_k^{-1}$ and $\mathfrak{m}_j$ preserves injectivity; hence each $(\iota_j^k)_*$ is injective. $\square$

For the remainder of this section, fix such a tower and set

$$K := \bigcap_{j \geq 1} V_j, \qquad M := S^n \setminus K.$$

The analytic arguments below use only the properties of K stated in the next proposition. The topological arguments additionally use the nesting and the markings.

Proposition 6.7 (The limit continuum). *K is a nondegenerate continuum with $K \subset V_{j+1} \subset \operatorname{int} V_j$ for all j, and $\operatorname{cap}(K) = 0$, $\mathcal{H}^1(K) > 0$, and $\dim_{\mathcal{H}} K = 1$.*

Proof. The sets V_j are nonempty compact connected sets nested by inclusion, so their intersection K is nonempty and compact. To prove connectedness, suppose that $K = A \sqcup A'$ is a separation into nonempty closed subsets. The compact sets

A and A' have disjoint open neighborhoods U and U' in S^n. The compact sets $V_j \setminus (U \cup U')$ decrease with j and have empty intersection, because $K \subset U \cup U'$. Hence the finite-intersection property gives $V_j \setminus (U \cup U') = \varnothing$ for some j, so $V_j \subset U \cup U'$. Since V_j contains K, it meets both U and U', contrary to the connectedness of V_j. Thus K is connected.

We next prove that K is nondegenerate. By Proposition 6.6, $x \in \bigcap_j \operatorname{int} V_j \subset K$. Suppose, to the contrary, that $K = \{x\}$. Choose a contractible coordinate ball Q such that $x \in Q \Subset \operatorname{int} V_1$. The compact sets $V_j \setminus Q$ decrease to $K \setminus Q = \varnothing$. By the finite intersection property, $V_k \setminus Q = \varnothing$ for some k. Replacing k by $\max\{k, 2\}$ and using the nesting of the V_j, we may assume that $k > 1$. Thus $V_k \subset Q$. The based inclusion $(V_k, x) \hookrightarrow (V_1, x)$ therefore factors through (Q, x) and hence induces the zero homomorphism on fundamental groups. This contradicts Proposition 6.6, because the induced homomorphism is injective and $\pi_1(V_k, x) \cong F_2 \neq 1$. Thus K is nondegenerate. Choose $z_0, z_1 \in K$ realizing its positive spherical diameter $D_K := \operatorname{diam}_{g_{st}} K$. The image of the connected set K under the 1-Lipschitz function $z \mapsto d_{g_{st}}(z_0, z)$ is a connected subset of $[0, D_K]$ containing both endpoints, and hence is $[0, D_K]$. Since Hausdorff measure is nonincreasing under 1-Lipschitz maps, $0 < D_K = \mathcal{H}^1([0, D_K]) \leq \mathcal{H}^1(K)$, so $\dim_{\mathcal{H}} K \geq 1$.

By monotonicity and Proposition 6.6(2), $0 \leq \operatorname{cap}(K) \leq \operatorname{cap}(V_j) < 2^{-j}$ for every j, and therefore $\operatorname{cap}(K) = 0$. Fix $\tau > 1$ and take j sufficiently large that $p_j := 1 + \frac{1}{j} < \tau$ and $2^{-j} < 1$. Put $d_{j,\ell} := \operatorname{diam} U_{j,\ell}$. Since $d_{j,\ell} < 2^{-j} < 1$, one has $d_{j,\ell}^{\tau} \leq d_{j,\ell}^{p_j}$. Proposition 6.6(3) therefore gives

$$\mathcal{H}^{\tau}_{2^{-j}}(K) \leq \sum_{\ell} d_{j,\ell}^{\tau} \leq \sum_{\ell} d_{j,\ell}^{p_j} < 2^{-j}.$$

Thus, for all sufficiently large j, $\mathcal{H}^{\tau}_{2^{-j}}(K) < 2^{-j}$. Since $2^{-j} \downarrow 0$, the definition of Hausdorff measure gives $\mathcal{H}^{\tau}(K) = 0$. This holds for every $\tau > 1$, and hence $\dim_{\mathcal{H}} K \leq 1$. Together with the preceding lower bound, this gives $\dim_{\mathcal{H}} K = 1$. $\square$

For $n \geq 5$, the capacity conclusion also follows from the dimension conclusion. Indeed, then $s = (n-2)/2 > 1$, and the Lipschitz map $\sigma|_B$ sends K to a set satisfying $\mathcal{H}^s(\sigma(K)) = 0$. The comparison $\Lambda_h \leq \mathcal{H}^s$ and the finite-critical-measure theorem used in Step 2 of Lemma 6.3 therefore give $\operatorname{cap}(K) = 0$. At the endpoint $n = 4$, where $s = 1$, the Hausdorff-content estimate above gives no upper bound for $\mathcal{H}^1(K)$ and, in particular, does not show that this measure is finite. Thus the identity $\dim_{\mathcal{H}} K = 1$ alone gives no capacity conclusion at the critical exponent. The capacity-thinning condition in Proposition 6.6(2) is therefore needed there.

6.2 Topology of the pair (K, M)

This subsection proves that M is contractible but not simply connected at infinity, as required in Theorem 6.1, and additionally records that K is Čech-acyclic. Two properties of the tower are used. Its nesting implies both the cofinality of the V_j among neighborhoods of K and the exhaustion of M by the C_j, as shown in Lemma 6.8. The two properties of ω established in Lemma 6.2 are used, through the marking identity in Proposition 6.6(4), for distinct purposes: the equality $H_1(\omega) = 0$

yields the vanishing results, whereas the injectivity of ω yields the obstruction to simple connectivity at infinity.

Lemma 6.8 (Cofinality and exhaustion). *Put $E_j := S^n \setminus V_j$ and $C_j := S^n \setminus \operatorname{int} V_j$. Then:* (a) *every open $U \supset K$ contains some V_j; the V_j are cofinal among compact neighborhoods of K;* (b) *the E_j are open and increase to M, and every compact $D \subset M$ satisfies $D \subset E_j \subset C_j$ for some j;* (c) *the C_j are compact and nested, exhaust M, and $C_j \subset E_{j+1} \subset \operatorname{int}_M C_{j+1}$.*

Proof. (a) Let $U \supset K$ be open. Each $V_j \setminus U = V_j \cap (S^n \setminus U)$ is a closed subset of the compact set V_j, hence compact, and these sets decrease in j because the V_j do; their intersection is $\left(\bigcap_j V_j\right) \setminus U = K \setminus U = \varnothing$. By Cantor's intersection theorem, a decreasing sequence of nonempty compacta has nonempty intersection, so some $V_{j_0} \setminus U = \varnothing$, that is, $V_{j_0} \subset U$. Since $K \subset V_{j+1} \subset \operatorname{int} V_j$, each V_j is a compact neighborhood of K, and every neighborhood of K contains an open set $U \supset K$ and hence some V_{j_0}. Thus the V_j are cofinal.

(b) Each E_j is open with $E_j \subset E_{j+1}$, and $\bigcup_j E_j = S^n \setminus \bigcap_j V_j = M$. Given compact $D \subset M$, apply (a) to the open set $U = S^n \setminus D \supset K$: some $V_j \subset S^n \setminus D$, that is, $D \subset E_j$; and $E_j \subset C_j$ because $\operatorname{int} V_j \subset V_j$.

(c) Each C_j is compact, $C_j \subset C_{j+1}$ because $\operatorname{int} V_{j+1} \subset \operatorname{int} V_j$, and $C_j \subset M$ because $K \subset \operatorname{int} V_j$; by part (b), the C_j exhaust M. Finally, $V_{j+1} \subset \operatorname{int} V_j$ gives $C_j \cap V_{j+1} = \varnothing$, that is, $C_j \subset E_{j+1}$; and E_{j+1} is open in S^n, contained in M (as $K \subset V_{j+1}$) and in C_{j+1}, whence $E_{j+1} \subset \operatorname{int}_M C_{j+1}$. $\square$

The next lemma is where $H_1(\omega) = 0$ enters the topology of actual spaces: the universal-coefficient isomorphism is natural, so the abstract algebra becomes a statement about restriction maps.

Lemma 6.9 (Cohomology of the stages). *$V_j \simeq S^1 \vee S^1$, so $\widetilde{H}^r(V_j;\mathbb{Z}) = 0$ for $r \neq 1$ and $H^1(V_j;\mathbb{Z}) \cong \mathbb{Z}^2$; and every restriction $(i_j)^* : H^1(V_j;\mathbb{Z}) \to H^1(V_{j+1};\mathbb{Z})$ is zero.*

Proof. Each V_j is a closed regular neighborhood of the embedded rank-two rose Γ_j, and a regular neighborhood collapses onto its spine [Bry02, Theorem 3.16]. The deformation retraction associated with the collapse fixes Γ_j pointwise, so the inclusion $\Gamma_j \hookrightarrow V_j$ is a homotopy equivalence. Since Γ_j is PL homeomorphic to the standard rose, $V_j \simeq S^1 \vee S^1$. Thus $H^1(V_j;\mathbb{Z}) \cong \mathbb{Z}^2$, $\widetilde{H}^0(V_j;\mathbb{Z}) = 0$ by connectedness, and $H^r(V_j;\mathbb{Z}) = 0$ for $r \geq 2$.

For a path-connected space, the Hurewicz homomorphism $h : \pi_1(X,x) \to H_1(X;\mathbb{Z})$ is the abelianization and is natural in based maps. Write h_j for this map on V_j. Since $H_1(V_j;\mathbb{Z})$ is abelian, the composite $h_j \circ \mathfrak{m}_j$ kills $[F_2,F_2]$ and descends to the isomorphism $\overline{\mathfrak{m}}_j : \mathbb{Z}^2 = F_2^{\mathrm{ab}} \xrightarrow{\cong} H_1(V_j;\mathbb{Z})$ induced on abelianizations by the group isomorphism $\mathfrak{m}_j$. Apply h_j to the marking identity $(i_j)_* \circ \mathfrak{m}_{j+1} = \mathfrak{m}_j \circ \omega$ of Proposition 6.6(4): by naturality of h, the left side becomes $(i_j)_* \circ h_{j+1} \circ \mathfrak{m}_{j+1}$, and both sides factor through the surjection $F_2 \to F_2^{\mathrm{ab}}$, giving $(i_j)_* \circ \overline{\mathfrak{m}}_{j+1} = \overline{\mathfrak{m}}_j \circ H_1(\omega)$, where $H_1(\omega) : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the map induced by ω on abelianizations. By Lemma 6.2, the images $\omega(a)$ and $\omega(b)$ are commutators, so $H_1(\omega) = 0$; since $\overline{\mathfrak{m}}_{j+1}$ is surjective, $(i_j)_* = 0$ on $H_1(V_{j+1};\mathbb{Z})$.

The universal-coefficient sequence

$$0 \to \operatorname{Ext}\big(H_0(X;\mathbb{Z}),\mathbb{Z}\big) \to H^1(X;\mathbb{Z}) \xrightarrow{\text{ev}} \operatorname{Hom}\big(H_1(X;\mathbb{Z}),\mathbb{Z}\big) \to 0$$

is natural in X; only the naturality of the evaluation map ev is used, not the (non-natural) splitting. Since $H_0(V_j;\mathbb{Z})$ and $H_0(V_{j+1};\mathbb{Z})$ are free, both Ext terms vanish and ev is an isomorphism for both spaces; under these identifications $(i_j)^*$ is precomposition with $(i_j)_*$,

$$(i_j)^* = \operatorname{Hom}\big((i_j)_*,\mathbb{Z}\big) : \operatorname{Hom}\big(H_1(V_j;\mathbb{Z}),\mathbb{Z}\big) \longrightarrow \operatorname{Hom}\big(H_1(V_{j+1};\mathbb{Z}),\mathbb{Z}\big).$$

Since $(i_j)_* = 0$, it follows that $(i_j)^* = 0$. $\square$

We next compute the cohomology groups of K. For a compact subset $A \subset S^n$, we use the standard neighborhood realization of Čech cohomology

$$\check{H}^r(A;\mathbb{Z}) \cong \varinjlim_{U\supset A} H^r(U;\mathbb{Z}),$$

where U ranges over the open neighborhoods of A in S^n, ordered by reverse inclusion, and the transition homomorphisms are restrictions; see [Hat02, p. 257]. Thus the relevant construction is a direct limit, and no inverse-limit derived term occurs.

Proposition 6.10 (Čech-acyclicity of the limit continuum)**.** *$\tilde{\check{H}}^r(K;\mathbb{Z}) = 0$ for all $r \geq 0$.*

Proof. Let $i_j : V_{j+1} \hookrightarrow V_j$ denote the bonding inclusion and let $u_j : \operatorname{int} V_{j+1} \hookrightarrow \operatorname{int} V_j$ be its restriction. By construction, $K \subset \operatorname{int} V_j$ for every j, and Lemma 6.8(a) shows that the decreasing sequence $(\operatorname{int} V_j)_{j\geq 1}$ is cofinal among the open neighborhoods of K: if $U \supset K$ is open, then some $V_j \subset U$, and hence $\operatorname{int} V_j \subset U$. Since V_j is a compact PL manifold with boundary, the PL collaring theorem [Bry02, Corollary 2.5] allows its boundary to be pushed inward along a collar. It follows that the inclusion $\lambda_j : \operatorname{int} V_j \hookrightarrow V_j$ is a homotopy equivalence. Consequently the restriction $\rho_j^r := \lambda_j^* : H^r(V_j;\mathbb{Z}) \xrightarrow{\cong} H^r(\operatorname{int} V_j;\mathbb{Z})$ is an isomorphism. Functoriality of restriction gives $u_j^* \circ \rho_j^r = \rho_{j+1}^r \circ i_j^*$. Thus the direct systems defined by the V_j and by their interiors are naturally isomorphic. The neighborhood realization above and cofinality therefore give

$$\check{H}^r(K;\mathbb{Z}) \cong \varinjlim_{j}\Big(H^r(V_1;\mathbb{Z}) \xrightarrow{i_1^*} H^r(V_2;\mathbb{Z}) \xrightarrow{i_2^*} \cdots\Big).$$

By Lemma 6.9, every V_j is connected, $H^r(V_j;\mathbb{Z}) = 0$ for $r \geq 2$, and $i_j^* : H^1(V_j;\mathbb{Z}) \to H^1(V_{j+1};\mathbb{Z})$ is the zero homomorphism. For $r \geq 2$ the direct system consists of zero groups, so $\check{H}^r(K;\mathbb{Z}) = 0$. For $r = 1$ the system is $\mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \cdots$, and its colimit is zero: the class in the colimit of an element at the jth stage equals the class of its image at the $(j+1)$st stage, which is zero. Thus $\check{H}^1(K;\mathbb{Z}) = 0$.

For $r = 0$, Proposition 6.7 shows that K is nonempty and connected. Since $\check{H}^0(K;\mathbb{Z})$ is the group of locally constant integer-valued functions on K, and on a connected space these are the constants, we have $\check{H}^0(K;\mathbb{Z}) \cong \mathbb{Z}$ and $\tilde{\check{H}}^0(K;\mathbb{Z}) = 0$. Combining the three cases proves $\tilde{\check{H}}^r(K;\mathbb{Z}) = 0$ for every $r \geq 0$. $\square$

Proposition 6.10 records an intrinsic property of the limit continuum K, but it is not needed for Theorem 6.1: the proof of Proposition 6.12 below instead uses natural Alexander duality at the finite stages and then passes to the direct limit. Alternatively, compactum Alexander duality [Hat02, pp. 255–257], applied directly to Proposition 6.10, would also imply the acyclicity of M. We retain the finite-stage argument because it makes the transition maps explicit.

In the next lemma the content is the naturality, not the isomorphism: under the Alexander-duality identifications, the complement map corresponds, up to the conventional sign, to the zero restriction map. Thus the complements inherit vanishing transitions from Lemma 6.9.

Lemma 6.11 (Natural Alexander duality). *Let $i_j : V_{j+1} \hookrightarrow V_j$ and $\jmath_j : E_j \hookrightarrow E_{j+1}$ denote the inclusions. Fix an orientation class $\mathfrak{o} \in H_n(S^n;\mathbb{Z})$. Alexander duality gives isomorphisms*

$$D_{j,i} : \widetilde{H}_i(E_j;\mathbb{Z}) \longrightarrow \widetilde{H}^{n-i-1}(V_j;\mathbb{Z}), \qquad i \geq 0,$$

such that, for signs $\varepsilon_i \in \{\pm 1\}$ depending only on the degree and the cap-product convention,

$$D_{j+1,i} \circ (\jmath_j)_* = \varepsilon_i \, i_j^* \circ D_{j,i}.$$

Consequently,

$$\widetilde{H}_i(E_j;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2, & i = n-2, \\ 0, & i \neq n-2, \end{cases} \tag{15}$$

and every transition $(\jmath_j)_* : \widetilde{H}_i(E_j;\mathbb{Z}) \to \widetilde{H}_i(E_{j+1};\mathbb{Z})$ vanishes for $i \geq 1$.

Proof. Each V_j is a nonempty, locally contractible compact polyhedron and a proper subset of S^n. Thus the reduced-complement form of Alexander duality [Hat02, Corollary 3.45] applies and, together with Lemma 6.9, gives (15), where reduced cohomology in negative degrees is understood to vanish. In particular, taking $i = 0$, each E_j is path-connected. No naturality is needed for this step, as the groups are computed from the isomorphism statement alone.

Fix $0 \leq i \leq n-2$. Since $n - i - 1 \geq 1$, ordinary and reduced cohomology agree canonically in the degree under consideration; we use this identification below. Since $E_j = S^n \setminus V_j \subset E_{j+1}$ and E_j is nonempty, the reduced long exact sequence of the pair (S^n, E_j) contains

$$\widetilde{H}_{i+1}(S^n) \longrightarrow H_{i+1}(S^n, E_j;\mathbb{Z}) \xrightarrow{\partial_{j,i}} \widetilde{H}_i(E_j;\mathbb{Z}) \longrightarrow \widetilde{H}_i(S^n).$$

Since both outer groups vanish, $\partial_{j,i}$ is an isomorphism. The cap-product construction in the proof of the compact-set pair-duality theorem, together with the cap-product naturality formula [Hat02, proof of Theorem 3.44 and pp. 248–249], supplies an isomorphism

$$\bar{\gamma}_{\mathfrak{o},j,i+1} : H_{i+1}(S^n, E_j;\mathbb{Z}) \longrightarrow \bar{H}^{n-i-1}(V_j;\mathbb{Z}),$$

determined by the fixed class $\mathfrak{o}$ and natural with respect to inclusions of compact pairs. Here

$$\bar{H}^r(V_j;\mathbb{Z}) := \varinjlim_{U \supset V_j} H^r(U;\mathbb{Z}),$$

where the open neighborhoods of V_j are ordered by reverse inclusion and, for $U' \subset U$, the transition homomorphism is the restriction $H^r(U;\mathbb{Z}) \to H^r(U';\mathbb{Z})$.

Choose a finite triangulation $\mathcal{S}_j^0$ of S^n in which V_j is the polyhedron of a subcomplex $\mathcal{L}_j^0$. Put

$$\mathcal{S}_j := \operatorname{sd}\mathcal{S}_j^0, \qquad \mathcal{L}_j := \operatorname{sd}\mathcal{L}_j^0.$$

Then $|\mathcal{L}_j| = V_j$, and $\mathcal{L}_j$ is full in $\mathcal{S}_j$. Indeed, a simplex of $\operatorname{sd}\mathcal{S}_j^0$ is represented by a chain of simplices of $\mathcal{S}_j^0$; if all its vertices belong to $\operatorname{sd}\mathcal{L}_j^0$, every simplex in the chain belongs to $\mathcal{L}_j^0$. The subcomplex $\mathcal{L}_j$ is nonempty because $\Gamma_j \subset V_j$, and it is proper because $V_j \subset \operatorname{int} B \subsetneq S^n$.

Let $f_{\mathcal{L}_j} : |\mathcal{S}_j| = S^n \to [0,1]$ be the level function of Lemma 6.4; thus $f_{\mathcal{L}_j}^{-1}(0) = V_j$. For $k \geq 1$, set $c_k := 2^{-k}$, $\qquad N_k := f_{\mathcal{L}_j}^{-1}([0,c_k])$. Lemma 6.4 shows that every N_k is a closed regular neighborhood of V_j, with $\operatorname{int} N_k = f_{\mathcal{L}_j}^{-1}([0,c_k))$. Since $c_{k+1} < c_k$, $V_j \subset \operatorname{int} N_{k+1} \subset N_{k+1} \subset \operatorname{int} N_k \subset N_k$. Moreover, $\bigcap_{k\geq 1} N_k = f_{\mathcal{L}_j}^{-1}\big(\bigcap_{k\geq 1}[0,c_k]\big) = f_{\mathcal{L}_j}^{-1}(0) = V_j$.

The open sets $\operatorname{int} N_k$ are cofinal among the open neighborhoods of V_j. To see this, let U be such a neighborhood. If $U \neq S^n$, then $S^n \setminus U$ is a nonempty compact set disjoint from $f_{\mathcal{L}_j}^{-1}(0)$, and hence $m_U := \min_{S^n\setminus U} f_{\mathcal{L}_j} > 0$. For every k with $c_k < m_U$, one has $N_k \subset U$, and therefore $\operatorname{int} N_k \subset U$; the case $U = S^n$ is immediate.

Each N_k collapses onto V_j by [Bry02, Theorem 3.16], so $V_j \hookrightarrow N_k$ is a homotopy equivalence. Since N_k is a compact PL manifold with boundary, a PL collar of ∂N_k [Bry02, Corollary 2.5] shows that $\operatorname{int} N_k \hookrightarrow N_k$ is also a homotopy equivalence. The two-out-of-three property applied to $V_j \hookrightarrow \operatorname{int} N_k \hookrightarrow N_k$ therefore shows that, for every $r \geq 0$, restriction induces an isomorphism $\rho_k^r : H^r(\operatorname{int} N_k;\mathbb{Z}) \xrightarrow{\cong} H^r(V_j;\mathbb{Z})$. For $k \geq 1$, denote the transition restriction by $\operatorname{res}^r_{k,k+1} : H^r(\operatorname{int} N_k;\mathbb{Z}) \to H^r(\operatorname{int} N_{k+1};\mathbb{Z})$. Functoriality gives $\rho^r_{k+1} \circ \operatorname{res}^r_{k,k+1} = \rho^r_k$. Since ρ^r_k and ρ^r_{k+1} are isomorphisms, $\operatorname{res}^r_{k,k+1}$ is an isomorphism. Cofinality identifies $\bar{H}^r(V_j;\mathbb{Z})$ with the colimit of the sequential system $\big(H^r(\operatorname{int} N_k;\mathbb{Z}), \operatorname{res}^r_{k,k+1}\big)_{k\geq 1}$. Under this identification, the canonical restriction map $\tau_j^r : \bar{H}^r(V_j;\mathbb{Z}) \to H^r(V_j;\mathbb{Z})$ is induced by the compatible family $(\rho_k^r)_k$ and is therefore an isomorphism. Taking $r = n-i-1$, write this map as $\tau_j : \bar{H}^{n-i-1}(V_j;\mathbb{Z}) \xrightarrow{\cong} H^{n-i-1}(V_j;\mathbb{Z})$. Define

$$D_{j,i} := \tau_j \circ \bar{\gamma}_{\mathfrak{o},j,i+1} \circ \partial_{j,i}^{-1}, \qquad 0 \leq i \leq n-2.$$

For $i \geq n-1$ both groups in the statement vanish, and we set $D_{j,i} := 0$, the unique homomorphism, which is an isomorphism of zero groups.

Let $\kappa_j : (S^n, E_j) \to (S^n, E_{j+1})$ denote the map of pairs that is the identity on S^n, and let $\bar{\imath}_j^* : \bar{H}^r(V_j;\mathbb{Z}) \to \bar{H}^r(V_{j+1};\mathbb{Z})$ be the canonical restriction: every neighborhood of V_j is a neighborhood of V_{j+1}. For $0 \leq i \leq n-2$,

$$\begin{aligned}
D_{j+1,i} \circ (\jmath_j)_* &= \tau_{j+1} \circ \bar{\gamma}_{\mathfrak{o},j+1,i+1} \circ \partial_{j+1,i}^{-1} \circ (\jmath_j)_* \\
&= \tau_{j+1} \circ \bar{\gamma}_{\mathfrak{o},j+1,i+1} \circ (\kappa_j)_* \circ \partial_{j,i}^{-1} && \text{(naturality of } \partial) \\
&= \varepsilon_i\, \tau_{j+1} \circ \bar{\imath}_j^* \circ \bar{\gamma}_{\mathfrak{o},j,i+1} \circ \partial_{j,i}^{-1} && \text{(cap-product naturality, same } \mathfrak{o}) \\
&= \varepsilon_i\, i_j^* \circ \tau_j \circ \bar{\gamma}_{\mathfrak{o},j,i+1} \circ \partial_{j,i}^{-1} && \text{(naturality of } \tau) \\
&= \varepsilon_i\, i_j^* \circ D_{j,i}.
\end{aligned}$$

Here the first nontrivial equality is $\partial_{j+1,i} \circ (\kappa_j)_* = (\jmath_j)_* \circ \partial_{j,i}$, the naturality of connecting homomorphisms for the map of pairs κ_j. For $i \geq n-1$ every square consists of zero groups and commutes trivially. This proves the signed naturality identity for all $i \geq 0$. Replacing $\mathfrak{o}$ by $-\mathfrak{o}$ multiplies $\bar{\gamma}_{\mathfrak{o}}$, and hence both vertical maps in each square, by -1, and therefore affects neither the signed identity nor the conclusions below.

For $i \geq 1$ with $i \neq n-2$ the source of $(\jmath_j)_*$ vanishes by (15). In the remaining degree the signed naturality identity gives $(\jmath_j)_* = \varepsilon_{n-2}\, D_{j+1,n-2}^{-1} \circ i_j^* \circ D_{j,n-2}$, and $i_j^* : H^1(V_j;\mathbb{Z}) \to H^1(V_{j+1};\mathbb{Z})$ is zero by Lemma 6.9. Hence $(\jmath_j)_* = 0$ in every degree $i \geq 1$. □

Proposition 6.12 (Acyclicity of the complement)**.** *For every $i \geq 0$, $\widetilde{H}_i(M;\mathbb{Z}) = 0$.*

Proof. By Lemma 6.8(b), the sets E_j form an increasing sequence such that $M = \bigcup_{j\geq 1} E_j$, and every compact subset of M is contained in some E_j. Hence the inclusions $E_j \hookrightarrow M$ and $\jmath_j : E_j \hookrightarrow E_{j+1}$ induce a canonical isomorphism

$$\Theta_i : \varinjlim_j \big(H_i(E_j;\mathbb{Z}), (\jmath_j)_*\big) \longrightarrow H_i(M;\mathbb{Z})$$

for every $i \geq 0$, by the direct-limit theorem for singular homology [Hat02, Proposition 3.33].

Now fix $i \geq 1$. In positive degrees reduced and unreduced homology agree, and Lemma 6.11 states that every transition $(\jmath_j)_* : H_i(E_j;\mathbb{Z}) \to H_i(E_{j+1};\mathbb{Z})$ is zero. Consequently, the direct limit vanishes. Hence $H_i(M;\mathbb{Z}) = 0$ for every $i \geq 1$.

It remains to treat degree zero. By (15), $\widetilde{H}_0(E_j;\mathbb{Z}) = 0$, so every E_j is path-connected. Choose $p \in E_1$. The inclusions are nested, so $p \in E_j$ for every j, and $H_0(E_j;\mathbb{Z}) = \mathbb{Z}[p]$. Since $\jmath_j$ fixes p, the induced map $(\jmath_j)_*$ sends $[p]$ to $[p]$. The degree-zero direct system is therefore literally

$$\mathbb{Z}[p] \xrightarrow{\text{id}} \mathbb{Z}[p] \xrightarrow{\text{id}} \mathbb{Z}[p] \xrightarrow{\text{id}} \cdots,$$

and its colimit is $\mathbb{Z}[p]$. The canonical isomorphism Θ_0 is induced by inclusions fixing p, so it identifies $H_0(M;\mathbb{Z})$ with $\mathbb{Z}[p]$. The augmentation $\varepsilon : H_0(M;\mathbb{Z}) \to \mathbb{Z}$ sends $[p]$ to 1 and is therefore an isomorphism. Hence its kernel, which by definition is $\widetilde{H}_0(M;\mathbb{Z})$, vanishes. In particular, M is path connected.

Combining degree zero with the positive-degree calculation proves $\widetilde{H}_i(M;\mathbb{Z}) = 0$ for every $i \geq 0$.

□

The assumption $n \geq 4$ enters sharply in the following lemma: pushing a singular 2-disk off a 1-dimensional spine by general position requires $2 + 1 - n < 0$. We use the lemma to prove that M is contractible.

Lemma 6.13 (Simply connected finite stages)**.** *Assume $n \geq 4$. For every j, the spaces $S^n \setminus \Gamma_j$ and C_j are path-connected and simply connected.*

Proof. Fix j and set $X_j := S^n \setminus \Gamma_j$. All distances are measured in the round metric on S^n.

We first record a compactly controlled retraction construction. By Proposition 6.6(6), there are a triangulation $\mathcal{T}_j$ of S^n in which Γ_j is the polyhedron of a nonempty, proper, full subcomplex, the simplicial map $f_j : |\mathcal{T}_j| = S^n \to [0,1]$

taking value 0 on the vertices of this subcomplex and value 1 on all other vertices, and a number $t_j \in (0,1)$ such that $f_j^{-1}(0) = \Gamma_j, \qquad V_j = f_j^{-1}([0,t_j])$, and, for every $c \in (0,1)$, the sublevel set $N_c := f_j^{-1}([0,c])$ is a closed regular neighborhood of Γ_j with boundary $f_j^{-1}(c)$. Lemma 6.4 also gives $\operatorname{int} N_c = f_j^{-1}([0,c)) \qquad (0 < c < 1)$. In particular, $\operatorname{int} V_j = f_j^{-1}([0,t_j)), \qquad C_j = f_j^{-1}([t_j,1])$. Fix $0 < \varepsilon < t_j$, so that $N_\varepsilon \subset f_j^{-1}([0,t_j)) = \operatorname{int} V_j$, with $\partial N_\varepsilon = f_j^{-1}(\varepsilon)$ and $\partial V_j = f_j^{-1}(t_j)$. Put $W_\varepsilon := f_j^{-1}\big([\varepsilon,t_j]\big)$, the compact slab bounded by the levels ε and t_j. Apply the homeomorphism Φ_{t_j} of (13) to f_j. Since its second coordinate at z is $f_j(z)/t_j$, its restriction gives a homeomorphism of triples

$$\big(W_\varepsilon;\, \partial N_\varepsilon,\, \partial V_j\big) \cong \big(\partial V_j \times [\varepsilon/t_j, 1];\, \partial V_j \times \{\varepsilon/t_j\},\, \partial V_j \times \{1\}\big).$$

Transporting the linear contraction of the second coordinate onto 1 through this homeomorphism gives a strong deformation retraction of W_ε onto ∂V_j, fixing ∂V_j pointwise. Since

$$S^n \setminus \operatorname{int} N_\varepsilon = f_j^{-1}\big([\varepsilon,1]\big) = W_\varepsilon \cup C_j, \qquad W_\varepsilon \cap C_j = f_j^{-1}(t_j) = \partial V_j,$$

we may glue this homotopy to the identity homotopy on C_j: the two agree on the overlap $(W_\varepsilon \cap C_j) \times I = \partial V_j \times I$, because the first fixes ∂V_j pointwise at every time; and $W_\varepsilon \times I$ and $C_j \times I$ are closed in $(S^n \setminus \operatorname{int} N_\varepsilon) \times I$, being products of preimages of closed intervals under f_j with I, so the pasting lemma yields a continuous homotopy $(R_{\varepsilon,u})_{u\in[0,1]}$ on $S^n \setminus \operatorname{int} N_\varepsilon$. It satisfies

$$R_{\varepsilon,0} = \mathrm{id}, \qquad R_{\varepsilon,u}|_{C_j} = \mathrm{id}_{C_j} \quad (0 \le u \le 1), \qquad R_{\varepsilon,1}(W_\varepsilon) = \partial V_j \subset C_j.$$

Hence $(R_{\varepsilon,u})_{u\in[0,1]}$ is a strong deformation retraction of $S^n \setminus \operatorname{int} N_\varepsilon$ onto C_j; denote its terminal retraction by r_ε, which fixes C_j pointwise. Finally, if $D \subset X_j$ is compact, then $f_j > 0$ on D, because $D \cap f_j^{-1}(0) = D \cap \Gamma_j = \varnothing$; compactness gives $m_D := \min_{x\in D} f_j(x) > 0$, and any $0 < \varepsilon < \min\{t_j, m_D\}$ yields $D \cap N_\varepsilon = \varnothing$, hence $D \subset S^n \setminus \operatorname{int} N_\varepsilon$.

We first prove path connectivity of X_j. Let $p_0, p_1 \in X_j$; if $p_0 = p_1$, there is nothing to prove. Otherwise choose a PL arc $\zeta : I \to S^n$ with $\zeta(0) = p_0$ and $\zeta(1) = p_1$, and put $A := \zeta(\partial I) = \{p_0, p_1\}$. Since $A \subset X_j$, one has $A \cap \Gamma_j = \varnothing$. Pass to a common subdivision in which $A \subset \zeta(I)$ and Γ_j are subpolyhedra, and apply the general-position theorem [Bry02, Theorem 4.2] in the ambient PL manifold S^n, with $X = \zeta(I), \qquad X_0 = A, \qquad Y = \Gamma_j$, and with the constant control function $\eta_0 \equiv 1$. Since $\dim(\zeta(I) \setminus A) = 1$ and $\dim \Gamma_j = 1$, it gives an ambient PL isotopy $\Psi_t : S^n \to S^n$, fixed on A, such that

$$\begin{aligned}\dim\big(\Psi_1(\zeta(I) \setminus A) \cap \Gamma_j\big) &\le \dim(\zeta(I) \setminus A) + \dim \Gamma_j - n \\ &\le 1 + 1 - n < 0.\end{aligned}$$

Hence $\Psi_1(\zeta(I) \setminus A) \cap \Gamma_j = \varnothing$. Because $\Psi_1|_A = \mathrm{id}_A$ and $A \cap \Gamma_j = \varnothing$, we obtain $\Psi_1(\zeta(I)) \cap \Gamma_j = \varnothing$. Thus $\widetilde{\zeta} := \Psi_1 \circ \zeta$ is an arc in X_j from p_0 to p_1. This part uses only $1 + 1 - n < 0$, hence only $n \ge 3$.

We next prove simple connectivity of X_j. Let $\gamma_0 : S^1 \to X_j$ be a loop and set $d := \operatorname{dist}\big(\gamma_0(S^1), \Gamma_j\big) > 0$, by compactness and disjointness. Choose triangulations of S^1 and S^n, with Γ_j a subcomplex of the latter. After subdivision, the controlled simplicial-approximation theorem [Bry02, p. 222], applied with control $d/2$, gives a free homotopy $\mathcal{A} : S^1 \times [0,1] \to S^n$ from γ_0 to a simplicial (hence PL) loop γ. Since every point track of $\mathcal{A}$ has diameter less than $d/2$, the homotopy lies in X_j, and $\operatorname{dist}\big(\gamma(S^1), \Gamma_j\big) \geq d/2 > 0$.

Since $n \geq 2$, $\pi_1(S^n) = 0$, so γ extends to a continuous map $F_0 : D^2 \to S^n$ with $F_0|_{\partial D^2} = \gamma$. With respect to the triangulations produced by the preceding step, γ is already simplicial. After extending the triangulation of S^1 over D^2, Zeeman's relative simplicial approximation theorem [Zee64] deforms F_0, relative to ∂D^2, to a PL map $F : D^2 \to S^n$ such that $F|_{\partial D^2} = \gamma$. No control on this homotopy is needed, because only the terminal map is used. Set $Q := F(D^2), \qquad Q_0 := \gamma(S^1)$. Then $Q \supseteq Q_0$, with both sets compact polyhedra, $\dim(Q \setminus Q_0) \leq 2$, and $Q_0 \cap \Gamma_j = \varnothing$. Pass to a common subdivision of the ambient PL structure in which $Q_0 \subset Q$ and Γ_j are subpolyhedra. If $Q = Q_0$, then $Q \cap \Gamma_j = \varnothing$ already and we take $\Psi_t := \mathrm{id}$ for all $t \in [0,1]$; otherwise apply [Bry02, Theorem 4.2] in the ambient PL manifold S^n, with $X = Q$, $X_0 = Q_0$, $Y = \Gamma_j$, and with the constant control function $\eta_0 \equiv 1$. It gives an ambient PL isotopy Ψ_t of S^n which, being a push rel Q_0, fixes Q_0 pointwise [Bry02, §4], and satisfies

$$\begin{aligned} \dim\big(\Psi_1(Q \setminus Q_0) \cap \Gamma_j\big) &\leq \dim(Q \setminus Q_0) + \dim \Gamma_j - n \\ &\leq 2 + 1 - n < 0. \end{aligned}$$

Hence $\Psi_1(Q \setminus Q_0) \cap \Gamma_j = \varnothing$. This is precisely where the hypothesis $n \geq 4$ is used. Since $\Psi_1(Q) = \Psi_1(Q \setminus Q_0) \cup Q_0$ and $Q_0 \cap \Gamma_j = \varnothing$, we conclude $\Psi_1(Q) \cap \Gamma_j = \varnothing$. Consequently, $G := \Psi_1 \circ F : D^2 \to X_j$ is well defined and satisfies $G|_{\partial D^2} = \Psi_1 \circ \gamma = \gamma$.

The controlled homotopy $\mathcal{A}$ constructed above takes values in X_j. Identify ∂D^2 with $S^1 \times \{1\}$ via the identification $S^1 = \partial D^2$ used in $F|_{\partial D^2} = \gamma$. The maps $\mathcal{A}$ and G then agree on their common domain. They therefore paste to a continuous map

$$\big(S^1 \times [0,1]\big) \cup_{S^1 \times \{1\} = \partial D^2} D^2 \longrightarrow X_j.$$

The adjunction space on the left is homeomorphic to D^2, with boundary corresponding to $S^1 \times \{0\}$, and the restriction of the pasted map to that boundary is γ_0. Hence γ_0 is null-homotopic in X_j. Since γ_0 was arbitrary, $\pi_1(X_j)$ is trivial; together with path connectivity, this proves that X_j is simply connected.

Finally, let $i : C_j \hookrightarrow X_j$ be the inclusion. The simplicial function f_j attains the value 1: it equals 1 at every vertex outside the full subcomplex triangulating Γ_j, and such vertices exist, since otherwise fullness would force $\Gamma_j = S^n$, which is impossible because $\dim \Gamma_j = 1 < n$. Hence $\varnothing \neq f_j^{-1}(1) \subset f_j^{-1}([t_j, 1]) = C_j$; fix a basepoint $x_0 \in C_j$.

Every compact subset of X_j lies in $S^n \setminus \operatorname{int} N_\varepsilon$ for some $0 < \varepsilon < t_j$. By the retraction construction established at the beginning of this proof, $(R_{\varepsilon,u})_{u \in [0,1]}$ is a strong deformation retraction of $S^n \setminus \operatorname{int} N_\varepsilon$ onto C_j, with terminal retraction r_ε, and $R_{\varepsilon,u}|_{C_j} = \mathrm{id}_{C_j}$ for every $u \in [0,1]$. In particular, $R_{\varepsilon,u}(x_0) = x_0$ for every $u \in [0,1]$.

Applying these retractions to representatives and homotopies, whose images are compact, yields the following. If $g : (S^k, *) \to (X_j, x_0)$ is a based map with $g(S^k) \subset S^n \setminus \operatorname{int} N_\varepsilon$, then $(z, u) \mapsto R_{\varepsilon,u}(g(z))$ is a based homotopy in X_j from g to $r_\varepsilon \circ g$, which maps into C_j; hence $i_* : \pi_k(C_j, x_0) \to \pi_k(X_j, x_0)$ is surjective for every $k \geq 1$. If a based map $h : (S^k, *) \to (C_j, x_0)$ is null-homotopic in X_j, composing the null-homotopy with r_ε, for ε so small that its compact image lies in $S^n \setminus \operatorname{int} N_\varepsilon$, gives a based null-homotopy of $r_\varepsilon \circ h = h$ in C_j; hence i_* is injective. The same two arguments, applied to points and paths, show that the inclusion induces a bijection $\pi_0(C_j) \to \pi_0(X_j)$ of sets of path components. Since X_j is path-connected and C_j is nonempty, this bijection implies that C_j is path-connected. The already established isomorphism $\pi_1(C_j, x_0) \to \pi_1(X_j, x_0)$ then implies that C_j is simply connected, because X_j is simply connected. □

Proposition 6.14 (Contractibility)**.** *The manifold M is contractible.*

Proof. Since $M = \bigcup_{j \geq 1} C_j$ and every C_j is path-connected by Lemma 6.13, the space M is path-connected. The image of every loop in M is compact and therefore lies in some C_j by Lemma 6.8(b). Since C_j is simply connected, the loop contracts in $C_j \subset M$. Hence $\pi_1(M) = 0$. Proposition 6.12 gives $\widetilde{H}_k(M; \mathbb{Z}) = 0$ for $k \geq 0$. If some higher homotopy group of M were nonzero, let $k \geq 2$ be the least index with $\pi_k(M) \neq 0$. Then M would be $(k-1)$-connected, and the Hurewicz theorem [Hat02, Theorem 4.32] would give an isomorphism $\pi_k(M) \cong H_k(M; \mathbb{Z}) = 0$, a contradiction. Thus M is weakly contractible. Since a smooth manifold has the homotopy type of a CW complex, choose a CW complex Z_M and a homotopy equivalence $Z_M \to M$. Then the constant map $Z_M \to *$ induces isomorphisms on every homotopy group, including π_0. Whitehead's theorem [Hat02, Theorem 4.5] makes this map a homotopy equivalence. Thus Z_M, and hence M, is contractible. □

We now distinguish M from $\mathbb{R}^n$ by examining its topology at infinity.

Proposition 6.15 (Not simply connected at infinity)**.** *The manifold M is not simply connected at infinity. Consequently, $M \ncong \mathbb{R}^n$.*

The proof constructs, beyond each compact subset of M, a loop on the boundary of a sufficiently deep neighborhood V_k. The injectivity of the bonding endomorphism ω implies that this loop remains essential in the fixed outer neighborhood V_1. Thus these loops cannot be contracted outside a fixed compact subset of M.

Proof. Our convention includes one-endedness in simple connectivity at infinity. Only the following necessary loop condition is used: for every compact set $C \subset X$, there is a compact set $D \subset X$, with $C \subset D$, such that every loop in $X \setminus D$ is null-homotopic in $X \setminus C$. To prove that M is not simply connected at infinity, it is enough to exhibit one compact set $C \subset M$ for which no such D exists.

Set $C := C_1 = S^n \setminus \operatorname{int} V_1$. Since $K \subset V_2 \subset \operatorname{int} V_1$, the sets C and K are disjoint. Hence C is a compact subset of $M = S^n \setminus K$. Moreover, $M \setminus C = \operatorname{int} V_1 \setminus K$. Let $D \subset M$ be an arbitrary compact set containing C. By Lemma 6.8(b), there is an index j such that $D \subset S^n \setminus V_j$. Because the sets $S^n \setminus V_j$ are increasing, we may increase j, if necessary, and choose $k \geq 2$ such that $D \subset S^n \setminus V_k$. Equivalently, $D \cap V_k = \varnothing$. In particular, $\partial V_k \cap D = \varnothing$; below we construct the required loop on ∂V_k.

Let $\alpha_k : (S^1, *) \to (\Gamma_k, x) \subset (V_k, x)$ parametrize the oriented a-petal of the rank-two rose Γ_k. By Proposition 6.6(5), $[\alpha_k] = \mathfrak{m}_k(a) \in \pi_1(V_k, x)$. The point x belongs to K, because $x \in V_j$ for every j. Consequently, $\alpha_k(S^1) \cap K \neq \varnothing$, so α_k is not a loop in M.

We therefore first use PL general position to move α_k away from the spine Γ_k, and then use the punctured-sublevel product structure of the regular neighborhood V_k to move the resulting loop onto ∂V_k, which is disjoint from K.

Set $Z := \alpha_k(S^1), \qquad Z_0 := \varnothing$. By Proposition 6.6(6), Γ_k is the polyhedron of a nonempty, proper, full subcomplex of $\mathcal{T}_k$. Hence Lemma 6.4 applies and gives $Z = \alpha_k(S^1) \subset \Gamma_k = f_k^{-1}(0) \subset f_k^{-1}([0, t_k)) = \operatorname{int} V_k$. The space $W := \operatorname{int} V_k$ is a PL n-manifold without boundary and therefore equals its own manifold interior. Moreover, $Z \supset Z_0$ and Γ_k are compact PL polyhedra in W, with $p_k := \dim(Z \setminus Z_0) \leq 1, \qquad q_k := \dim \Gamma_k = 1$.

For clarity, we record the special case of the controlled PL general-position theorem used here. Let $\mathsf{Z} \supset \mathsf{Z}_0$ and Y be compact polyhedra contained in the interior of a PL n-manifold W, and fix a metric d_{W} inducing the topology of W. For every $\varepsilon > 0$, there is an ε-push $(\Psi_t)_{0 \leq t \leq 1}$ of Z in W, relative to Z_0, such that

$$\dim\bigl(\Psi_1(\mathsf{Z} \setminus \mathsf{Z}_0) \cap \mathsf{Y}\bigr) \leq \dim(\mathsf{Z} \setminus \mathsf{Z}_0) + \dim \mathsf{Y} - n$$

[Bry02, Theorem 4.2]. Bryant states the theorem for arbitrary polyhedra; compactness is imposed here only because it holds in the application. By the definition of a push [Bry02, §4], $(\Psi_t)_{0 \leq t \leq 1}$ is an ambient PL isotopy of W satisfying

$$\Psi_0 = \mathrm{id}_{\mathsf{W}}, \qquad \Psi_t|_{\mathsf{Z}_0} = \mathrm{id}_{\mathsf{Z}_0} \quad (0 \leq t \leq 1).$$

For every $w \in \mathsf{W}$, the track $\{\Psi_t(w) : 0 \leq t \leq 1\}$ has d_{W}-diameter less than ε, and Ψ_t is the identity outside $\bigl\{w \in \mathsf{W} : d_{\mathsf{W}}(w, \mathsf{Z}) < \varepsilon\bigr\}$. In particular, the theorem does not require Z and Y to be disjoint.

Apply this statement with

$$\begin{aligned} \mathsf{W} &= \operatorname{int} V_k, \\ d_{\mathsf{W}} &= d_{g_{st}}|_{\operatorname{int} V_k \times \operatorname{int} V_k}, \\ \mathsf{Z} &= \alpha_k(S^1), \qquad \mathsf{Z}_0 = \varnothing, \\ \mathsf{Y} &= \Gamma_k, \qquad \varepsilon = 1. \end{aligned}$$

Let $(\Psi_t)_{0 \leq t \leq 1}$ denote the resulting ambient PL isotopy of $\operatorname{int} V_k$. Then $\Psi_0 = \mathrm{id}$ and $\dim\bigl(\Psi_1(\alpha_k(S^1)) \cap \Gamma_k\bigr) \leq p_k + q_k - n \leq 2 - n < 0$. Thus $\Psi_1(\alpha_k(S^1)) \cap \Gamma_k = \varnothing$.

Let $\beta_k := \Psi_1 \circ \alpha_k$. Then $\beta_k(S^1) \subset \operatorname{int} V_k \setminus \Gamma_k$. The map $(z, t) \mapsto \Psi_t(\alpha_k(z))$ is a free homotopy in $\operatorname{int} V_k$ from α_k to β_k. Consequently, the free homotopy class of β_k in V_k is the conjugacy class of $\mathfrak{m}_k(a)$.

Proposition 6.6(6) gives $V_k = f_k^{-1}([0, t_k])$, $\Gamma_k = f_k^{-1}(0)$, and $\partial V_k = f_k^{-1}(t_k)$ for the 0–1 simplicial level function f_k. Applying the punctured-sublevel homeomorphism (13) from Lemma 6.4, with $c = t_k$, gives a homeomorphism of pairs

$$(V_k \setminus \Gamma_k, \partial V_k) \cong (\partial V_k \times (0, 1], \partial V_k \times \{1\}).$$

Transporting the factor homotopy $(z, \vartheta) \mapsto (z, (1-u)\vartheta + u)$ therefore gives a strong deformation retraction

$$R_{k,u} : V_k \setminus \Gamma_k \longrightarrow V_k \setminus \Gamma_k, \qquad 0 \leq u \leq 1,$$

such that

$$R_{k,0} = \mathrm{id}, \qquad R_{k,u}\big|_{\partial V_k} = \mathrm{id}_{\partial V_k}, \qquad R_{k,1}(V_k \setminus \Gamma_k) = \partial V_k.$$

Define $\delta_k := R_{k,1} \circ \beta_k$. Then $\delta_k(S^1) \subset \partial V_k$. Concatenating the free homotopy $\Psi_t \circ \alpha_k$ from α_k to β_k with the homotopy $R_{k,t} \circ \beta_k$ from β_k to δ_k, we conclude that δ_k is freely homotopic to α_k in V_k. Hence the free homotopy class of δ_k in V_k is the conjugacy class of $\mathfrak{m}_k(a)$.

We next verify that δ_k lies in $M \setminus D$. Since $\delta_k(S^1) \subset \partial V_k \subset V_k$ and $D \cap V_k = \varnothing$, we have $\delta_k(S^1) \cap D = \varnothing$. On the other hand, the nesting of the tower gives $K \subset V_{k+1} \subset \operatorname{int} V_k$, whereas $\delta_k(S^1) \subset \partial V_k$. Therefore $\delta_k(S^1) \cap K = \varnothing$. Combining the two conclusions yields $\delta_k(S^1) \subset S^n \setminus (D \cup K) = M \setminus D$.

Suppose, toward a contradiction, that δ_k were null-homotopic in $M \setminus C$. Since $M \setminus C = \operatorname{int} V_1 \setminus K \subset V_1$, the same null-homotopy would make the free homotopy class of δ_k trivial in V_1. Let $\iota_1^k : V_k \hookrightarrow V_1$ denote the inclusion. By Proposition 6.6, $(\iota_1^k)_* \circ \mathfrak{m}_k = \mathfrak{m}_1 \circ \omega^{k-1}$. It follows that the free homotopy class of δ_k, when viewed in V_1, is the conjugacy class of $\mathfrak{m}_1\big(\omega^{k-1}(a)\big)$. By Lemma 6.2, the endomorphism $\omega : F_2 \to F_2$ is injective. Since $a \neq 1$, injectivity gives $\omega^{k-1}(a) \neq 1$. As $\mathfrak{m}_1$ is an isomorphism, it follows that $\mathfrak{m}_1\big(\omega^{k-1}(a)\big) \neq 1$. The identity element is conjugate only to itself, so the conjugacy class of this element is nontrivial. Hence the free homotopy class of δ_k in V_1 is nontrivial. This contradicts the assumed null-homotopy of δ_k in $M \setminus C$.

Therefore, for the fixed compact set $C = C_1$, every compact set $D \subset M$ containing C admits a loop $\delta_k : S^1 \to M \setminus D$ that is not null-homotopic in $M \setminus C$. Consequently, M is not simply connected at infinity.

Simple connectivity at infinity, with the convention fixed above, is invariant under homeomorphism. Indeed, a homeomorphism carries compact sets to compact sets and restricts to a homeomorphism of the corresponding complements; hence it preserves both the inverse system of components of complements, and therefore one-endedness, and the loop null-homotopy condition. The closed balls $\overline{B}(0, m)$ form a cofinal compact exhaustion of $\mathbb{R}^n$, and $\mathbb{R}^n \setminus \overline{B}(0, m)$ is connected for $n \geq 2$; thus $\mathbb{R}^n$ is one-ended. If $n \geq 3$ and $C' \subset \mathbb{R}^n$ is compact, choose a closed ball D' containing C'. Since $\mathbb{R}^n \setminus D'$ deformation retracts onto S^{n-1} and $\pi_1(S^{n-1}) = 0$, every loop in $\mathbb{R}^n \setminus D'$ is null-homotopic there, hence also in $\mathbb{R}^n \setminus C'$. Therefore $\mathbb{R}^n$ is simply connected at infinity for $n \geq 3$. Since $n \geq 4$ here, it follows that $M \not\cong \mathbb{R}^n$. $\square$

6.3 The metric and its shifts

Our metric construction follows the scheme of Karakhanyan [Kar24, Theorem 1.1], who shows that $S^n \setminus K$ carries a complete scalar-flat metric conformal to the round metric if and only if the critical Bessel capacity of K vanishes. In the fixed chart of this section, this condition is $\operatorname{cap}(K) = 0$. After stereographic projection, our topological construction yields a compact set $F \subset \mathbb{R}^n$ with $\mathcal{C}^{\mathbb{R}^n}_{1+2/n,\, n/2}(F) = 0$. To construct the metric in Theorem 6.1, we need a probability measure on F whose Newtonian potential gives infinite conformal length to every curve approaching F. This is governed by the Wolff potential of the measure: its divergence at every point of F will imply completeness, whereas capacity zero initially provides only averaged divergence.

The following Evans-type theorem converts this averaged information into pointwise divergence. For the parameters of the conformal problem, the statement appears in [Kar24, Proposition 2.1]. The decisive compactness step there is only sketched; since pointwise divergence of the Wolff potential is not, in general, preserved by weak-$*$ convergence (Remark 6.17), we give a complete argument for arbitrary (d, α, q) with $q \geq 2$, the range needed here.

Theorem 6.16. *Let $d \geq 1$, $\alpha > 0$, and $2 \leq q < \infty$, with $\alpha q \leq d$. Set $p := q/(q-1)$, $\theta := p - 1 = 1/(q-1) \in (0, 1]$, and $s := d - \alpha q \geq 0$. For a finite positive Radon measure ν on $\mathbb{R}^d$ write*

$$\mathcal{W}^\nu(x) := \int_0^1 \left(\frac{\nu(B(x,r))}{r^s} \right)^\theta \frac{dr}{r}.$$

If $E \subset \mathbb{R}^d$ is nonempty and compact with $\mathcal{C}^{\mathbb{R}^d}_{\alpha,q}(E) = 0$, then there is a Radon probability measure μ with

$$\operatorname{supp} \mu = E, \qquad \mathcal{W}^\mu(x) = +\infty \quad \text{for every } x \in E.$$

The full-support conclusion is included as part of the sharp measure-theoretic statement. The metric construction below uses only $\mu(\mathbb{R}^d \setminus E) = 0$ and $\mathcal{W}^\mu(x) = +\infty$ for every $x \in E$.

The idea of the proof is as follows. By the Hedberg–Wolff inequality [HW83, Theorem 1], zero capacity forces every probability measure on E to have infinite Wolff self-energy. Since the full Wolff potential is not stable under weak-$*$ convergence, we replace it by finite jointly continuous smoothed truncations. Compactness then promotes pointwise divergence of the truncated self-energies to uniform divergence of their minima. Sion's minimax theorem [Sio58, Theorem 3.4], using the concavity in the measure afforded by $\theta \leq 1$, produces measures whose truncated potentials are uniformly large on E. A convex series of these measures then yields a probability measure whose full Wolff potential diverges at every point of E.

Proof. Throughout, $B(x, r)$ denotes the open ball in $\mathbb{R}^d$, and $\mathcal{P}(E)$ denotes the set of Borel probability measures on the compact set E; each $\nu \in \mathcal{P}(E)$ is also regarded as a measure on $\mathbb{R}^d$ vanishing off E.

Step 1 (the space $\mathcal{P}(E)$ and the self-energy). We equip $\mathcal{P}(E)$ with the weak topology of [Par67, Ch. II, §6]: $\nu_k \to \nu$ means $\int_E g \, d\nu_k \to \int_E g \, d\nu$ for every $g \in C(E)$. By [Par67, Ch. II, Thm. 5.8], $\nu \mapsto \left(g \mapsto \int_E g \, d\nu\right)$ identifies $\mathcal{P}(E)$ with the set of positive linear functionals ℓ on $C(E)$ with $\ell(1) = 1$, and under this identification the weak topology is the restriction of the weak-$*$ topology of $C(E)^*$. Thus $\mathcal{P}(E)$ is a convex subset of the locally convex Hausdorff space $C(E)^*$ endowed with its weak-$*$ topology, and since E is a compact metric space, it is compact and metrizable [Par67, Ch. II, Thm. 6.4]. In particular, every sequence in $\mathcal{P}(E)$ has a weak-$*$ convergent subsequence, and continuity of functions on $\mathcal{P}(E)$ may be tested along sequences.

For a finite positive Radon measure ν on $\mathbb{R}^d$, the map $(x, r) \longmapsto \nu(B(x, r))$ is lower semicontinuous on $\mathbb{R}^d \times (0, \infty)$: if $(x_k, r_k) \to (x, r)$ and $L \subset B(x, r)$ is compact, then $L \subset B(x_k, r_k)$ for all large k, so $\liminf_k \nu(B(x_k, r_k)) \geq \nu(L)$, and inner regularity gives $\liminf_k \nu(B(x_k, r_k)) \geq \nu(B(x, r))$. Consequently the integrand of $\mathcal{W}^\nu$ is Borel on $\mathbb{R}^d \times (0, 1]$, $\mathcal{W}^\nu : \mathbb{R}^d \to [0, \infty]$ is Borel by Tonelli's theorem, and the self-energy $J(\nu) := \int_E \mathcal{W}^\nu \, d\nu \in [0, \infty]$ is well defined. Three elementary properties of $\nu \mapsto \mathcal{W}^\nu$ are used repeatedly. It is *monotone*: $\nu \leq \nu'$ as measures implies $\nu(B(x, r)) \leq \nu'(B(x, r))$ for every ball, hence $\mathcal{W}^\nu \leq \mathcal{W}^{\nu'}$ pointwise. It is θ-*homogeneous*: $\mathcal{W}^{c\nu} = c^\theta \mathcal{W}^\nu$ for $c > 0$. Finally, it has *finite tails*: since $\nu(B(x, r)) \leq \nu(\mathbb{R}^d)$, for $0 < a \leq 1$

$$\int_a^1 \Big(\frac{\nu(B(x, r))}{r^s}\Big)^\theta \frac{dr}{r} \;\leq\; \nu(\mathbb{R}^d)^\theta \, c(a), \qquad c(a) := \int_a^1 r^{-s\theta - 1} \, dr < \infty. \tag{16}$$

Step 2 (capacity zero forces infinite self-energy). Suppose that $J(\nu) < \infty$ for some $\nu \in \mathcal{P}(E)$. In the notation of Hedberg–Wolff [HW83, pp. 161, 164], their capacity exponent q is our q, and their conjugate exponent p, determined by $\frac{1}{p} + \frac{1}{q} = 1$, is our p. In that notation,

$$W^\nu_{\alpha,q}(x) = \int_0^1 \Big(\frac{\nu(B(x, r))}{r^{\,d - \alpha q}}\Big)^{p-1} \frac{dr}{r};$$

since $s = d - \alpha q$ and $\theta = p - 1$, this is $W^\nu_{\alpha,q} = \mathcal{W}^\nu$. The Hedberg–Wolff inequality [HW83, Theorem 1, p. 165] applies, because ν is a positive Radon measure and $1 < q \leq d/\alpha$, and gives a constant $A = A(d, \alpha, q)$ with

$$\|G_\alpha * \nu\|^p_{L^p(\mathbb{R}^d)} \leq A \int_{\mathbb{R}^d} W^\nu_{\alpha,q} \, d\nu = A \int_E \mathcal{W}^\nu \, d\nu = A \, J(\nu) < \infty,$$

the middle equality because ν vanishes off E. Since $G_\alpha > 0$ and $\nu \neq 0$, also $\|G_\alpha * \nu\|_{L^p} > 0$.

Now let f be admissible in the definition of $\mathcal{C}^{\mathbb{R}^d}_{\alpha,q}(E)$, that is, $f \geq 0$, $f \in L^q(\mathbb{R}^d)$, and $G_\alpha * f \geq 1$ everywhere on E. By Tonelli's theorem (all integrands are nonnegative), the symmetry $G_\alpha(x - y) = G_\alpha(y - x)$, and Hölder's inequality,

$$\begin{aligned} 1 = \nu(E) \leq \int_E (G_\alpha * f) \, d\nu &= \int_E \int_{\mathbb{R}^d} G_\alpha(x - y) f(y) \, dy \, d\nu(x) \\ &= \int_{\mathbb{R}^d} f \, (G_\alpha * \nu) \, dy \leq \|f\|_{L^q} \, \|G_\alpha * \nu\|_{L^p}. \end{aligned}$$

Consequently, $\|f\|^q_{L^q} \geq \|G_\alpha * \nu\|^{-q}_{L^p}$, and taking the infimum over all admissible f, $\mathcal{C}^{\mathbb{R}^d}_{\alpha,q}(E) \geq \|G_\alpha * \nu\|^{-q}_{L^p} > 0$, contrary to the hypothesis. Hence

$$\mathcal{C}^{\mathbb{R}^d}_{\alpha,q}(E) = 0 \quad\Longrightarrow\quad J(\nu) = +\infty \qquad \text{for every } \nu \in \mathcal{P}(E). \tag{17}$$

The restriction $\alpha q \leq d$ is invoked here to apply the Hedberg–Wolff inequality; in the remaining steps we use only its consequence $s = d - \alpha q \geq 0$.

Step 3 (smoothed truncated potentials). Fix once and for all a continuous $\chi : [0, \infty) \to [0, 1]$ with $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $[2, \infty)$; being continuous and constant outside a compact set, χ is uniformly continuous. For $\nu \in \mathcal{P}(E)$, $x \in \mathbb{R}^d$ and $r > 0$ put

$$P_r\nu(x) := \int_E \chi\Big(\frac{|x - y|}{r}\Big) \, d\nu(y).$$

Then:

(3a) $\nu(B(x,r)) \le P_r\nu(x) \le \nu(B(x,2r)) \le 1$, because $\chi(|x-y|/r) = 1$ for $|x-y|<r$, $=0$ for $|x-y|\ge 2r$, and $0\le\chi\le 1$.

(3b) $\nu \mapsto P_r\nu(x)$ is the restriction to $\mathcal{P}(E)$ of a linear functional, hence affine.

(3c) $(\nu,x,r)\mapsto P_r\nu(x)$ is jointly continuous on $\mathcal{P}(E)\times\mathbb{R}^d\times(0,\infty)$. Indeed, let $(\nu_k,x_k,r_k)\to(\nu,x,r)$ and put $g_k(y) := \chi(|x_k-y|/r_k)$, $g(y) := \chi(|x-y|/r)$. Since E is bounded and $r_k\to r>0$, $\sup_{y\in E}\big||x_k-y|/r_k - |x-y|/r\big|\to 0$, hence $\|g_k-g\|_{C(E)}\to 0$ by uniform continuity of χ, and

$$|P_{r_k}\nu_k(x_k) - P_r\nu(x)| \le \|g_k-g\|_{C(E)}\,\nu_k(E) + \Big|\int_E g\,d\nu_k - \int_E g\,d\nu\Big| \longrightarrow 0,$$

the second term by weak-$*$ convergence, as $g\in C(E)$.

For $m\in\mathbb{N}$, $\nu,\lambda\in\mathcal{P}(E)$ and $x\in\mathbb{R}^d$ define

$$T_m^\nu(x) := \int_{2^{-m-2}}^{1/2}\left(\frac{P_r\nu(x)}{r^s}\right)^\theta\frac{dr}{r},\qquad F_m(\nu,\lambda) := \int_E T_m^\nu\,d\lambda,\qquad I_m(\nu) := F_m(\nu,\nu).$$

These have the following properties.

(3d) *Boundedness:* $0\le T_m^\nu(x)\le\int_{2^{-m-2}}^{1/2} r^{-s\theta-1}dr<\infty$, by (3a).

(3e) *Monotonicity in m:* $T_m^\nu\le T_{m+1}^\nu$ pointwise, since the interval of integration grows and the integrand is nonnegative; hence also $F_m\le F_{m+1}$ and $I_m\le I_{m+1}$.

(3f) *Joint continuity:* $(\nu,x)\mapsto T_m^\nu(x)$ is continuous on $\mathcal{P}(E)\times\mathbb{R}^d$. Indeed, the integrand $(P_r\nu(x)/r^s)^\theta r^{-1}$ is jointly continuous in (ν,x,r) by (3c) and the continuity of $t\mapsto t^\theta$ on $[0,\infty)$, and it is bounded by $r^{-s\theta-1}\le 2^{(m+2)(s\theta+1)}$ on the compact interval $[2^{-m-2},\frac12]$; dominated convergence applies.

(3g) *Concavity in ν:* for fixed x and m, the map $\nu\mapsto T_m^\nu(x)$ is concave on $\mathcal{P}(E)$. Indeed, since $0<\theta\le 1$, the map $t\mapsto t^\theta$ is concave and nondecreasing on $[0,\infty)$; this is the only point at which $q\ge 2$ is used. Hence, by (3b), its composition with the affine nonnegative map $\nu\mapsto P_r\nu(x)$ is concave in ν, and integration against the positive measure dr/r preserves concavity.

(3h) *Comparison with $\mathcal{W}$:* for all ν, m, x,

$$T_m^\nu(x) \;\le\; 2^{s\theta}\,\mathcal{W}^\nu(x). \tag{18}$$

Indeed, by the upper half of (3a) and the substitution $\rho=2r$ (so that $r^s=2^{-s}\rho^s$ and $dr/r = d\rho/\rho$),

$$T_m^\nu(x)\le\int_{2^{-m-2}}^{1/2}\Big(\frac{\nu(B(x,2r))}{r^s}\Big)^\theta\frac{dr}{r} = 2^{s\theta}\int_{2^{-m-1}}^{1}\Big(\frac{\nu(B(x,\rho))}{\rho^s}\Big)^\theta\frac{d\rho}{\rho}\le 2^{s\theta}\,\mathcal{W}^\nu(x).$$

This is the reason the truncation stops at $r=\frac12$: after doubling, the range of ρ stays inside $(0,1]$, the range of integration of $\mathcal{W}^\nu$.

(3i) *Lower bound in the limit:* for every ν and x, by the lower half of (3a), monotone convergence in m, and (16) with $a=\frac12$,

$$\mathcal{W}^\nu(x)\;\le\;\int_0^{1/2}\Big(\frac{\nu(B(x,r))}{r^s}\Big)^\theta\frac{dr}{r}+c(\tfrac12)\;\le\;\lim_{m\to\infty}T_m^\nu(x)+c(\tfrac12).$$

Step 4 (joint continuity of F_m; the minima of I_m diverge). We first show that F_m is jointly continuous on $\mathcal{P}(E)\times\mathcal{P}(E)$. Let $(\nu_k,\lambda_k)\to(\nu,\lambda)$. Then $T_m^{\nu_k}\to T_m^\nu$ uniformly on E: otherwise there would be $\varepsilon > 0$ and $x_k \in E$ with $|T_m^{\nu_k}(x_k) - T_m^{\nu}(x_k)| \geq \varepsilon$; after passing to a subsequence with $x_k \to x \in E$, we obtain from (3f) that $T_m^{\nu_k}(x_k) \to T_m^{\nu}(x)$ and $T_m^{\nu}(x_k) \to T_m^{\nu}(x)$, a contradiction. Hence

$$|F_m(\nu_k, \lambda_k) - F_m(\nu, \lambda)| \leq \|T_m^{\nu_k} - T_m^{\nu}\|_{C(E)} + \Big|\int_E T_m^{\nu}\, d\lambda_k - \int_E T_m^{\nu}\, d\lambda\Big| \longrightarrow 0,$$

the last term by weak-$*$ convergence, since $T_m^{\nu} \in C(E)$ by (3f). By (3g), $\nu \mapsto F_m(\nu, \lambda)$ is concave for fixed λ; by definition, $\lambda \mapsto F_m(\nu, \lambda)$ is affine for fixed ν. In particular, I_m is continuous on the compact set $\mathcal{P}(E)$, so $a_m := \min_{\nu \in \mathcal{P}(E)} I_m(\nu)$ is attained, and $a_m \leq a_{m+1}$ by (3e). We claim that

$$a_m \longrightarrow +\infty \qquad (m \to \infty). \tag{19}$$

For each fixed $\nu \in \mathcal{P}(E)$, integration of (3i) against ν gives $J(\nu) \leq \int_E \lim_{m\to\infty} T_m^{\nu}\, d\nu + c(\frac{1}{2})$. Since $J(\nu) = +\infty$ by (17), it follows that $\int_E \lim_m T_m^{\nu}\, d\nu = +\infty$, and the monotone convergence theorem together with (3e) yields $I_m(\nu) \nearrow \int_E \lim_{m\to\infty} T_m^{\nu}\, d\nu = +\infty$; thus $I_m \to +\infty$ pointwise on $\mathcal{P}(E)$. Suppose (19) fails. Since (a_m) is nondecreasing, it is then bounded, say $a_m \leq A < \infty$ for all m, and there are $\nu_m \in \mathcal{P}(E)$ with $I_m(\nu_m) \leq A$. By compactness and metrizability, a subsequence (ν_{m_j}) converges weak-$*$ to some $\nu \in \mathcal{P}(E)$. Fix l. For $m_j \geq l$, (3e) gives $I_l(\nu_{m_j}) \leq I_{m_j}(\nu_{m_j}) \leq A$, and the continuity of I_l gives $I_l(\nu) = \lim_j I_l(\nu_{m_j}) \leq A$. Letting $l \to \infty$ contradicts $I_l(\nu) \to +\infty$. This proves (19). Observe that compactness is applied only to the continuous functionals I_l with l fixed, never to $\mathcal{W}$.

Step 5 (minimax: from the diagonal to a uniform lower bound). For fixed $\lambda \in \mathcal{P}(E)$ the map $\nu \mapsto F_m(\nu, \lambda)$ is concave: $P_r\nu(x)$ depends affinely on ν, $z \mapsto z^{\theta}$ is concave on $[0, \infty)$ because $0 < \theta = \frac{1}{q-1} \leq 1$, and concavity is preserved by the integrations against dr/r and $d\lambda$ (this is (3g)). For fixed ν the map $\lambda \mapsto F_m(\nu, \lambda) = \int_E T_m^{\nu}\, d\lambda$ is affine, hence convex. Moreover, F_m is real-valued and jointly continuous on the compact convex set $\mathcal{P}(E) \times \mathcal{P}(E)$ (Step 4). In particular, F_m is upper semicontinuous and quasi-concave in ν and lower semicontinuous and quasi-convex in λ, so Sion's minimax theorem [Sio58, Theorem 3.4] applies with ν as the maximizing and λ as the minimizing variable:

$$\sup_{\nu \in \mathcal{P}(E)} \inf_{\lambda \in \mathcal{P}(E)} F_m(\nu, \lambda) = \inf_{\lambda \in \mathcal{P}(E)} \sup_{\nu \in \mathcal{P}(E)} F_m(\nu, \lambda).$$

All extrema are attained: the inner ones because F_m is continuous on the compact set $\mathcal{P}(E)$. For completeness, joint continuity on the compact product implies uniform continuity, and

$$\Big|\min_{\lambda} F_m(\nu, \lambda) - \min_{\lambda} F_m(\nu', \lambda)\Big| \leq \sup_{\lambda} |F_m(\nu, \lambda) - F_m(\nu', \lambda)|.$$

By uniform continuity, the right-hand side tends to zero as $\nu' \to \nu$. The analogous estimate, with $\max_{\nu}$ and the roles of the variables reversed, proves continuity of the other value function. Compactness therefore gives the outer extrema as well. Thus

$$\max_{\nu \in \mathcal{P}(E)} \min_{\lambda \in \mathcal{P}(E)} F_m(\nu, \lambda) = \min_{\lambda \in \mathcal{P}(E)} \max_{\nu \in \mathcal{P}(E)} F_m(\nu, \lambda).$$

For fixed ν, continuity of T_m^ν and compactness of E give a point $x_\nu \in E$ with $T_m^\nu(x_\nu) = \min_{x\in E} T_m^\nu(x)$. For every $\lambda \in \mathcal{P}(E)$, $F_m(\nu,\lambda) = \int_E T_m^\nu \, d\lambda \geq \min_{x\in E} T_m^\nu(x)$, with equality for $\lambda = \delta_{x_\nu}$. Consequently $\min_{\lambda\in\mathcal{P}(E)} F_m(\nu,\lambda) = \min_{x\in E} T_m^\nu(x)$, and therefore

$$\begin{aligned} \max_{\nu\in\mathcal{P}(E)} \min_{x\in E} T_m^\nu(x) &= \min_{\lambda\in\mathcal{P}(E)} \max_{\nu\in\mathcal{P}(E)} F_m(\nu,\lambda) \\ &\geq \min_{\lambda\in\mathcal{P}(E)} F_m(\lambda,\lambda) = \min_{\lambda\in\mathcal{P}(E)} I_m(\lambda) = a_m, \end{aligned} \tag{20}$$

the inequality because $\max_\nu F_m(\nu,\lambda) \geq F_m(\lambda,\lambda)$ for each λ. Thus the diagonal quantity a_m, which diverges by (19), bounds from below a quantity that is uniform in $x \in E$.

Step 6 (assembly by a convex series). For each integer $k \geq 1$, choose by (19) an index $m(k)$ with $a_{m(k)} \geq 2^{s\theta} k\, 2^{k\theta}$, and let $\nu_k \in \mathcal{P}(E)$ attain the maximum in (20) for $m = m(k)$. Then $T_{m(k)}^{\nu_k}(x) \geq 2^{s\theta} k\, 2^{k\theta}$ for every $x \in E$, and (18) gives

$$\mathcal{W}^{\nu_k}(x) \;\geq\; 2^{-s\theta}\, T_{m(k)}^{\nu_k}(x) \;\geq\; k\, 2^{k\theta} \qquad \text{for every } x \in E. \tag{21}$$

Set $\mu := \sum_{k\geq1} 2^{-k}\nu_k$. The series converges in total variation, so μ is a positive Radon measure with $\mu(E) = \sum_{k\geq1} 2^{-k} = 1$ and $\mu(\mathbb{R}^d \setminus E) = 0$; thus $\mu \in \mathcal{P}(E)$ and $\operatorname{supp}\mu \subset E$. Since $\mu \geq 2^{-k}\nu_k$ as measures, the monotonicity and θ-homogeneity of $\nu \mapsto \mathcal{W}^\nu$ (Step 1) and (21) give, for every $x \in E$ and every integer $k \geq 1$,

$$\mathcal{W}^\mu(x) \;\geq\; \mathcal{W}^{2^{-k}\nu_k}(x) = 2^{-k\theta}\, \mathcal{W}^{\nu_k}(x) \;\geq\; 2^{-k\theta}\, k\, 2^{k\theta} = k.$$

Hence $\mathcal{W}^\mu \equiv +\infty$ on E. This termwise lower bound avoids attempting to pass pointwise divergence through weak-$*$ convergence.

Finally, $\operatorname{supp}\mu = E$. If $x \in E \setminus \operatorname{supp}\mu$, there is $\rho > 0$ with $\mu(B(x,\rho)) = 0$, hence $\mu(B(x,r)) = 0$ for $0 < r \leq \rho$, and (16) with $a = \min\{\rho, 1\}$ gives $\mathcal{W}^\mu(x) \leq c(\min\{\rho,1\}) < \infty$, the integral over $(0,a]$ being zero; this contradicts $\mathcal{W}^\mu(x) = +\infty$. The proof is complete. □

Remark 6.17 (Why compactness alone cannot work). If $\nu_k \to \nu$ weak-$*$ in $\mathcal{P}(E)$, then $\nu(B(x,r)) \leq \liminf_k \nu_k(B(x,r))$ for every open ball [Par67, Ch. II, Thm. 6.1(d)], hence by Fatou's lemma $\mathcal{W}^\nu(x) \leq \liminf_k \mathcal{W}^{\nu_k}(x)$. This one-sided inequality does not transfer pointwise divergence to the weak-$*$ limit. For instance, suppose that $s \geq 1$ and let $E = [0,1] \times \{0\}^{d-1} \subset \mathbb{R}^d$. Since $\mathcal{H}^s(E) < \infty$, the finite-critical-measure theorem used in the proof of Lemma 6.3 gives $\mathcal{C}_{\alpha,q}^{\mathbb{R}^d}(E) = 0$. Let λ be the normalized length measure on E and ν the normalized length measure on $[\frac{1}{2}, 1] \times \{0\}^{d-1}$. Since $\lambda(B(x,r)) \geq r$ for $x \in E$ and $r \leq \frac{1}{2}$, $\mathcal{W}^\lambda \geq \int_0^{1/2} r^{(1-s)\theta-1} dr = +\infty$ on E, so $\nu_k := (1-2^{-k})\nu + 2^{-k}\lambda$ satisfies $\mathcal{W}^{\nu_k} \equiv +\infty$ on E for every k; but $\nu_k \to \nu$, and $\mathcal{W}^\nu(0) \leq \int_{1/2}^1 r^{-s\theta-1} dr < \infty$ because $\nu(B(0,r)) = 0$ for $r \leq \frac{1}{2}$. This is why Steps 3–6 apply compactness only to the truncated functionals T_m^ν and assemble the final measure by a convex series rather than by a limit.

Theorem 6.16 is the sole measure-theoretic input: in the fixed stereographic chart it yields a probability measure μ carried by $F := \sigma(K)$ such that $\mathcal{W}^\mu \equiv +\infty$ on F. This divergence gives both the blow-up of the Newtonian potential on F and the completeness of the conformal metric, while $\mu(F) = 1$ controls the potential at infinity, yielding the smooth extension across the pole and the global positive lower bound needed for negative shifts. For the remainder of this section we use the sign convention $\Delta_g := \operatorname{div}_g \nabla$. Thus the conformal Laplacian is

$$L_g := -\frac{4(n-1)}{n-2}\Delta_g + \mathrm{Sc}_g,$$

and its conformal covariance is

$$\mathrm{Sc}_{w^{4/(n-2)}g} = w^{-(n+2)/(n-2)} L_g w \qquad (w > 0 \text{ smooth}). \tag{22}$$

Karakhanyan's existence theorem [Kar24, Theorem 1.1] does not provide these controls, and its existence direction relies on [Kar24, Proposition 2.1], whose decisive compactness step is only sketched (Remark 6.17). The following lemma therefore adapts [Kar24, Theorem 1.1, implication (ii)⇒(i), equations (4.1)–(4.7)], retaining the arguments for regularity at the pole, uniform blow-up at the singular set, and completeness; the lemma after it bounds the conformal factor from below through its potential representation and uses the linearity of L_g to construct the shifted family. It is stated for an arbitrary compact set E and pole p_∞; it is applied with $(\Omega, E, p_\infty) = (M, K, p_\sigma)$, where $\sigma_{p_\sigma} = \sigma$ is the fixed chart.

Lemma 6.18 (Controlled one-chart scalar-flat metric)**.** *Let $n \geq 4$, let $E \subset S^n$ be nonempty and compact, and assume that $\Omega := S^n \setminus E$ is connected. Let $p_\infty \in \Omega$, and let $\sigma_{p_\infty} : S^n \setminus \{p_\infty\} \to \mathbb{R}^n$ be the stereographic projection with pole p_∞. Suppose that $\mathcal{C}^{\mathbb{R}^n}_{1+2/n,\,n/2}\big(\sigma_{p_\infty}(E)\big) = 0$. Then there is a smooth function $\phi : \Omega \to (0,\infty)$ such that $h := \phi^{4/(n-2)} g_{st}\big|_\Omega$ is complete and scalar-flat. Moreover,*

$$\phi(x) \longrightarrow +\infty \quad \text{uniformly as } d_{g_{st}}(x, E) \longrightarrow 0, \qquad \text{and} \qquad \phi(p_\infty) = 2^{-(n-2)/2}.$$

Proof. Put $F := \sigma_{p_\infty}(E)$, $\quad R_F := \max_{y\in F} |y|$, $\quad \rho_F := \frac{1}{1+2R_F} > 0$. Then F is a nonempty compact subset of $\mathbb{R}^n$. Recall from (11) and (12) that

$$s = \frac{n-2}{2}, \qquad \theta = \frac{2}{n-2}, \qquad s\theta = 1, \qquad (n-2)\theta = 2, \qquad n-2-s = s,$$

and put $\beta_0 := \frac{n-2}{2}$ for the exponent of the round conformal factor. For $(d,\alpha,q) = \big(n, 1+\frac{2}{n}, \frac{n}{2}\big)$, the assumption $n \geq 4$ gives $q = \frac{n}{2} \geq 2$ and $\alpha q = \frac{n+2}{2} \leq n$. Thus Theorem 6.16 applies and yields a Radon probability measure μ such that

$$\mu(\mathbb{R}^n \setminus F) = 0, \qquad \mathcal{W}^\mu(x) = +\infty \quad \text{for every } x \in F.$$

Define

$$u(x) := \int_F |x-y|^{2-n}\, d\mu(y) \in (0,\infty], \qquad x \in \mathbb{R}^n.$$

Step 1: the Newtonian potential. Let $Q \subset \mathbb{R}^n \setminus F$ be compact and put $\delta := \operatorname{dist}(Q,F) > 0$. Every x-derivative of order j of the kernel $|x-y|^{2-n}$ is bounded on $Q \times F$ by a constant depending only on j and δ, so differentiation under the integral sign is legitimate. Hence u is positive, smooth, and harmonic on $\mathbb{R}^n \setminus F$.

We first prove that $u(x) = +\infty$ for every $x \in F$. Fix $x_0 \in F$ and suppose, to the contrary, that $u(x_0) < \infty$. For $0 < r \leq 1$, since $|x_0 - y|^{2-n} > r^{2-n}$ on $B(x_0,r)$,

$$\mu(B(x_0,r))\, r^{2-n} \leq \int_{B(x_0,r)} |x_0-y|^{2-n}\, d\mu(y) \leq u(x_0),$$

$$\text{and hence} \qquad \mu(B(x_0,r)) \leq u(x_0)\, r^{n-2}.$$

Using $(n-2-s)\theta = s\theta = 1$, we obtain

$$\mathcal{W}^\mu(x_0) = \int_0^1 \left(\frac{\mu(B(x_0,r))}{r^s}\right)^\theta \frac{dr}{r} \le u(x_0)^\theta \int_0^1 r^{(n-2-s)\theta}\frac{dr}{r} = u(x_0)^\theta \int_0^1 dr < \infty,$$

contradicting the choice of μ.

The function u is lower semicontinuous on $\mathbb{R}^n$ by Fatou's lemma, so $\{u > A\}$ is open for every $A > 0$. Since $u \equiv +\infty$ on F, this open set contains the compact set F, and hence it contains a uniform neighborhood of F: there exists $\varepsilon_A > 0$ such that $\{x : \operatorname{dist}(x,F) < \varepsilon_A\} \subset \{u > A\}$. Thus

$$u(x) \longrightarrow +\infty \quad \text{uniformly as} \quad \operatorname{dist}(x,F) \longrightarrow 0. \tag{23}$$

We also record the behavior of u at infinity. If $|x| \ge 1 + 2R_F$ and $y \in F$, then $|x-y| \ge |x| - R_F \ge \frac{|x|}{2}$, and hence $u(x) \le 2^{n-2}|x|^{2-n}$. Moreover, $(|x|/|x-y|)^{n-2} \to 1$ uniformly for $y \in F$ as $|x| \to \infty$, and the preceding estimate bounds this integrand by 2^{n-2} for all sufficiently large $|x|$. Since $\mu(F) = 1$, dominated convergence gives

$$\lim_{|x|\to\infty} |x|^{n-2}u(x) = \lim_{|x|\to\infty} \int_F \left(\frac{|x|}{|x-y|}\right)^{n-2} d\mu(y) = 1. \tag{24}$$

Step 2: construction of the metric and extension across the pole. Set $U_{\mathrm{rd}}(\xi) := \left(\frac{2}{1+|\xi|^2}\right)^{\beta_0}$. Then $0 < U_{\mathrm{rd}} \le U_{\mathrm{rd}}(0) = 2^{\beta_0}$, and $(\sigma_{p_\infty}^{-1})^* g_{st} = U_{\mathrm{rd}}^{4/(n-2)} g_{\mathrm{E}}$, where g_{E} is the Euclidean metric. Define ϕ on $\Omega \setminus \{p_\infty\}$ by

$$\phi \circ \sigma_{p_\infty}^{-1} := \frac{u}{U_{\mathrm{rd}}} \qquad \text{on } \sigma_{p_\infty}(\Omega \setminus \{p_\infty\}) = \mathbb{R}^n \setminus F. \tag{25}$$

It follows that

$$(\sigma_{p_\infty}^{-1})^* \left(\phi^{4/(n-2)} g_{st}\right) = u^{4/(n-2)} g_{\mathrm{E}} =: \widetilde{h}.$$

Since u is positive and harmonic on $\mathbb{R}^n \setminus F$ and $\mathrm{Sc}_{g_{\mathrm{E}}} = 0$, formula (22) gives $\mathrm{Sc}_{\widetilde{h}} = 0$. Thus h is smooth and scalar-flat on $\Omega \setminus \{p_\infty\}$.

It remains to extend the metric across p_∞. Define $I(z) := \frac{z}{|z|^2}$, $\quad \psi(z) := \sigma_{p_\infty}^{-1}(I(z)) \quad (z \ne 0), \quad \psi(0) := p_\infty$. Viewing $S^n \subset \mathbb{R}^{n+1}$ as the unit sphere, let σ_{-p_∞} denote stereographic projection from the antipode $-p_\infty$, using the same orthogonal identification of $p_\infty^\perp$ with $\mathbb{R}^n$. Since $I \circ \sigma_{p_\infty} = \sigma_{-p_\infty}$ on $S^n \setminus \{\pm p_\infty\}$, we have $\psi = \sigma_{-p_\infty}^{-1}$, including at $z = 0$; hence ψ is a smooth inverse chart about p_∞. For $0 < |z| < \rho_F$, put $\widehat{\phi} := \phi \circ \psi$; this is well defined because $|I(z)| > 1 + 2R_F > R_F$, and hence $I(z) \notin F$. Since

$$I^* g_{\mathrm{E}} = |z|^{-4} g_{\mathrm{E}} \qquad \text{and} \qquad U_{\mathrm{rd}}(I(z)) = \left(\frac{2|z|^2}{1+|z|^2}\right)^{\beta_0} = |z|^{n-2} U_{\mathrm{rd}}(z),$$

we have $\psi^* g_{st} = U_{\mathrm{rd}}(z)^{4/(n-2)} g_{\mathrm{E}}$: the round metric has the same conformal factor in the z-coordinate. Furthermore,

$$U_{\mathrm{rd}}(z)\widehat{\phi}(z) = \frac{U_{\mathrm{rd}}(z)\, u(I(z))}{U_{\mathrm{rd}}(I(z))} = |z|^{2-n} u(I(z)) =: v(z).$$

Since $\Delta v(z) = |z|^{-n-2}(\Delta u)(I(z))$, the function v is harmonic on the punctured ball $0 < |z| < \rho_F$. For $0 < |z| \le \rho_F$ we have $|I(z)| \ge 1 + 2R_F$, so Step 1 gives

$0 < v(z) \leq |z|^{2-n}\, 2^{n-2}|I(z)|^{2-n} = 2^{n-2}$. Hence v is bounded near the origin and extends harmonically across $z = 0$, and by (24),

$$v(0) = \lim_{z \to 0} |z|^{2-n} u(I(z)) = \lim_{|x| \to \infty} |x|^{n-2} u(x) = 1.$$

Consequently, $\widehat{\phi} = v/U_{\mathrm{rd}}$ extends smoothly and positively across $z = 0$, with $\widehat{\phi}(0) = v(0)/U_{\mathrm{rd}}(0) = 2^{-\beta_0}$. Since ψ is a smooth inverse chart about p_∞, this defines a smooth positive extension of ϕ across p_∞, with $\phi(p_\infty) = 2^{-\beta_0} = 2^{-(n-2)/2}$. In the z-coordinate,

$$\psi^* h = \widehat{\phi}^{4/(n-2)} U_{\mathrm{rd}}^{4/(n-2)} g_{\mathrm{E}} = v^{4/(n-2)} g_{\mathrm{E}} \qquad \text{on } |z| < \rho_F,$$

which is smooth and scalar-flat by (22), because v is positive and harmonic. Thus h is smooth and scalar-flat on all of Ω.

We next transfer (23) to ϕ. Since $U_{\mathrm{rd}} \leq 2^{\beta_0}$, equation (25) gives $\phi \geq 2^{-\beta_0}\, u \circ \sigma_{p_\infty}$ on $\Omega \setminus \{p_\infty\}$. Fix $A > 0$ and apply (23) with the threshold $2^{\beta_0} A$: there is $\varepsilon_A > 0$ such that $\mathrm{dist}(x, F) < \varepsilon_A$ implies $u(x) > 2^{\beta_0} A$. The set

$$Q_A := \{p_\infty\} \cup \{x \in S^n \setminus \{p_\infty\} : \mathrm{dist}(\sigma_{p_\infty}(x), F) \geq \varepsilon_A\}$$

is closed in S^n because its complement is the open set

$$\sigma_{p_\infty}^{-1}(\{z \in \mathbb{R}^n : \mathrm{dist}(z, F) < \varepsilon_A\}).$$

Thus Q_A is compact, and it is disjoint from E. Therefore $\eta_A := d_{g_{st}}(Q_A, E) > 0$. If $d_{g_{st}}(x, E) < \eta_A$, then $x \notin Q_A$, so $x \neq p_\infty$ and $\mathrm{dist}(\sigma_{p_\infty}(x), F) < \varepsilon_A$, whence $\phi(x) \geq 2^{-\beta_0} u(\sigma_{p_\infty}(x)) > A$. It follows that

$$\phi(x) \longrightarrow +\infty \quad \text{uniformly as } \; d_{g_{st}}(x, E) \longrightarrow 0. \tag{26}$$

Step 3: completeness. For a Riemannian manifold (Z, g), call a locally Lipschitz map $c : [0, T) \to Z$, $0 < T \leq \infty$, *divergent* if, for every compact $Q \subset Z$, there is $t_Q \in (0, T)$ such that $c(t) \notin Q$ for all $t > t_Q$. Here local Lipschitz continuity is measured with respect to the Riemannian distance d_g. By continuity and the Lindelöf property of $[0, T)$, there is a countable cover by parameter intervals, each of which is mapped by c into a single coordinate neighborhood on which the coordinate distance and d_g are bi-Lipschitz equivalent. Rademacher's theorem applied to the corresponding coordinate representatives therefore gives a vector $c'(t) \in T_{c(t)}Z$ for almost every t. On chart overlaps, the chain rule for the smooth transition maps shows that this vector is independent of the chosen chart. For every $0 < T' < T$, the restriction $c|_{[0,T']}$ is Lipschitz and hence absolutely continuous. We define its Riemannian length by

$$\mathrm{Len}_g(c) := \sup_{0 < T' < T} \int_0^{T'} |c'(t)|_g \, dt = \int_0^T |c'(t)|_g \, dt \in [0, \infty].$$

We use the same notation, with the integral taken over the given parameter interval, for locally Lipschitz curves defined on other intervals and for their restrictions. We first record the criterion that if (Z, g) is connected and every divergent locally Lipschitz curve in Z has infinite g-length, then (Z,g) is complete. Indeed, if (Z,g) were incomplete, Hopf–Rinow would produce a maximal unit-speed geodesic $c:[0,T)\to Z$ with $T<\infty$. This geodesic is divergent. Otherwise there would be a compact set $Q\subset Z$ and times $t_\ell\uparrow T$ such that $c(t_\ell)\in Q$. The vectors $(c(t_\ell),c'(t_\ell))$ lie in the compact unit-sphere bundle over Q, so, after passage to a subsequence, they converge to some $v\in TZ$. Local existence and continuous dependence for the smooth geodesic vector field give an open neighborhood $W_v\subset TZ$ of v and a number $\delta>0$ such that its flow is defined on $[0,\delta]\times W_v$. For all sufficiently large ℓ, $(c(t_\ell),c'(t_\ell))\in W_v$ and $T-t_\ell<\delta$. By uniqueness, the projected flow line with this initial datum agrees with c on $[t_\ell,T)$ and remains defined beyond T, contradicting maximality. Thus an incomplete connected Riemannian manifold admits a divergent locally Lipschitz curve of finite length. The contrapositive is the criterion used below; compare [Kar24, Theorem 2.3].

By this criterion, applied to $(Z,g)=(\Omega,h)$, it suffices to prove that every divergent locally Lipschitz curve in Ω has infinite h-length.

Since $E\neq\varnothing$ and $p_\infty\in\Omega$, the manifold Ω is a nonempty proper open subset of S^n. It is noncompact, since otherwise it would be both open and closed in the connected sphere S^n.

For the completeness step specifically, the following dyadic-annulus argument is adapted from [Kar24, proof of Theorem 1.1, implication (ii)⇒(i), equations (4.3)–(4.7)]. Let $c:[0,T)\to\Omega$ be a divergent locally Lipschitz curve. Choose $\rho_0>0$ with $\overline{B}_{S^n}(p_\infty,\rho_0)\subset\Omega$. Since this closed ball is compact and c is divergent, there is $t_0<T$ such that $c(t)\notin\overline{B}_{S^n}(p_\infty,\rho_0)$ for $t\geq t_0$. Therefore $\gamma:=\sigma_{p_\infty}\circ c|_{[t_0,T)}$ is a bounded locally Lipschitz curve in $\mathbb{R}^n\setminus F$, because $S^n\setminus B_{S^n}(p_\infty,\rho_0)$ is a compact subset of the stereographic chart. The curve γ is also divergent there: if $Q\subset\mathbb{R}^n\setminus F$ is compact, then $\sigma_{p_\infty}^{-1}(Q)$ is a compact subset of Ω, which the tail of c eventually avoids. Since $(\sigma_{p_\infty}^{-1})^*h=\widetilde{h}=u^{4/(n-2)}g_{\mathrm{E}}$, the $\widetilde{h}$-length element is $ds_{\widetilde{h}}=u^{2/(n-2)}\,ds_{\mathrm{E}}=u^\theta\,ds_{\mathrm{E}}$, and it suffices to show that $\mathrm{Len}_{\widetilde{h}}(\gamma)=\infty$.

Choose times $\varpi_m\uparrow T$. Since γ is bounded, after passing to a subsequence we may assume that $\gamma(\varpi_m)\to x_0$ for some $x_0\in\mathbb{R}^n$. Necessarily $x_0\in F$: otherwise a closed Euclidean ball of sufficiently small positive radius centered at x_0 would be a compact subset of $\mathbb{R}^n\setminus F$ visited by γ at arbitrarily late times, contradicting divergence. Set $r_k:=2^{-k}$ and choose $k_0\geq 0$ such that $r_{k_0}\leq|\gamma(t_0)-x_0|$, which is possible because $\gamma(t_0)\notin F\ni x_0$. For $k\geq k_0$, define

$$
\begin{aligned}
b_k &:= \inf\{t\in[t_0,T):|\gamma(t)-x_0|\leq r_{k+1}\}\,,\\
a_k &:= \sup\{t\in[t_0,b_k]:|\gamma(t)-x_0|\geq r_k\}\,.
\end{aligned}
$$

The defining set for b_k is nonempty because $\gamma(\varpi_m)\to x_0$. Since $|\gamma(t_0)-x_0|\geq r_k>r_{k+1}$, continuity at t_0 gives $t_0<b_k$, while nonemptiness of the defining set gives $b_k<T$. Choose a sequence in that set converging to b_k. Continuity gives $|\gamma(b_k)-x_0|\leq r_{k+1}$. On the other hand, every $t\in[t_0,b_k)$ satisfies $|\gamma(t)-x_0|>r_{k+1}$; letting $t\uparrow b_k$ gives the reverse inequality. Hence $|\gamma(b_k)-x_0|=r_{k+1}$.

The defining set for a_k is a nonempty closed subset of $[t_0,b_k]$, so its supremum is attained. Since $|\gamma(b_k)-x_0|=r_{k+1}<r_k$, one has $a_k<b_k$. Moreover, $|\gamma(a_k)-x_0|\geq r_k$. If this inequality were strict, continuity would give a point $t\in(a_k,b_k)$ with

$|\gamma(t) - x_0| \geq r_k$, contrary to the definition of a_k. Thus $|\gamma(a_k) - x_0| = r_k$. The definitions now give

$$|\gamma(a_k) - x_0| = r_k, \qquad |\gamma(b_k) - x_0| = r_{k+1},$$
$$\gamma((a_k, b_k)) \subset A_k := \{x : r_{k+1} < |x - x_0| < r_k\}.$$

The annuli A_k are pairwise disjoint, so the intervals (a_k, b_k) are pairwise disjoint. Indeed, if $t \in (a_k, b_k) \cap (a_\ell, b_\ell)$ for $k \neq \ell$, then $\gamma(t) \in A_k \cap A_\ell$, a contradiction. Consequently, the closed intervals $[a_k, b_k]$ have pairwise disjoint interiors and can intersect only at endpoints. Since finitely many endpoints have Lebesgue measure zero, for every $K \geq k_0$,

$$\mathrm{Len}_{\widetilde{h}}(\gamma) \geq \sum_{k=k_0}^{K} \mathrm{Len}_{\widetilde{h}}\big(\gamma|_{[a_k,b_k]}\big). \tag{27}$$

Each crossing has Euclidean length at least $|\gamma(a_k) - \gamma(b_k)| \geq r_k - r_{k+1} = r_k/2$. If $x \in A_k$ and $y \in B(x_0, r_{k+2})$, then

$$|x - y| \leq |x - x_0| + |x_0 - y| \leq r_k + r_{k+2} = \frac{5}{4} r_k,$$

and hence $\qquad u(x) \geq \left(\frac{4}{5}\right)^{n-2} \mu(B(x_0, r_{k+2}))\, r_k^{2-n}.$

Using $(n-2)\theta = 2$ and $\gamma((a_k, b_k)) \subset A_k \setminus F$, we obtain the following estimate. The endpoints of the crossing lie on ∂A_k and do not affect the length integral, so the infimum over $A_k \setminus F$ is valid:

$$\mathrm{Len}_{\widetilde{h}}\big(\gamma|_{[a_k,b_k]}\big) \geq \Big(\inf_{A_k \setminus F} u\Big)^{\theta} \frac{r_k}{2} \geq \frac{8}{25} \frac{\mu(B(x_0, r_{k+2}))^{\theta}}{r_k}.$$

Letting $K \to \infty$ in (27) gives

$$\mathrm{Len}_{\widetilde{h}}(\gamma) \geq \frac{8}{25} \sum_{k \geq k_0} \frac{\mu(B(x_0, r_{k+2}))^{\theta}}{r_k} = \frac{2}{25} \sum_{m \geq k_0+2} \frac{\mu(B(x_0, r_m))^{\theta}}{r_m}. \tag{28}$$

On the other hand, since $r \mapsto \mu(B(x_0, r))$ is nondecreasing and $s\theta = 1$,

$$\begin{aligned} \mathcal{W}^{\mu}(x_0) &= \sum_{m \geq 0} \int_{r_{m+1}}^{r_m} \left(\frac{\mu(B(x_0, r))}{r^s}\right)^{\theta} \frac{dr}{r} \\ &\leq \sum_{m \geq 0} \mu(B(x_0, r_m))^{\theta} \int_{r_{m+1}}^{r_m} \frac{dr}{r^2} \\ &= \sum_{m \geq 0} \frac{\mu(B(x_0, r_m))^{\theta}}{r_m}. \end{aligned} \tag{29}$$

Because $x_0 \in F$, Theorem 6.16 gives $\mathcal{W}^{\mu}(x_0) = +\infty$. Every term in the series on the right-hand side of (29) is finite, since $\mu(B) \leq 1$ and $r_m > 0$. Hence the series, and therefore each of its tails, diverges, and (28) yields $\mathrm{Len}_{\widetilde{h}}(\gamma) = +\infty$. Thus every divergent locally Lipschitz curve in Ω has infinite h-length, and the criterion established at the beginning of Step 3 shows that h is complete. $\square$

The preceding metric yields all three scalar-curvature signs after constant shifts. The only additional inputs are the lower bound $m_0 > 0$, which follows from the fact that μ is a probability measure supported on a bounded set, and the linearity of the conformal Laplacian, together with $L_{g_{st}}\phi = 0$ pointwise on Ω and $L_{g_{st}}1 = n(n-1)$. The completeness comparison and the scalar-curvature computation are independent parts of the argument.

Lemma 6.19 (Complete shifts of every sign). *In the setting of Lemma 6.18, let ϕ be the function constructed there and put $m_0 := \inf_\Omega \phi$. Then $m_0 > 0$, and for every $t \in (-m_0, \infty)$ the metric $g^{(t)} := (\phi + t)^{4/(n-2)}\, g_{st}\big|_\Omega$ is smooth, complete, and locally conformally flat, with*

$$\mathrm{Sc}_{g^{(t)}} = n(n-1)\, t\, (\phi + t)^{-(n+2)/(n-2)}, \qquad \lim_{t\to 0} \int_\Omega |\mathrm{Sc}_{g^{(t)}}|^{n/2}\, dV_{g^{(t)}} = 0. \tag{30}$$

In particular, $\mathrm{Sc}_{g^{(t)}}$ is negative, zero, or positive according as $t < 0$, $t = 0$, or $t > 0$. For $t > 0$, the rescaled metric $\bar{g}^{(t)} := t^{-4/(n-2)} g^{(t)} = \left(1 + \frac{\phi}{t}\right)^{4/(n-2)} g_{st}\big|_\Omega$ satisfies

$$0 < \mathrm{Sc}_{\bar{g}^{(t)}} < n(n-1), \qquad \lim_{\delta \downarrow 0} \sup_{\substack{x\in\Omega \\ d_{g_{st}}(x,E)<\delta}} \mathrm{Sc}_{\bar{g}^{(t)}}(x) = 0.$$

Proof. Retain the notation F, μ, and R_F from the construction in Lemma 6.18, and put $\beta_0 := \frac{n-2}{2}$.

The lower bound. In the chart, by (25),

$$\phi \circ \sigma_{p_\infty}^{-1}(x) = 2^{-\beta_0}(1 + |x|^2)^{\beta_0} \int_F |x - y|^{-2\beta_0}\, d\mu(y).$$

For $y \in F$, $|x - y| \le |x| + R_F \le (1 + R_F)(1 + |x|)$, and $1 + |x|^2 \ge \frac{1}{2}(1 + |x|)^2$. Since $\mu(F) = 1$, for every $x \in \mathbb{R}^n \setminus F$ one has

$$\begin{aligned} \phi \circ \sigma_{p_\infty}^{-1}(x) &\ge 2^{-\beta_0} \cdot 2^{-\beta_0}(1 + |x|)^{2\beta_0}(1 + R_F)^{-2\beta_0}(1 + |x|)^{-2\beta_0} \\ &= 2^{-(n-2)}(1 + R_F)^{-(n-2)}. \end{aligned} \tag{31}$$

Equation (31) applies on $\sigma_{p_\infty}^{-1}(\mathbb{R}^n \setminus F) = \Omega \setminus \{p_\infty\}$. At the remaining point, $\phi(p_\infty) = 2^{-\beta_0}$ and $2^{-\beta_0} = (\sqrt{2})^{-(n-2)} > [2(1 + R_F)]^{-(n-2)} = 2^{-(n-2)}(1 + R_F)^{-(n-2)}$, because $2(1 + R_F) \ge 2 > \sqrt{2}$. Thus the same lower bound holds at p_∞, and hence $m_0 \ge 2^{-(n-2)}(1 + R_F)^{-(n-2)} > 0$.

Smoothness and completeness. For $t > -m_0$, $\phi + t \ge m_0 + t > 0$, so $g^{(t)}$ is a smooth metric conformal to g_{st}, hence locally conformally flat. With $a_t := \min\{1, 1 + t/m_0\} > 0$ and $b_t := \max\{1, 1 + t/m_0\}$, the bound $\phi \ge m_0$ gives $a_t \le 1 + t/\phi \le b_t$; that is, $a_t^{4/(n-2)} h \le g^{(t)} = (1 + t/\phi)^{4/(n-2)} h \le b_t^{4/(n-2)} h$. Taking lengths and then infima over curves gives $a_t^{2/(n-2)} d_h \le d_{g^{(t)}} \le b_t^{2/(n-2)} d_h$. Thus d_h and $d_{g^{(t)}}$ are bi-Lipschitz equivalent, so the completeness of h implies that of $g^{(t)}$.

Curvature. Since $\mathrm{Sc}_{g_{st}} = n(n-1)$ and $\mathrm{Sc}_{w^{4/(n-2)}g_{st}} = w^{-(n+2)/(n-2)} L_{g_{st}} w$ for smooth $w > 0$, scalar-flatness of h means $L_{g_{st}}\phi = 0$ pointwise on Ω, and $L_{g_{st}}1 = n(n-1)$. By linearity, $L_{g_{st}}(\phi + t) = n(n-1)\, t$, and therefore $\mathrm{Sc}_{g^{(t)}} = n(n-1)\, t\, (\phi + t)^{-(n+2)/(n-2)}$, which is the first part of (30); since $\phi + t > 0$, the sign of $\mathrm{Sc}_{g^{(t)}}$

is the sign of t. For the second part, note that $dV_{g^{(t)}} = (\phi + t)^{2n/(n-2)}\, dV_{g_{st}}$ and $\frac{2n}{n-2} - \frac{n(n+2)}{2(n-2)} = -\frac{n}{2}$, so that

$$\begin{aligned}\int_\Omega |\mathrm{Sc}_{g^{(t)}}|^{n/2}\, dV_{g^{(t)}} &= [n(n-1)]^{n/2} \int_\Omega \Big(\frac{|t|}{\phi + t}\Big)^{n/2} dV_{g_{st}} \\ &\le [n(n-1)]^{n/2}\, \mathrm{Vol}_{g_{st}}(S^n) \Big(\frac{|t|}{m_0 + t}\Big)^{n/2},\end{aligned}$$

where we use $\phi + t \ge m_0 + t > 0$. The right-hand side tends to 0 as $t \to 0$.

The rescaled family. Under constant rescaling, scalar curvature transforms by $\mathrm{Sc}_{\varrho g} = \varrho^{-1}\mathrm{Sc}_g$ for $\varrho > 0$. For $t > 0$, taking $\varrho = t^{-4/(n-2)}$ and using $\frac{4}{n-2} + 1 = \frac{n+2}{n-2}$, we obtain

$$\mathrm{Sc}_{\bar{g}^{(t)}} = t^{\frac{4}{n-2}}\, \mathrm{Sc}_{g^{(t)}} = n(n-1)\, t^{\frac{n+2}{n-2}} (\phi + t)^{-\frac{n+2}{n-2}} = n(n-1)\Big(1 + \frac{\phi}{t}\Big)^{-\frac{n+2}{n-2}}.$$

Since $\phi > 0$, this lies in $(0, n(n-1))$, and the uniform divergence (26) of ϕ at E gives $\mathrm{Sc}_{\bar{g}^{(t)}}(x) \to 0$ uniformly as $d_{g_{st}}(x, E) \to 0$. □

Finally, we assemble the preceding topological and analytic constructions.

Proof of Theorem 6.1. Fix $n \ge 4$, fix the tower of Proposition 6.6, and put $K := \bigcap_{j \ge 1} V_j$ and $M := S^n \setminus K$.

The set K. By Proposition 6.7, K is a nondegenerate continuum with $\dim_{\mathcal{H}} K = 1$, $K \subset V_1 \subset \operatorname{int} B \Subset S^n \setminus \{p_\sigma\}$, and $\mathrm{cap}(K) = \mathcal{C}^{\mathbb{R}^n}_{1+2/n,\, n/2}\big(\sigma(K)\big) = 0$. In particular, $p_\sigma \in M$, and M, being an open subset of S^n, is a smooth manifold. It is noncompact: indeed, M is nonempty and proper because $p_\sigma \in M$ and $K \neq \varnothing$, and if M were compact, then it would be both open and closed in the connected sphere S^n.

Assertion (1). M is contractible by Proposition 6.14 and not simply connected at infinity by Proposition 6.15.

Assertion (2): the scalar-flat metric. We apply Lemma 6.18 with $(\Omega, E, p_\infty) := (M, K, p_\sigma)$. Its hypotheses hold: $n \ge 4$; $E = K$ is nonempty and compact; $\Omega = M$ is connected, being contractible; $p_\infty = p_\sigma \in M$; and the stereographic chart with pole p_σ is the fixed chart σ, so the capacity assumption $\mathcal{C}^{\mathbb{R}^n}_{1+2/n,\, n/2}(\sigma(K)) = 0$ is exactly $\mathrm{cap}(K) = 0$. The lemma provides a smooth function $\phi : M \to (0, \infty)$ such that $h := \phi^{4/(n-2)} g_{st}|_M$ is complete and scalar-flat, with $\phi(x) \to +\infty$ uniformly as $d_{g_{st}}(x, K) \to 0$.

Assertion (2): the shifted family. We apply Lemma 6.19 to this ϕ. It gives $m_0 = \inf_M \phi > 0$; for every $t \in (-m_0, \infty)$ the metric $g^{(t)} = (\phi + t)^{4/(n-2)} g_{st}|_M$ is smooth, complete, and locally conformally flat, with $\mathrm{Sc}_{g^{(t)}} = n(n-1)\, t\, (\phi + t)^{-(n+2)/(n-2)}$, which has the sign of t because $\phi + t > 0$ on M (and $g^{(0)} = h$). Moreover, $\int_M |\mathrm{Sc}_{g^{(t)}}|^{n/2}\, dV_{g^{(t)}} \to 0$ as $t \to 0$ by (30). Finally, for $t > 0$, Lemma 6.19 gives $\mathrm{Sc}_{\bar{g}^{(t)}} = n(n-1)(1 + \phi/t)^{-(n+2)/(n-2)} \in (0, n(n-1))$ for $\bar{g}^{(t)} = t^{-4/(n-2)} g^{(t)}$, and the uniform divergence of ϕ at K yields

$$\lim_{\delta \downarrow 0} \sup_{\substack{x \in M \\ d_{g_{st}}(x, K) < \delta}} \mathrm{Sc}_{\bar{g}^{(t)}}(x) = 0.$$

This proves (2) and completes the proof. □

Corollary 6.20 (Failure of Euclidean rigidity without the additional hypotheses)**.** *Fix $n \geq 4$, and let K, $M = S^n \setminus K$, and $(g^{(t)})_{t\in(-m_0,\infty)}$ be as in Theorem 6.1. The manifold M is connected, smooth, open, and simply connected, and $\widetilde{H}_i(M;\mathbb{Z}) = 0$ for $i \geq 0$. For every $t \geq 0$, the metric $g^{(t)}$ is complete and locally conformally flat, with $\mathrm{Sc}_{g^{(t)}} \geq 0$. Moreover, the inclusion $\iota : M \hookrightarrow S^n$ may be chosen as a developing map of $(M, g^{(t)})$. The omitted set, and also the topological boundary of the developing image, is $\Lambda := S^n \setminus \iota(M) = \partial\iota(M) = K$, which is a nondegenerate continuum. Nevertheless, M is not homeomorphic to $\mathbb{R}^n$.*

Consequently, the conclusion of Theorem E(ii) *can fail when its two additional topological hypotheses are omitted simultaneously. The disjunction of conditions* (i)–(v) *in Corollary 5.4 likewise cannot be omitted, even if its conclusion is weakened from conformal equivalence to homeomorphism.*

Proof. Theorem 6.1(1) states that M is a contractible smooth open manifold. Hence M is connected and simply connected and has vanishing reduced integral homology; in particular, $\widetilde{H}_{n-1}(M;\mathbb{Z}) = 0$, as required in Corollary 5.4. The same part of the theorem states that M is not simply connected at infinity. Since $\mathbb{R}^n$ is simply connected at infinity for $n \geq 3$, and simple connectivity at infinity is invariant under homeomorphism, M is not homeomorphic to $\mathbb{R}^n$.

For $t \geq 0$, Theorem 6.1(2) gives completeness, local conformal flatness, and $\mathrm{Sc}_{g^{(t)}} \geq 0$. If $\iota : M \hookrightarrow S^n$ is the inclusion, then $\iota^* g_{st} = (\phi + t)^{-4/(n-2)} g^{(t)}$. Thus ι is a conformal local diffeomorphism and is a developing map for the locally conformally flat structure of $g^{(t)}$. Its image is $M = S^n \setminus K$. Since $\dim_{\mathcal{H}} K = 1 < n$, the set K has empty interior in S^n; hence the omitted set and the topological boundary of the developing image are both exactly K. Since M is simply connected, the holonomy is trivial; accordingly, K is not the classical Kleinian limit set of the holonomy group (which is empty).

After both additional hypotheses are removed from Theorem E(ii), the preceding facts verify every remaining hypothesis, whereas the conclusion $M \cong \mathbb{R}^n$ fails. Moreover, $\lfloor (n-2)/2 \rfloor \geq 1$, so the failure of simple connectivity at infinity implies that M is not $\lfloor (n-2)/2 \rfloor$-connected at infinity; as stated at the beginning of this section, no assertion is made about the small loops condition. Finally, all standing hypotheses of Corollary 5.4 hold, including the required homology vanishing, while its conclusion fails even topologically. This proves both sharpness assertions. □

6.4 Uniformly PSC for $n \geq 6$

The construction underlying Theorem 6.1 can be modified to produce locally conformally flat metrics with uniformly positive scalar curvature on contractible manifolds not homeomorphic to $\mathbb{R}^n$. The rose is replaced by a 2-dimensional Newman spine. The topological construction requires only $n \geq 5$, corresponding to codimension at least three for a 2-complex, while the analytic argument requires the spine dimension $\mathfrak{d} = 2$ to satisfy $\mathfrak{d} \leq a = \frac{n-2}{2}$, and hence $n \geq 6$.

Theorem 6.21. *For every integer $n \geq 6$ there exists a contractible smooth open n-manifold M which is not homeomorphic to $\mathbb{R}^n$ and which admits a smooth complete* *locally conformally flat metric* $\widehat{g}$ *such that*

$$0 < \inf_M \mathrm{Sc}_{\widehat{g}} \le \sup_M \mathrm{Sc}_{\widehat{g}} < \infty.$$

Strategy of the proof. We refine Newman's construction to obtain a finite pure PL 2-complex $P \subset S^n$ whose complement is contractible but not homeomorphic to $\mathbb{R}^n$, and whose area measure is Ahlfors 2-regular. Setting $a := s = (n-2)/2$, we define $\widehat{g} = (1+V)^{4/(n-2)} g_{st}$ using the potential of this measure with chordal kernel $\mathfrak{q}^{-(2+a)}$ when $a > 2$, and its logarithmically corrected analogue when $a = 2$. Indeed, for a power kernel $\mathfrak{q}^{-b}$ with $2 < b < n-2$, the 2-regular potential estimate gives $V \asymp \rho^{2-b}$, while the positive $\mathfrak{q}^{-b-2}$ term in $L_{g_{st}} \mathfrak{q}^{-b}$ gives $L_{g_{st}} V \asymp \rho^{-b}$. Thus the matching condition $L_{g_{st}} V \asymp V^{(a+2)/a}$ requires $-b = (2-b)\frac{a+2}{a}$, equivalently $b = a+2$. Thus $b = a+2 > 2$, whereas $b < n-2 = 2a$ exactly when $a > 2$. At the endpoint $a = 2$, the matching exponent is $b = a+2 = n-2$, the conformal-Laplacian-harmonic exponent, and the logarithmic correction restores a positive leading term. In this critical case the logarithmic exponent is forced by the same matching: if $V \asymp \rho^{-a}\ell(\rho)^{-\lambda}$ and $L_{g_{st}} V \asymp \rho^{-a-2}\ell(\rho)^{-\lambda-1}$, then $L_{g_{st}} V \asymp V^{\kappa}$, where $\kappa = (a+2)/a$, requires $\lambda\kappa = \lambda + 1$, and hence $\lambda = 1/(\kappa - 1) = a/2$. Ahlfors-regular potential estimates, combined with the conformal scalar-curvature calculation, then prove completeness and the required two-sided scalar-curvature bound; the condition $2 \le a$ is exactly $n \ge 6$.

The construction in this subsection is independent of the chart σ, its pole p_σ, the ball B, and the PL structure fixed in the initial setup of this section. For the proof, fix $p_\infty \in S^n$ and a stereographic chart $\sigma_\infty : S^n \setminus \{p_\infty\} \to \mathbb{R}^n$. Whenever a compact set or measure is first constructed in $\mathbb{R}^n$, we use the same symbol for the image of the set or the push-forward of the measure under σ_∞^{-1}; in particular, inclusions such as $\mathbb{R}^5 \subset \mathbb{R}^n \subset S^n$ below are understood through this identification. Viewing S^n as the unit sphere in $\mathbb{R}^{n+1}$, define the chordal distance by $\mathfrak{q}(x,y) := |x-y|_{\mathbb{R}^{n+1}} = 2\sin\Big(\frac{d_{g_{st}}(x,y)}{2}\Big) \in [0,2]$. Then $\frac{2}{\pi} d_{g_{st}}(x,y) \le \mathfrak{q}(x,y) \le d_{g_{st}}(x,y)$, and $\mathfrak{q}(x,y) > 0$ precisely when $x \ne y$. Recall that $a = s = \frac{n-2}{2}$, and set $\kappa = \frac{n+2}{n-2} = \frac{a+2}{a}$ and $\ell(u) = \log\frac{8}{u}$, so that $\ell(\mathfrak{q}) \ge \log 4$ for $0 < \mathfrak{q} \le 2$. For positive functions on a specified set, $f \asymp g$ means $cg \le f \le Cg$ for positive constants independent of the variables under consideration; dependence on fixed parameters is understood. When a limiting regime is specified, the comparison is asserted in that regime; otherwise it holds throughout the indicated set.

A $\mathfrak{d}$*-regular pair* in S^n is a compact set $\Lambda \subset S^n$ together with a Borel probability measure μ with $\operatorname{supp}\mu = \Lambda$ such that, for some $r_0, c, C > 0$,

$$c r^{\mathfrak{d}} \le \mu(B_{\mathfrak{q}}(p,r)) \le C r^{\mathfrak{d}} \qquad (p \in \Lambda,\ 0 < r < r_0). \tag{32}$$

The next lemma is Newman's classical construction [New48], whose general form, starting from a balanced presentation of a perfect group, is described in [Gui16, Example 3.2.2]; the group used below is Newman's own. We record only the refinements needed later: a pure PL spine in an affine chart, its Ahlfors-regular area measure, and the identification of its complement with the corresponding open Newman manifold.

Lemma 6.22 (A 2-regular Newman spine)**.** *For every $n \ge 5$ there is a finite connected pure simplicial 2-complex $P \subset \mathbb{R}^5 \subset \mathbb{R}^n \subset S^n$ such that $\widetilde{H}_*(P;\mathbb{Z}) = 0$ and*

$\pi_1(P) \neq 1$, the complement $M = S^n \setminus P$ is contractible but not homeomorphic to $\mathbb{R}^n$, and the push-forward of the probability measure obtained by normalizing Euclidean area on the Euclidean realization of P makes (P, μ) a 2-regular pair.

Proof. Following Newman [New48], let

$$G = \langle x, y \mid r_1, r_2 \rangle, \qquad r_1 = y^{-3}(xy)^2, \qquad r_2 = x^{-5}(xy)^2.$$

Let X be the associated presentation 2-complex, with unique 0-cell v_0, oriented generator 1-cells x, y, and one 2-cell attached along each of r_1, r_2. Under the substitution $(x, y) = (B, A)$, the relators become Newman's words, $A^{-3}(BA)^2, \qquad B^{-5}(BA)^2$, so X is his complex Z. The only group-theoretic fact imported from Newman at this point is that $G \neq 1$. As Newman observes, this follows from the fact that the relations hold in the icosahedral group A_5. Indeed, the assignments $x \longmapsto (1\,2\,3\,4\,5), \qquad y \longmapsto (1\,4\,2)$ satisfy $x^5 = y^3 = (xy)^2 = 1$, since xy is a product of two disjoint transpositions, regardless of the convention for composition. They therefore define a homomorphism $G \to A_5$ with nontrivial image. The exponent-sum matrix $E = \left(\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right)$ has $\det E = 1$. The map E^T is both the map whose cokernel is the abelianization of G and, by the cellular boundary formula [Hat02, Section 2.2, pp. 140–141], the cellular boundary ∂_2 of X. Since X has a single 0-cell and both 1-cells are loops at that cell, $\partial_1 = 0$. Moreover, the cellular augmentation $\varepsilon : C_0(X) = \mathbb{Z} \to \mathbb{Z}$ is an isomorphism. Thus the augmented cellular chain complex of X is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{E^T} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Because $\det E = 1$, the map E^T is an isomorphism; the augmentation is an isomorphism as well. Hence the cokernel, which is G^{ab}, is zero, and all reduced cellular homology groups of X vanish. Thus G is perfect, X is acyclic, and $\pi_1(X) = G \neq 1$. In particular, X is path connected.

The metric part of the lemma requires a pure simplicial realization of X. Subdivide the attaching circle of each relator 2-cell into one edge for each letter of the relator, map that edge affinely onto the generator edge x or y with the orientation prescribed by the sign of the letter, and cone the resulting polygonal circle to a new interior vertex w, chosen separately for each relator. Give every radial edge the order from w to v_0. For a cone triangle over a letter edge, label its two boundary vertices q_0, q_1 so that the restriction to the ordered outer face $[q_0, q_1]$ is the chosen characteristic map of the corresponding generator edge, rather than its reverse, and order the triangle as $[w, q_0, q_1]$. This choice is also available for a letter with exponent -1, since both boundary vertices are identified with v_0. The restrictions to the radial faces $[w, q_0]$ and $[w, q_1]$ are then the chosen radial characteristic maps. Moreover, a radial edge common to two adjacent triangles receives, from both triangles, the order from w to v_0. Consequently, the restriction of every triangle characteristic map to each face is exactly the characteristic map of the corresponding 1-simplex under the order-preserving identification of that face with Δ^1. The quotient maps embed the interiors of all the vertex, edge, and triangle simplices just described, and these open simplices partition X; moreover, X carries the quotient topology by construction. The quotient is therefore a finite 2-dimensional Δ-complex. Repeated occurrences of a generator edge in an attaching word are permitted in a Δ-complex.

This Δ-complex is pure, because every radial edge is a face of a cone triangle and both generator edges occur in r_1.

The second barycentric subdivision of a Δ-complex is a simplicial complex; see [Hat02, Exercise 23, p. 133]. Indeed, after the first barycentric subdivision the vertices of each simplex correspond to a strictly increasing flag of faces and are therefore distinct; in the next barycentric subdivision each simplex is uniquely determined by the set of vertices corresponding to its flag. Barycentric subdivision preserves finiteness, dimension, and purity. Denote the resulting finite pure simplicial 2-complex by P; its underlying polyhedron is canonically homeomorphic to X, so P is acyclic and $\pi_1(P) = G \neq 1$. Since P is a finite polyhedron of dimension $p = 2$ and the target is $\mathbb{R}^5$,

$$2p + 1 = 5 = \dim \mathbb{R}^5,$$

the dimension hypothesis in [Bry02, Corollary 4.4, p. 235] holds at equality. Apply that corollary to the constant map $P \to \mathbb{R}^5$, with $X_0 = \varnothing$ and the constant control function $\varepsilon \equiv 1$. The restriction to X_0 is vacuously a PL embedding, so the corollary gives a PL embedding $P \hookrightarrow \mathbb{R}^5$. After subdivision, identify P with its linear image in $\mathbb{R}^5 \subset \mathbb{R}^n$ and, according to the convention above, with its inverse-stereographic image in S^n.

We make the PL structure used below explicit. Choose a large affine n-simplex $\mathcal{B} \subset \mathbb{R}^n$ with $P \subset \operatorname{int} \mathcal{B}$. A common subdivision gives a finite triangulation of $\mathcal{B}$ in which both P and $\partial\mathcal{B}$ are subcomplexes. Transport this triangulation to $\sigma_\infty^{-1}(\mathcal{B})$ and triangulate the complementary topological n-ball by coning the induced boundary triangulation to p_∞; equivalently, the required homeomorphism is obtained by compactifying the rays from an interior point of $\mathcal{B}$. We use the resulting PL structure on S^n. In particular, the specified inverse-stereographic image of P is a subpolyhedron of this PL sphere.

After a derived subdivision relative to P, assume that P is the polyhedron of a full subcomplex $\mathcal{L}_P$ of a finite triangulation $\mathcal{T}_P$ of S^n. The subcomplex $\mathcal{L}_P$ is nonempty and proper, because P is nonempty and $P \subset S^n \setminus \{p_\infty\}$. Let $\xi_{\mathcal{N}} : |\mathcal{T}_P| \to [0, 1]$ be the simplicial map taking value 0 on the vertices of $\mathcal{L}_P$ and value 1 on all other vertices, and choose the half-neighborhood model $\mathcal{N} := \xi_{\mathcal{N}}^{-1}([0, 1/2])$. Lemma 6.4, with $c = 1/2$, shows that $\mathcal{N}$ is a compact regular neighborhood of P and a PL n-manifold, with $\operatorname{int} \mathcal{N} = \xi_{\mathcal{N}}^{-1}([0, 1/2))$, $\quad \partial\mathcal{N} = \xi_{\mathcal{N}}^{-1}(1/2)$. Thus the half-neighborhood is chosen from the outset; no arbitrary regular neighborhood is being identified silently with Bryant's model. Set $\mathcal{L}_P^{\mathrm{c}} := C(\mathcal{L}_P, \mathcal{T}_P)$. By Lemma 6.4, $\mathcal{L}_P^{\mathrm{c}}$ is a nonempty proper full subcomplex of $\mathcal{T}_P$, and its level function is $1 - \xi_{\mathcal{N}}$. Since the preceding interior formula gives $S^n \setminus \operatorname{int} \mathcal{N} = \xi_{\mathcal{N}}^{-1}([1/2, 1])$, we may write

$$C := S^n \setminus \operatorname{int} \mathcal{N} = \xi_{\mathcal{N}}^{-1}([1/2, 1]) = (1 - \xi_{\mathcal{N}})^{-1}([0, 1/2]).$$

A second application of Lemma 6.4, now to $(\mathcal{T}_P, \mathcal{L}_P^{\mathrm{c}}, 1 - \xi_{\mathcal{N}})$ with $c = 1/2$, shows that C is a closed regular neighborhood of $|\mathcal{L}_P^{\mathrm{c}}|$ and a compact PL n-manifold, with

$$\partial C = (1 - \xi_{\mathcal{N}})^{-1}(1/2) = \xi_{\mathcal{N}}^{-1}(1/2) = \partial\mathcal{N},$$
$$\operatorname{int} C = (1 - \xi_{\mathcal{N}})^{-1}([0, 1/2)) = \xi_{\mathcal{N}}^{-1}((1/2, 1]).$$

Consequently,

$$\mathcal{N} \cap C = \xi_{\mathcal{N}}^{-1}(1/2) = \partial\mathcal{N} = \partial C. \tag{33}$$

Thus $\partial\mathcal{N}$ and ∂C are the same subpolyhedron $\xi_{\mathcal{N}}^{-1}(1/2)$ of the fixed PL sphere S^n and hence carry the same PL structure induced from S^n. In particular, the one-sided PL collars used below have the same PL base. We give Newman's argument for this chosen embedding; see [New48, §§ 2–3, pp. 193–195] and [Gui16, Example 3.2.2].

By the punctured-sublevel conclusion of Lemma 6.4, applied to $(\mathcal{T}_P, \mathcal{L}_P, \xi_{\mathcal{N}})$, there is a homeomorphism of pairs

$$(\mathcal{N} \setminus P, \partial\mathcal{N}) \cong (\partial\mathcal{N} \times (0,1], \partial\mathcal{N} \times \{1\}).$$

The corresponding factor homotopy is a strong deformation retraction of $\mathcal{N} \setminus P$ onto $\partial\mathcal{N}$; denote its terminal retraction by $r : \mathcal{N} \setminus P \to \partial\mathcal{N}$. Every map in this deformation fixes $\partial\mathcal{N}$ pointwise. Since $P = \xi_{\mathcal{N}}^{-1}(0)$ is contained in $\operatorname{int}\mathcal{N}$, it is disjoint from C. Hence the two sets $\mathcal{N} \setminus P$ and C are closed in $S^n \setminus P$, cover it, and intersect in $\partial\mathcal{N}$ by (33). The closed-cover pasting lemma, applied to the products of these two sets with the parameter interval, therefore pastes this deformation to the identity deformation on C and gives a strong deformation retraction

$$S^n \setminus P \longrightarrow C. \tag{34}$$

Finally, $\mathcal{N}$ collapses onto P by [Bry02, Theorem 3.16]; hence the associated collapse deformation is a strong deformation retraction fixing P pointwise.

The finite polyhedron P is nonempty, compact, locally contractible, and proper in S^n. Since P is acyclic, the universal coefficient theorem gives $\widetilde{H}^*(P;\mathbb{Z}) = 0$. The reduced-complement form of Alexander duality [Hat02, Corollary 3.45] and (34) therefore give

$$\widetilde{H}_i(C;\mathbb{Z}) \cong \widetilde{H}_i(S^n \setminus P;\mathbb{Z}) \cong \widetilde{H}^{n-i-1}(P;\mathbb{Z}) = 0 \qquad (i \geq 0).$$

Thus C is acyclic. The boundary $\partial\mathcal{N}$ is nonempty. Indeed, $\mathcal{N}$ is connected because it deformation retracts onto the connected polyhedron P; if its boundary were empty, the codimension-zero submanifold $\mathcal{N} \subset S^n$ would be both open and closed and hence equal to S^n, which is impossible because $\mathcal{N} \simeq P$ and $H_n(P;\mathbb{Z}) = 0$. In particular, C is nonempty. The equality $\widetilde{H}_0(C;\mathbb{Z}) = 0$ also shows that C is path connected.

We next prove simple connectivity. Let $\gamma_0 : S^1 \to S^n \setminus P$ be a loop and set $d := \operatorname{dist}_{g_{st}}(\gamma_0(S^1), P) > 0$. Choose triangulations of S^1 and S^n with P a subcomplex. After subdivision, controlled simplicial approximation [Bry02, p. 222], with control $d/2$, gives a free homotopy from γ_0 to a PL loop γ for which every point track has diameter less than $d/2$. The homotopy therefore lies in $S^n \setminus P$. It is enough to contract γ, because the homotopy annulus can then be pasted to a filling disk for γ to give a filling disk for γ_0. Since S^n is simply connected, γ extends to a continuous map $F_0 : D^2 \to S^n$. Extend the boundary triangulation over D^2. Relative simplicial approximation [Zee64] gives a PL map $F : D^2 \to S^n$ with $F|_{\partial D^2} = \gamma$. Put $Q := F(D^2)$ and $Q_0 := \gamma(S^1)$, and choose a common subdivision in which $Q_0 \subset Q$ and P are subpolyhedra. If $Q = Q_0$, the map F already misses P and is the required null-homotopy. Assume therefore that $Q \neq Q_0$. The relative PL general-position theorem [Bry02, Theorem 4.2], applied to $(Q, Q_0, P) \subset S^n$, gives an ambient PL isotopy Ψ_t, fixed on Q_0, such that

$$\dim\bigl(\Psi_1(Q \setminus Q_0) \cap P\bigr) \leq \dim(Q \setminus Q_0) + \dim P - n \leq 2 + 2 - n < 0.$$

Since $Q_0 \cap P = \varnothing$, the map $\Psi_1 \circ F$ is a null-homotopy of γ in $S^n \setminus P$. Hence $S^n \setminus P$, and therefore C by (34), is simply connected. Beyond the embedding $P \hookrightarrow \mathbb{R}^5 \subset \mathbb{R}^n$, the restriction $n \geq 5$ is used precisely through the inequality $2 + 2 - n < 0$, both in the preceding simple-connectivity argument and in the injectivity argument below.

If some $\pi_k(C)$ were nonzero, choose the least such $k \geq 2$. Then C would be $(k-1)$-connected, and the Hurewicz theorem [Hat02, Theorem 4.32] would give $\pi_k(C) \cong H_k(C; \mathbb{Z}) = 0$, a contradiction. Thus C is weakly contractible. As a compact PL manifold, C has the homotopy type of a finite CW complex. Choose a finite CW complex Z_C and a homotopy equivalence $Z_C \to C$. The constant map $Z_C \to *$ is a weak homotopy equivalence, so Whitehead's theorem [Hat02, Theorem 4.5] makes it a homotopy equivalence. Consequently, C is contractible.

It remains to identify the fundamental group of the boundary. Put $\Sigma := \partial \mathcal{N} = \partial C$. By the half-neighborhood construction above, the two occurrences of Σ carry the same PL structure. The PL collaring theorem [Bry02, Corollary 2.5], applied separately to $\mathcal{N}$ and C, gives, after reparametrizing the collar coordinates, PL collar embeddings

$$c_{\mathcal{N}} : \Sigma \times [-2, 0] \longrightarrow \mathcal{N}, \qquad c_C : \Sigma \times [0, 2] \longrightarrow C$$

such that $c_{\mathcal{N}}(z, 0) = c_C(z, 0) = z \qquad (z \in \Sigma)$. The normalization at parameter 0 and injectivity of the collar embeddings imply

$$c_{\mathcal{N}}^{-1}(\Sigma) = \Sigma \times \{0\}, \qquad c_C^{-1}(\Sigma) = \Sigma \times \{0\}. \tag{35}$$

Indeed, if $c_{\mathcal{N}}(z, t) = w \in \Sigma$, then $c_{\mathcal{N}}(w, 0) = w$ by the normalization, so injectivity of $c_{\mathcal{N}}$ gives $(z, t) = (w, 0)$. The reverse inclusion follows from the normalization, and the proof for c_C is identical. Define

$$\overline{\chi}_\Sigma(z, t) := \begin{cases} c_{\mathcal{N}}(z, t), & -2 \leq t \leq 0, \\ c_C(z, t), & 0 \leq t \leq 2. \end{cases}$$

The two formulas agree on the common subpolyhedron $\Sigma \times \{0\}$, so the PL pasting lemma shows that $\overline{\chi}_\Sigma$ is PL. We next verify its injectivity. If the two parameters both lie in $[-2, 0]$, or both lie in $[0, 2]$, this follows from injectivity of $c_{\mathcal{N}}$ or c_C, respectively. In the only remaining case, after interchanging the two points, the parameters satisfy $t < 0 < t'$. Equality of their images would put the common image in $\mathcal{N} \cap C = \Sigma$ by (33), contrary to (35). Hence $\overline{\chi}_\Sigma$ is injective. Since $\Sigma \times [-2, 2]$ is compact and S^n is Hausdorff, $\overline{\chi}_\Sigma$ is an embedding. Therefore $\chi_\Sigma := \overline{\chi}_\Sigma\big|_{\Sigma \times (-2,2)} : \Sigma \times (-2, 2) \to S^n$ is a PL embedding. Since $\Sigma = \partial \mathcal{N}$ and $\mathcal{N}$ is a compact PL n-manifold, Σ is a closed PL $(n-1)$-manifold. PL invariance of domain therefore shows that the image of χ_Σ is open. Thus χ_Σ is a PL bicollar of Σ, with

$$\chi_\Sigma(\Sigma \times (-2, 0]) \subset \mathcal{N}, \qquad \chi_\Sigma(\Sigma \times [0, 2)) \subset C.$$

More precisely, (35) and $\partial \mathcal{N} = \partial C = \Sigma$ give

$$\chi_\Sigma(\Sigma \times (-2, 0)) \subset \operatorname{int} \mathcal{N}, \qquad \chi_\Sigma(\Sigma \times (0, 2)) \subset \operatorname{int} C. \tag{36}$$

The PL bicollar comes from the collaring theorem. The homeomorphism Φ_c of Lemma 6.4 is not used to construct χ_Σ. Define

$$\mathcal{O}_{\mathcal{N}} := \operatorname{int} \mathcal{N} \cup \chi_\Sigma(\Sigma \times (-2, 1)), \qquad \mathcal{O}_C := \operatorname{int} C \cup \chi_\Sigma(\Sigma \times (-1, 2)).$$

They are open because χ_Σ is a homeomorphism onto an open subset of S^n, and they cover S^n because $S^n = \operatorname{int}\mathcal{N} \sqcup \Sigma \sqcup \operatorname{int} C$. For $0 \le u \le 1$, let $H_u^{\mathcal{N}}$ be the identity on $\mathcal{N}$ and set

$$H_u^{\mathcal{N}}\big(\chi_\Sigma(z,t)\big) := \chi_\Sigma\big(z,(1-u)t\big) \qquad (0 \le t < 1).$$

Likewise, let H_u^C be the identity on C and set

$$H_u^C\big(\chi_\Sigma(z,t)\big) := \chi_\Sigma\big(z,(1-u)t\big) \qquad (-1 < t \le 0).$$

Since $\mathcal{N} = \operatorname{int}\mathcal{N} \cup \Sigma$ and $C = \operatorname{int} C \cup \Sigma$, the bicollar-side inclusions give the covers

$$\mathcal{O}_{\mathcal{N}} = \mathcal{N} \cup \chi_\Sigma(\Sigma \times [0,1)), \qquad \mathcal{O}_C = C \cup \chi_\Sigma(\Sigma \times (-1,0]).$$

These are closed covers in their respective ambient open sets. Indeed, $\mathcal{N}$ and C are compact and hence closed in S^n. Moreover, (33) and (36) give

$$\mathcal{N} \cap \chi_\Sigma(\Sigma \times [0,1)) = \Sigma,$$
$$C \cap \chi_\Sigma(\Sigma \times (-1,0]) = \Sigma.$$

Consequently,

$$\mathcal{O}_{\mathcal{N}} \setminus \chi_\Sigma(\Sigma \times [0,1)) = \mathcal{N} \setminus \Sigma = \operatorname{int}\mathcal{N},$$
$$\mathcal{O}_C \setminus \chi_\Sigma(\Sigma \times (-1,0]) = C \setminus \Sigma = \operatorname{int} C.$$

The right-hand sides are open, so the two collar strips are closed relative to $\mathcal{O}_{\mathcal{N}}$ and $\mathcal{O}_C$, respectively. The formulas agree on the common set Σ in each cover. Applied to the products of these closed covers with $[0,1]$, the pasting lemma therefore shows that $\{H_u^{\mathcal{N}}\}_{0\le u\le 1}$ and $\{H_u^C\}_{0\le u\le 1}$ are strong deformation retractions of $\mathcal{O}_{\mathcal{N}}$ onto $\mathcal{N}$ and of $\mathcal{O}_C$ onto C, respectively. The same side-separation identities also give $\mathcal{O}_{\mathcal{N}} \cap \mathcal{O}_C = \chi_\Sigma(\Sigma \times (-1,1))$ and the homotopy $\chi_\Sigma(z,t) \mapsto \chi_\Sigma(z,(1-u)t)$ is a strong deformation retraction of this intersection onto Σ. Both $\mathcal{N}$ and C are path connected: $\mathcal{N}$ deformation retracts onto the connected complex P, while C is contractible. The reduced Mayer–Vietoris sequence for the open cover $S^n = \mathcal{O}_{\mathcal{N}} \cup \mathcal{O}_C$ therefore contains

$$0 = \widetilde{H}_1(S^n;\mathbb{Z}) \longrightarrow \widetilde{H}_0(\mathcal{O}_{\mathcal{N}} \cap \mathcal{O}_C;\mathbb{Z}) \longrightarrow \widetilde{H}_0(\mathcal{O}_{\mathcal{N}};\mathbb{Z}) \oplus \widetilde{H}_0(\mathcal{O}_C;\mathbb{Z}) = 0.$$

Thus $\mathcal{O}_{\mathcal{N}} \cap \mathcal{O}_C$, and hence Σ, is path connected.

Fix $z \in \partial\mathcal{N}$ and let $\iota : (\partial\mathcal{N}, z) \hookrightarrow (\mathcal{N}, z)$ be the inclusion. We claim that ι_* is an isomorphism on fundamental groups. Choose a PL collar $c : \partial\mathcal{N} \times [0,2] \to \mathcal{N}$ [Bry02, Corollary 2.5]. Because P is compact and disjoint from $\partial\mathcal{N}$, restrict and reparametrize this collar so that its image is disjoint from P. Put $z' = c(z,1)$, and let $\delta(t) = c(z,t)$, $0 \le t \le 1$. Pushing a collar inward shows that $\operatorname{int}\mathcal{N} \hookrightarrow \mathcal{N}$ is a homotopy equivalence. Consequently, every element of $\pi_1(\mathcal{N}, z)$ is represented by a loop $\delta * \beta * \overline{\delta}$, where β is a continuous loop in $\operatorname{int}\mathcal{N}$ based at z'. After relative simplicial approximation fixing z', we may and do take β to be PL. Since z' belongs to the collar, it is disjoint from P. Pass to a common subdivision in which $\{z'\} \subset \beta(S^1)$ and P are subpolyhedra. Apply the relative PL general-position theorem [Bry02, Theorem 4.2] in $\operatorname{int}\mathcal{N}$ to the pair $(\beta(S^1), \{z'\})$ and the polyhedron P. It gives, in the nonconstant case, an ambient PL isotopy Υ_t fixed at z' and satisfying

$$\dim\big(\Upsilon_1(\beta(S^1) \setminus \{z'\}) \cap P\big) \le \dim(\beta(S^1) \setminus \{z'\}) + \dim P - n \le 1 + 2 - n < 0.$$

Thus $\Upsilon_1(\beta(S^1))$ misses P. Since Υ_t fixes z', replacing β by $\Upsilon_1 \circ \beta$ does not change its based homotopy class. The constant case already misses P. Hence $\delta * \beta * \overline{\delta}$ may be chosen in $\mathcal{N} \setminus P$. The punctured-sublevel deformation retraction $r : \mathcal{N} \setminus P \to \partial\mathcal{N}$ gives a based homotopy from this loop to $\iota \circ r \circ (\delta * \beta * \overline{\delta})$. Hence its class lies in the image of ι_*, proving surjectivity.

For injectivity, let a PL loop $\alpha : S^1 \to \partial\mathcal{N}$, based at z, be null-homotopic in $\mathcal{N}$, and put $\alpha'(w) = c(\alpha(w), 1)$ for $w \in S^1$. The collar homotopy $(w, t) \mapsto c(\alpha(w), t)$ has basepoint track δ, and hence $[\alpha'] = [\overline{\delta} * \alpha * \delta]$ in $\pi_1(\mathcal{N}, z')$. Since α is null-homotopic in $\mathcal{N}$, this class is trivial. The isomorphism induced by inclusion $\pi_1(\operatorname{int}\mathcal{N}, z') \xrightarrow{\cong} \pi_1(\mathcal{N}, z')$ therefore shows that α' is null-homotopic in $\operatorname{int}\mathcal{N}$. After extending the boundary triangulation over D^2, relative simplicial approximation [Zee64] supplies a PL null-homotopy $F : (D^2, \partial D^2) \to (\operatorname{int}\mathcal{N}, \alpha'(S^1))$ whose boundary map is α'. Since the collar is disjoint from P, $\alpha'(S^1) \cap P = \varnothing$. Put $Q := F(D^2)$ and $Q_0 := \alpha'(S^1)$. If $Q = Q_0$, then $F(D^2)$ already misses P; set $\widetilde{F} := F$. If $Q \neq Q_0$, pass to a common subdivision in which $Q_0 \subset Q$ and P are subpolyhedra. Relative PL general position in $\operatorname{int}\mathcal{N}$ [Bry02, Theorem 4.2], applied to (Q, Q_0, P), gives an ambient PL isotopy Θ_t fixed on Q_0 and satisfying

$$\dim\bigl(\Theta_1(Q \setminus Q_0) \cap P\bigr) \leq \dim(Q \setminus Q_0) + \dim P - n \leq 2 + 2 - n < 0.$$

Since $Q_0 \cap P = \varnothing$, set $\widetilde{F} := \Theta_1 \circ F$; this is a null-homotopy of α' in $\operatorname{int}\mathcal{N} \setminus P$. In either case, attach to $\widetilde{F}$ the collar annulus $(w, t) \mapsto c(\alpha(w), t)$, $w \in S^1$ and $0 \leq t \leq 1$. Since the collar misses P, the resulting disk is a null-homotopy of α in $\mathcal{N} \setminus P$. Composing it with r, which fixes $\partial\mathcal{N}$ pointwise, gives a null-homotopy in $\partial\mathcal{N}$. Hence ι_* is injective. Since $\mathcal{N}$ deformation retracts onto P, we have proved

$$\pi_1(\partial C) = \pi_1(\partial\mathcal{N}) \cong \pi_1(\mathcal{N}) \cong \pi_1(P) = G \neq 1. \tag{37}$$

Finally, after replacing the factor coordinate $\vartheta \in (0, 1]$ by $1 - \vartheta$, the punctured-sublevel homeomorphism above writes $M = S^n \setminus P$ as C with an open collar $\partial C \times [0, 1)$ attached along $\partial C \times \{0\}$. Choose an inward PL collar $c : \partial C \times [0, 1] \to C$ with $c(z, 0) = z$ [Bry02, Corollary 2.5]. Give the union of the inward and attached collars the coordinate $u \in [-1, 1)$, with $u = -t$ on $c(\partial C \times [0, 1])$ and $u = t$ on the attached collar. The homeomorphism $u \mapsto (u - 1)/2$ from $[-1, 1)$ onto $[-1, 0)$ fixes the endpoint -1 and therefore extends by the identity off the inward collar. This gives a homeomorphism $M \cong \operatorname{int} C$. In particular, M is contractible, as also follows directly from (34). On the punctured inward collar, introduce a new coordinate $t \in (0, 1]$, with $t \downarrow 0$ corresponding to approach to ∂C, hence to infinity in $\operatorname{int} C$. Choose a homeomorphism $(0, 1] \to [0, \infty)$ with this orientation. This reparametrization identifies a cofinal sequence of connected neighborhoods of infinity in $\operatorname{int} C$ with $\partial C \times (j, \infty)$, $j = 1, 2, \ldots$. With base points chosen along the ray $t \mapsto (z, t)$, projection onto ∂C identifies the corresponding fundamental-group system with

$$G \xleftarrow{\mathrm{id}} G \xleftarrow{\mathrm{id}} G \xleftarrow{\mathrm{id}} \cdots.$$

By (37) this system is not pro-trivial, so M is not simply connected at infinity [Gui16, Example 3.2.16 and Proposition 3.4.36(a)]. Simple connectivity at infinity is a homeomorphism invariant, whereas $\mathbb{R}^n$ is simply connected at infinity for $n \geq 3$; it follows that $M \not\cong \mathbb{R}^n$; this is precisely the conclusion of [Gui16, Theorem 3.5.2].

It remains to verify (32) with $\mathfrak{d} = 2$. Let $P_{\mathrm{E}} \subset \mathbb{R}^5 \subset \mathbb{R}^n$ be the Euclidean realization of P, and let $\Delta_1, \ldots, \Delta_m$ be its 2-simplices, each a nondegenerate planar triangle. In accordance with the unnormalized Hausdorff convention fixed at the beginning of this section, put $\mathcal{A}^2 := \frac{\pi}{4}\mathcal{H}^2$. Then $\mathcal{A}^2$ agrees with Euclidean planar area on every affine 2-plane. Distinct Δ_i meet in common faces, which are $\mathcal{A}^2$-null, so $\mathcal{A}^2(P_{\mathrm{E}}) = \sum_i \operatorname{area}\Delta_i \in (0,\infty)$. Put $\mu_{\mathrm{E}} := \frac{\mathcal{A}^2|_{P_{\mathrm{E}}}}{\mathcal{A}^2(P_{\mathrm{E}})}$, $\quad \mu := (\sigma_\infty^{-1})_{\#}\mu_{\mathrm{E}}$. Thus the spherical complex already denoted by P is $\sigma_\infty^{-1}(P_{\mathrm{E}})$, and μ is its spherical probability measure. The map σ_∞^{-1} is smooth and bi-Lipschitz on the compact set P_{E} with respect to Euclidean and chordal distances: there is $C_{\mathrm{bi}} \geq 1$ with $C_{\mathrm{bi}}^{-1}|x-y| \leq \mathfrak{q}(\sigma_\infty^{-1}(x), \sigma_\infty^{-1}(y)) \leq C_{\mathrm{bi}}|x-y|$ for $x, y \in P_{\mathrm{E}}$. (For the chart obtained by projecting onto the equatorial hyperplane, $\mathfrak{q}(\sigma_\infty^{-1}(x), \sigma_\infty^{-1}(y)) = 2|x-y|\,\big((1+|x|^2)(1+|y|^2)\big)^{-1/2}$, so $C_{\mathrm{bi}} = 1 + R^2$ works if $P_{\mathrm{E}} \subset B(0,R)$, $R \geq 1$.) Writing B_{E} for Euclidean balls, we take $p_{\mathrm{E}} \in P_{\mathrm{E}}$, set $p = \sigma_\infty^{-1}(p_{\mathrm{E}})$, and let $r > 0$; then we obtain

$$P_{\mathrm{E}} \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r/C_{\mathrm{bi}}) \subseteq \sigma_\infty\big(P \cap B_{\mathfrak{q}}(p,r)\big) \subseteq P_{\mathrm{E}} \cap B_{\mathrm{E}}(p_{\mathrm{E}}, C_{\mathrm{bi}} r), \tag{38}$$

so it suffices to find $c_0, C_0, r_0 > 0$ with

$$c_0 r^2 \leq \mathcal{A}^2\big(P_{\mathrm{E}} \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)\big) \leq C_0 r^2 \qquad (p_{\mathrm{E}} \in P_{\mathrm{E}},\ 0 < r \leq r_0).$$

Upper estimate. For each i, the set $\Delta_i \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)$ lies in the intersection of $B_{\mathrm{E}}(p_{\mathrm{E}}, r)$ with the affine plane of Δ_i, which is a planar disc of radius at most r. Hence $\mathcal{A}^2(\Delta_i \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)) \leq \pi r^2$, and summing over i gives $\mathcal{A}^2(P_{\mathrm{E}} \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)) \leq m\pi r^2$ for all $r > 0$.

Lower estimate. Since P_{E} is pure, p_{E} lies in some $\Delta = \Delta_i$. For $0 < t < 1$ the homothety $h_t(y) = p_{\mathrm{E}} + t(y - p_{\mathrm{E}})$ maps the convex set Δ into itself, satisfies $|h_t(y) - p_{\mathrm{E}}| \leq t \operatorname{diam}\Delta$, and scales area by t^2. Thus $h_t(\Delta) \subseteq \Delta \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)$ whenever $t \operatorname{diam}\Delta < r$, and letting $t \uparrow r/\operatorname{diam}\Delta$ yields

$$\mathcal{A}^2\big(P_{\mathrm{E}} \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)\big) \geq \mathcal{A}^2\big(\Delta \cap B_{\mathrm{E}}(p_{\mathrm{E}}, r)\big) \geq \frac{\operatorname{area}\Delta}{(\operatorname{diam}\Delta)^2}\, r^2 \qquad (0 < r \leq \operatorname{diam}\Delta).$$

Since there are finitely many simplices, this is the required lower estimate with $c_0 = \min_i \operatorname{area}\Delta_i/(\operatorname{diam}\Delta_i)^2 > 0$ and $r_0 = \min_i \operatorname{diam}\Delta_i > 0$.

Dividing by $\mathcal{A}^2(P_{\mathrm{E}})$ and using (38), we obtain

$$\frac{c_0}{C_{\mathrm{bi}}^2\, \mathcal{A}^2(P_{\mathrm{E}})}\, r^2 \leq \mu\big(B_{\mathfrak{q}}(p,r)\big) \leq \frac{m\pi C_{\mathrm{bi}}^2}{\mathcal{A}^2(P_{\mathrm{E}})}\, r^2 \qquad (p \in P,\ 0 < r \leq r_0/C_{\mathrm{bi}}),$$

which is (32) with $\mathfrak{d} = 2$. In particular, $\mu(B_{\mathfrak{q}}(p,r)) > 0$ for every $p \in P$ and $0 < r \leq r_0/C_{\mathrm{bi}}$, and the same holds for larger r by monotonicity. Thus $P \subset \operatorname{supp}\mu$. Conversely, by construction $\mu(S^n \setminus P) = 0$, and P is closed, so $\operatorname{supp}\mu \subset P$. Hence $\operatorname{supp}\mu = P$, and (P, μ) is a 2-regular pair. $\square$

Two lemmas provide the analytic input for Proposition 6.25. Lemma 6.23 gives the blow-up estimates needed for completeness, while Lemma 6.24 supplies the positivity and matching growth needed for the scalar-curvature bounds. In the critical case $\mathfrak{d} = a$, the logarithmic correction compensates for the fact that the pure power $\mathfrak{q}^{2-n}$ is $L_{g_{st}}$-harmonic off the diagonal.

Lemma 6.23 (Ahlfors-regular potential estimate)**.** *Let (Λ,μ) be a $\mathfrak{d}$-regular pair, put $\rho=\operatorname{dist}_{\mathfrak q}(\cdot,\Lambda)$, and let $k:(0,2]\to(0,\infty)$ be continuous with*

$$k(u)\asymp u^{-\alpha}\ell(u)^{-\lambda}\quad (u\downarrow 0),\qquad \alpha>\mathfrak{d},\quad \lambda\ge 0.$$

Then, for $x\in S^n\setminus\Lambda$,

$$\int_\Lambda k(\mathfrak q(x,y))\,d\mu(y)\asymp \rho(x)^{\mathfrak d-\alpha}\ell(\rho(x))^{-\lambda}\qquad (\rho(x)\downarrow 0). \tag{39}$$

Idea of proof. At distance $\delta=\rho(x)$ from Λ, the portion of Λ within distance comparable to δ has measure comparable to $\delta^{\mathfrak d}$, while the kernel there has size $\delta^{-\alpha}\ell(\delta)^{-\lambda}$. The remaining contribution is controlled by dyadic annuli, whose estimates form a convergent geometric series precisely because $\alpha>\mathfrak d$.

Proof. Recall that $\ell(u)=\log\frac{8}{u}$, so ℓ is decreasing, $\ell\ge\log 4$ on $(0,2]$, and

$$\ell(2^m u)=\ell(u)-m\log 2\qquad (m\in\mathbb N_0,\ 2^m u\le 2). \tag{40}$$

By hypothesis, there are $r_1\in(0,2]$ and $0<c_1\le C_1$ with

$$c_1u^{-\alpha}\ell(u)^{-\lambda}\le k(u)\le C_1u^{-\alpha}\ell(u)^{-\lambda}\qquad (0<u\le r_1). \tag{41}$$

Set $r_2=\frac12\min\{r_0,r_1\}$ and assume $x\in S^n\setminus\Lambda$ and $0<\delta:=\rho(x)<r_2/4$. Choose $p\in\Lambda$ with $\mathfrak q(x,p)=\delta$ (Λ is compact) and put $\ell_\delta:=\ell(\delta)$. All constants below may depend on the fixed regular pair, on $\mathfrak d,\alpha,\lambda$, and on k, but not on x.

Lower bound. For $y\in B_{\mathfrak q}(p,\delta)\cap\Lambda$ the triangle inequality gives $\mathfrak q(x,y)\le\mathfrak q(x,p)+\mathfrak q(p,y)<2\delta\le r_1$, while $\mathfrak q(x,y)\ge\rho(x)=\delta$; hence $\mathfrak q(x,y)^{-\alpha}\ge(2\delta)^{-\alpha}$ and, since ℓ is decreasing and $\lambda\ge0$, $\ell(\mathfrak q(x,y))^{-\lambda}\ge\ell_\delta^{-\lambda}$. Since $\delta<r_0$, (32) gives $\mu(B_{\mathfrak q}(p,\delta))\ge c\delta^{\mathfrak d}$, so (41) yields

$$\int_\Lambda k(\mathfrak q(x,y))\,d\mu(y)\ge\int_{B_{\mathfrak q}(p,\delta)\cap\Lambda}k(\mathfrak q(x,y))\,d\mu(y)\ge c_1c\,2^{-\alpha}\delta^{\mathfrak d-\alpha}\ell_\delta^{-\lambda}.$$

Upper bound. Since $\mathfrak q(x,\cdot)\ge\delta$ on Λ, the annuli

$$A_j=\{y\in\Lambda:2^j\delta\le\mathfrak q(x,y)<2^{j+1}\delta\},\qquad j\ge0,$$

are pairwise disjoint and, for every $J\ge0$,

$$\bigcup_{j=0}^J A_j=\{y\in\Lambda:\mathfrak q(x,y)<2^{J+1}\delta\}.$$

Let J be maximal with $(2^{J+1}+1)\delta<r_2$; such a finite J exists because $3\delta<r_2$, whereas $(2^{j+1}+1)\delta\to\infty$ as $j\to\infty$. For $0\le j\le J$ the triangle inequality gives $A_j\subset B_{\mathfrak q}\big(p,(2^{j+1}+1)\delta\big)$, a ball of radius less than $r_2<r_0$, so (32) and $2^{j+1}+1\le 3\cdot 2^j$ give $\mu(A_j)\le 3^{\mathfrak d}C(2^j\delta)^{\mathfrak d}$. On A_j one has $2^j\delta\le\mathfrak q(x,y)<2^{j+1}\delta\le r_1$, so (41) and (40) give $k(\mathfrak q(x,y))\le C_1(2^j\delta)^{-\alpha}[\ell_\delta-(j+1)\log 2]^{-\lambda}$, where the bracket equals $\ell(2^{j+1}\delta)>\ell(r_2)>0$. Hence

$$\int_{A_j}k(\mathfrak q(x,y))\,d\mu(y)\le C'\delta^{\mathfrak d-\alpha}2^{-j\gamma}\,\ell(2^{j+1}\delta)^{-\lambda}\qquad (0\le j\le J), \tag{42}$$

where $\gamma = \alpha - \mathfrak{d} > 0$. Split $\{0, \dots, J\}$ according to whether $(j+1)\log 2 \le \ell_\delta/2$. For the first range, the bracket is at least $\ell_\delta/2$, and the geometric series $\sum_{j\ge 0} 2^{-j\gamma} = (1-2^{-\gamma})^{-1}$ (this is where $\alpha > \mathfrak{d}$ enters) bounds that part of the sum of (42) by $C\delta^{\mathfrak{d}-\alpha}\ell_\delta^{-\lambda}$. For the remaining indices, $(j+1)\log 2 > \ell_\delta/2$ and $\ell(2^{j+1}\delta) \ge \ell(r_2) > 0$; therefore

$$\sum_{\substack{0\le j\le J\\ (j+1)\log 2>\ell_\delta/2}} 2^{-j\gamma}\,\ell(2^{j+1}\delta)^{-\lambda} \le C \sum_{j>\ell_\delta/(2\log 2)-1} 2^{-j\gamma} \le Ce^{-\gamma\ell_\delta/2} \le C\ell_\delta^{-\lambda},$$

the last inequality because $\sup_{u\ge\log 4} u^\lambda e^{-\gamma u/2} < \infty$. Thus this part also contributes at most $C\delta^{\mathfrak{d}-\alpha}\ell_\delta^{-\lambda}$. Finally, maximality of J gives $(2^{J+2}+1)\delta \ge r_2$, hence $2^{J+1}\delta \ge (r_2-\delta)/2 \ge 3r_2/8$; the remaining set $\{y \in \Lambda : \mathfrak{q}(x,y) \ge 2^{J+1}\delta\}$ lies at chordal distance at least $3r_2/8$ from x, where the continuous function k is bounded by $C_{\mathrm{far}} := \max_{u\in[3r_2/8,2]} k(u)$, so it contributes at most $C_{\mathrm{far}}\mu(\Lambda) = C_{\mathrm{far}}$, which is absorbed because $\delta^{\mathfrak{d}-\alpha}\ell_\delta^{-\lambda} \to \infty$ as $\delta \downarrow 0$. Together the three parts give the upper bound in (39). ☐

Lemma 6.24 (Kernel identities). *For $b > 0$ and $x, y \in S^n$ with $x \ne y$,*

$$L_{g_{st},x}\mathfrak{q}(x,y)^{-b} = A_{n,b}\mathfrak{q}(x,y)^{-b} + B_{n,b}\mathfrak{q}(x,y)^{-b-2}, \tag{43}$$

where

$$A_{n,b} = \frac{(n-1)(n-2-b)(n-b)}{n-2}, \qquad B_{n,b} = \frac{4(n-1)b(n-2-b)}{n-2}.$$

In particular, both coefficients are positive when $0 < b < n-2$. Moreover, if $\lambda > 0$, then

$$\begin{aligned} L_{g_{st},x}\big(\mathfrak{q}^{2-n}\ell(\mathfrak{q})^{-\lambda}\big) &= \frac{4(n-1)\lambda}{n-2}\mathfrak{q}^{-n}\frac{(n-2)\ell(\mathfrak{q})-(\lambda+1)}{\ell(\mathfrak{q})^{\lambda+2}} \\ &\quad + \frac{(n-1)\lambda}{n-2}\mathfrak{q}^{2-n}\frac{2\ell(\mathfrak{q})+\lambda+1}{\ell(\mathfrak{q})^{\lambda+2}}. \end{aligned} \tag{44}$$

For $\lambda = a/2 = (n-2)/4$, this expression is positive and

$$L_{g_{st},x}\big(\mathfrak{q}^{2-n}\ell(\mathfrak{q})^{-a/2}\big) \asymp \mathfrak{q}^{-n}\ell(\mathfrak{q})^{-a/2-1} \qquad (0 < \mathfrak{q} \le 2). \tag{45}$$

Proof. Fix $y \in S^n$ and write $r = d_{g_{st}}(x,y)$, $\quad \mathfrak{q} = 2\sin\frac{r}{2}$. For $0 < r < \pi$,

$$\mathfrak{q}_r = \cos\frac{r}{2}, \qquad \mathfrak{q}_r^2 = 1 - \frac{\mathfrak{q}^2}{4}, \qquad \mathfrak{q}_{rr} = -\frac{\mathfrak{q}}{4}, \qquad \cot r\,\mathfrak{q}_r = \frac{1}{\mathfrak{q}} - \frac{\mathfrak{q}}{2},$$

the last identity because $\sin r = \mathfrak{q}\,\mathfrak{q}_r$ and $\cos r = 1 - \frac{\mathfrak{q}^2}{2}$. Hence, if $f = f(\mathfrak{q})$ is radial about y, then

$$\begin{aligned} \Delta_{g_{st}} f &= f''\mathfrak{q}_r^2 + f'\big(\mathfrak{q}_{rr} + (n-1)\cot r\,\mathfrak{q}_r\big) \\ &= \left(1 - \frac{\mathfrak{q}^2}{4}\right) f'' + \left(\frac{n-1}{\mathfrak{q}} - \frac{(2n-1)\mathfrak{q}}{4}\right) f'. \end{aligned}$$

Although geodesic polar coordinates degenerate at $r = \pi$, both sides extend continuously there, so the resulting identities hold for every $0 < \mathfrak{q} \le 2$.

For $f(\mathfrak{q}) = \mathfrak{q}^{-b}$, one has $f' = -b\mathfrak{q}^{-b-1}, \qquad f'' = b(b+1)\mathfrak{q}^{-b-2}$. Substitution into the preceding formula gives

$$\Delta_{g_{st}}\mathfrak{q}^{-b} = b(b-n+2)\mathfrak{q}^{-b-2} + \frac{b(2n-2-b)}{4}\mathfrak{q}^{-b}.$$

Since $L_{g_{st}} = -\frac{4(n-1)}{n-2}\Delta_{g_{st}} + n(n-1)$, we obtain

$$\begin{aligned} L_{g_{st}}\mathfrak{q}^{-b} = {} & \frac{4(n-1)b(n-2-b)}{n-2}\mathfrak{q}^{-b-2} \\ & + \frac{n-1}{n-2}\big[n(n-2) - b(2n-2-b)\big]\mathfrak{q}^{-b}. \end{aligned}$$

The factorization $n(n-2) - b(2n-2-b) = (n-2-b)(n-b)$ proves (43). If $0 < b < n-2$, then $n-2-b > 0$, $n-b > 0$, and $b > 0$, so both coefficients are positive. At $b = n-2$, both coefficients vanish; equivalently, $\mathfrak{q}^{2-n}$ is annihilated by $L_{g_{st}}$ off the diagonal. This is the Green-kernel degeneration underlying the critical case.

For the critical kernel, put $m = n-2, \qquad t = \ell(\mathfrak{q}), \qquad f(\mathfrak{q}) = \mathfrak{q}^{-m}t^{-\lambda}$. Since $t' = -\mathfrak{q}^{-1}$, direct differentiation gives

$$f' = \mathfrak{q}^{-m-1}t^{-\lambda-1}(\lambda - mt)$$

and

$$f'' = \mathfrak{q}^{-m-2}t^{-\lambda-2}\big[m(m+1)t^2 - \lambda(2m+1)t + \lambda(\lambda+1)\big].$$

Substituting these derivatives into the radial Laplacian formula and using $n = m+2$ yields

$$\Delta_{g_{st}}f = \lambda\mathfrak{q}^{-n}\frac{(\lambda+1) - mt}{t^{\lambda+2}} + \frac{1}{4}\mathfrak{q}^{2-n}\frac{mn\,t^2 - 2\lambda t - \lambda(\lambda+1)}{t^{\lambda+2}}.$$

When $L_{g_{st}}$ is applied, the term containing $mn\,\mathfrak{q}^{2-n}t^{-\lambda}$ cancels exactly with $n(n-1)f$. The remaining terms are

$$\begin{aligned} L_{g_{st}}f = {} & \frac{4(n-1)\lambda}{n-2}\mathfrak{q}^{-n}\frac{(n-2)t - (\lambda+1)}{t^{\lambda+2}} \\ & + \frac{(n-1)\lambda}{n-2}\mathfrak{q}^{2-n}\frac{2t+\lambda+1}{t^{\lambda+2}}, \end{aligned}$$

which is (44). Finally, take $\lambda = \frac{a}{2} = \frac{n-2}{4}$. Since $t = \ell(\mathfrak{q}) \geq \log 4$, $(n-2)t - (\lambda+1) \geq (n-2)\log 4 - \frac{n+2}{4} > 0$. Indeed, $(n-2)\log 4 - \frac{n+2}{4}$ is increasing in n and is already positive at $n = 3$, where it equals $\log 4 - \frac{5}{4} > 0$. Thus both terms in (44) are positive. Moreover, for $t \geq \log 4$, $(n-2)t - (\lambda+1) \geq \Big(n - 2 - \frac{n+2}{4\log 4}\Big)t$, and the coefficient in parentheses is positive because it equals $\frac{(n-2)\log 4 - (n+2)/4}{\log 4} > 0$. Hence the first term in (44) is comparable to $\mathfrak{q}^{-n}t^{-\lambda-1}$. The second term is positive and satisfies

$$\mathfrak{q}^{2-n}\frac{2t+\lambda+1}{t^{\lambda+2}} \leq C\mathfrak{q}^{2-n}t^{-\lambda-1} = C\mathfrak{q}^2\mathfrak{q}^{-n}t^{-\lambda-1} \leq C\mathfrak{q}^{-n}t^{-\lambda-1}.$$

The first term supplies the corresponding lower bound, proving (45). $\square$

Proposition 6.25 (Metrics from regular pairs). *Let $n \geq 3$, let $0 < \mathfrak{d} \leq a = (n-2)/2$, and let (Λ, μ) be a $\mathfrak{d}$-regular pair in S^n with $\varnothing \neq \Lambda \subsetneq S^n$. Define*

$$\mathcal{K}_{\mathfrak{d}}(u) = \begin{cases} u^{-(\mathfrak{d}+a)}, & \mathfrak{d} < a, \\ u^{-2a}\ell(u)^{-a/2}, & \mathfrak{d} = a, \end{cases} \qquad V(x) = \int_\Lambda \mathcal{K}_{\mathfrak{d}}(\mathfrak{q}(x,y))\, d\mu(y), \quad x \in S^n \setminus \Lambda.$$

Then

$$\widehat{g} = (1+V)^{4/(n-2)} g_{st} \tag{46}$$

is smooth and locally conformally flat on $S^n \setminus \Lambda$; each connected component is geodesically complete; and its scalar curvature satisfies

$$0 < \inf_{S^n \setminus \Lambda} \mathrm{Sc}_{\widehat{g}} \leq \sup_{S^n \setminus \Lambda} \mathrm{Sc}_{\widehat{g}} < \infty.$$

Proof. Smoothness. Let $Q_{\mathrm{sm}} \subset S^n \setminus \Lambda$ be a nonempty compact set, so that $\varepsilon := \min\{\mathfrak{q}(x,y) : x \in Q_{\mathrm{sm}},\, y \in \Lambda\} > 0$. On $Q_{\mathrm{sm}} \times \Lambda$ one has $\mathfrak{q} \geq \varepsilon$. Since $\mathfrak{q}^2$ is smooth, $\mathfrak{q}$ is smooth on a neighborhood of this compact set. As $\mathcal{K}_{\mathfrak{d}}$ is smooth on $[\varepsilon, 2]$, the integrand $\mathcal{K}_{\mathfrak{d}}(\mathfrak{q}(x,y))$ and all its x-derivatives are continuous, hence uniformly bounded on $Q_{\mathrm{sm}} \times \Lambda$. As μ is finite, dominated convergence justifies differentiation under the integral: V is smooth and positive on $S^n \setminus \Lambda$ and

$$H := L_{g_{st}} V = \int_\Lambda L_{g_{st},x} \mathcal{K}_{\mathfrak{d}}(\mathfrak{q}(x,y))\, d\mu(y).$$

Here $x \notin \Lambda = \operatorname{supp} \mu$, so the kernel identities are used only off the diagonal; no distributional mass at $x = y$ enters this calculation.

Positivity and asymptotics. Suppose first that $\mathfrak{d} < a$ and set $b = \mathfrak{d} + a$, so that $0 < b < 2a = n-2$. With $I_2 = \int_\Lambda \mathfrak{q}(x,y)^{-b-2}\, d\mu(y)$, identity (43) has positive coefficients and gives $H = A_{n,b} V + B_{n,b} I_2 > 0$. The kernels u^{-b} and u^{-b-2} are continuous and positive on $(0,2]$ and satisfy the hypotheses of Lemma 6.23 with $(\alpha, \lambda) = (b, 0)$ and $(b+2, 0)$, respectively; hence, as $\rho \downarrow 0$, $V \asymp \rho^{-a}$ and $I_2 \asymp \rho^{-a-2}$. In particular, for ρ sufficiently small, $V \leq C\rho^{-a} = C\rho^2 \rho^{-a-2} \leq C' I_2$. Since $H = A_{n,b} V + B_{n,b} I_2$ and $B_{n,b} > 0$, it follows that

$$V \asymp \rho^{-a}, \qquad H \asymp I_2 \asymp \rho^{-a-2}. \tag{47}$$

If $\mathfrak{d} = a$, then $\mathcal{K}_a(\mathfrak{q}) = \mathfrak{q}^{2-n}\ell(\mathfrak{q})^{-a/2}$, and by Lemma 6.24 its $L_{g_{st}}$-image is again a continuous positive function of $\mathfrak{q}$ alone, comparable to $\mathfrak{q}^{-n}\ell(\mathfrak{q})^{-a/2-1}$ on $(0,2]$ by (45). Hence $H > 0$, and Lemma 6.23, applied with $(\alpha, \lambda) = (2a, \frac{a}{2})$ to V and with $(\alpha, \lambda) = (n, \frac{a}{2}+1)$ to H (both admissible, as $\alpha > \mathfrak{d} = a$), gives

$$V \asymp \rho^{-a}\ell(\rho)^{-a/2}, \qquad H \asymp \rho^{-a-2}\ell(\rho)^{-a/2-1}. \tag{48}$$

The comparison $H \asymp V^\kappa$. Since $\kappa = \frac{a+2}{a}$, one has $a\kappa = a+2$, $\quad \frac{a}{2}\kappa = \frac{a}{2} + 1$, so raising the V-asymptotics in (47)–(48) to the power κ reproduces exactly the H-asymptotics: in both cases there is $\rho_1 > 0$ with $H \asymp V^\kappa$ on $\{0 < \rho \leq \rho_1\}$. The set $\{\rho \geq \rho_1\}$ is a compact subset of $S^n \setminus \Lambda$, on which H and V^κ are continuous and positive; hence H/V^κ is bounded between positive constants there as well, and $H \asymp V^\kappa$ holds on all of $S^n \setminus \Lambda$.

Curvature. Set $U=1+V\ge 1$. Since $L_{g_{st}}1=n(n-1)$, $L_{g_{st}}U=n(n-1)+H\asymp 1+V^\kappa$. Since $\kappa>1$, $1+V^\kappa\le(1+V)^\kappa\le 2^{\kappa-1}\big(1+V^\kappa\big)$, the second inequality by convexity of $x\mapsto x^\kappa$. Consequently, $L_{g_{st}}U\asymp 1+V^\kappa\asymp U^\kappa$. As U is smooth and $U\ge 1$, (46) defines a smooth metric conformal to g_{st}, and hence it is locally conformally flat. Since $a=\frac{n-2}{2}$ and $\kappa=\frac{n+2}{n-2}=\frac{a+2}{a}$, we have $\widehat{g}=U^{4/(n-2)}g_{st}=U^{2/a}g_{st}$. For $L_{g_{st}}=-\frac{4(n-1)}{n-2}\Delta_{g_{st}}+n(n-1)$, the conformal scalar-curvature formula gives $\mathrm{Sc}_{\widehat{g}}=U^{-(n+2)/(n-2)}L_{g_{st}}U=U^{-\kappa}L_{g_{st}}U$. The comparison $L_{g_{st}}U\asymp U^\kappa$ means that, for some constants $0<c\le C<\infty$, $cU^\kappa\le L_{g_{st}}U\le CU^\kappa$ on $S^n\setminus\Lambda$. Since $U>0$, multiplying by $U^{-\kappa}$ yields $c\le \mathrm{Sc}_{\widehat{g}}(x)\le C$ for every $x\in S^n\setminus\Lambda$. Thus $0<c\le\inf_{S^n\setminus\Lambda}\mathrm{Sc}_{\widehat{g}}\le\sup_{S^n\setminus\Lambda}\mathrm{Sc}_{\widehat{g}}\le C<\infty$.

Completeness. The inequality $U\ge V$ and the lower bounds in (47)–(48) show that, after shrinking ρ_1, there is $c>0$ such that on $\{0<\rho\le\rho_1\}$,

$$U^{1/a}\ge\begin{cases} c\rho^{-1}, & \mathfrak{d}<a,\\ \dfrac{c}{\rho\sqrt{\ell(\rho)}}, & \mathfrak{d}=a.\end{cases}$$

Note that $\widehat{g}=U^{2/a}g_{st}$, so $ds_{\widehat{g}}=U^{1/a}ds_{g_{st}}$, and that ρ is 1-Lipschitz for g_{st}, because $|\rho(x)-\rho(x')|\le\mathfrak{q}(x,x')\le d_{g_{st}}(x,x')$.

Let $\gamma:[0,T)\to S^n\setminus\Lambda$ be a divergent locally Lipschitz curve; thus, for every compact $Q\subset S^n\setminus\Lambda$, there is $t_Q\in[0,T)$ such that $\gamma(t)\notin Q$ for all $t>t_Q$. For every $\varepsilon>0$, the set $K_\varepsilon=\{x\in S^n:\rho(x)\ge\varepsilon\}$ is a compact subset of $S^n\setminus\Lambda$. Since γ is divergent, it eventually avoids every K_ε; hence $\rho(\gamma(t))\to 0$ as $t\to T$, and there is t_0 with $\rho(\gamma(t))\le\rho_1$ for $t\ge t_0$. Write $\rho(t):=\rho(\gamma(t))$. This function is locally Lipschitz, and $|\rho'(t)|\le|\gamma'(t)|_{g_{st}}$ almost everywhere. For each $t<T$, the positive continuous function ρ has a positive minimum on $[t_0,t]$; hence the absolutely continuous chain rule applies to $\log\rho$ and to $\sqrt{\ell(\rho)}$ on this interval.

In the subcritical case, for $t_0\le t<T$,

$$\begin{aligned}\mathrm{Len}_{\widehat{g}}\big(\gamma|_{[t_0,t]}\big)&=\int_{t_0}^t U(\gamma(\tau))^{1/a}|\gamma'(\tau)|_{g_{st}}\,d\tau\\ &\ge c\int_{t_0}^t\frac{|\rho'(\tau)|}{\rho(\tau)}\,d\tau\\ &\ge c\left|\int_{t_0}^t\frac{\rho'(\tau)}{\rho(\tau)}\,d\tau\right|\\ &=c\big|\log\rho(t)-\log\rho(t_0)\big|\underset{t\uparrow T}{\longrightarrow}+\infty.\end{aligned}$$

In the critical case, since $\frac{d}{dr}\big(-2\sqrt{\ell(r)}\big)=\frac{1}{r\sqrt{\ell(r)}}$, the same argument gives

$$\begin{aligned}\mathrm{Len}_{\widehat{g}}\big(\gamma|_{[t_0,t]}\big)&=\int_{t_0}^t U(\gamma(\tau))^{1/a}|\gamma'(\tau)|_{g_{st}}\,d\tau\\ &\ge c\int_{t_0}^t\frac{|\rho'(\tau)|}{\rho(\tau)\sqrt{\ell(\rho(\tau))}}\,d\tau\\ &\ge c\left|\int_{t_0}^t\frac{\rho'(\tau)}{\rho(\tau)\sqrt{\ell(\rho(\tau))}}\,d\tau\right|\\ &=2c\big|\sqrt{\ell(\rho(t))}-\sqrt{\ell(\rho(t_0))}\big|\underset{t\uparrow T}{\longrightarrow}+\infty.\end{aligned}$$

In both cases the terminal limit follows from $\rho(t) \to 0$ as $t \uparrow T$: one has $\log \rho(t) \to -\infty$ and $\ell(\rho(t)) = \log(8/\rho(t)) \to +\infty$. Thus every divergent locally Lipschitz curve has infinite $\widehat{g}$-length.

Suppose that some connected component were not geodesically complete. By the unit-speed formulation of geodesic completeness, one could, after translating and reversing the parameter if necessary, choose a unit-speed geodesic $\gamma : [0, T) \to S^n \setminus \Lambda$, with $T < \infty$, that is not extendible past T. This geodesic must be divergent. Indeed, otherwise there would be a compact set $Q \subset S^n \setminus \Lambda$ and a sequence of times tending to T whose images under γ lie in Q. The corresponding unit tangent vectors lie in the compact unit tangent bundle over Q and therefore have a convergent subsequence. Local existence for the smooth geodesic vector field then supplies a common positive existence time for all sufficiently late terms of this subsequence, and uniqueness extends γ beyond T, contradicting its nonextendibility. Thus γ is a divergent smooth, hence locally Lipschitz, curve, but its $\widehat{g}$-length is $T < \infty$, contradicting the preceding paragraph. Every connected component is therefore geodesically complete. □

Proof of Theorem 6.21. Let $P \subset S^n$, $M = S^n \setminus P$, and μ be supplied by Lemma 6.22. Then (P, μ) is 2-regular, and the measure has support equal to the nonempty proper subset $P \subset S^n$. Moreover, the analytic hypothesis $\mathfrak{d} = 2 \leq a = (n-2)/2$ holds because $n \geq 6$. The same lemma shows that M is contractible, and hence connected, and that $M \not\approx \mathbb{R}^n$. Thus M is the unique connected component of $S^n \setminus P$. Since the finite complex P is closed, M is an open subset of the smooth sphere and hence is a smooth open manifold. Proposition 6.25 therefore gives the required geodesically complete metric on M. □

Question 6.26. *For $n \in \{4, 5\}$, is there a contractible open n-manifold $M \not\approx \mathbb{R}^n$ carrying a complete locally conformally flat metric with* $\mathrm{Sc} \geq 1$*?*

Proof of Theorem F. Fix $n \geq 4$ and choose any $t > 0$ as in Theorem 6.1. The metric $g^{(t)}$ is complete and locally conformally flat, and its scalar curvature is strictly positive; its underlying manifold is contractible and is not homeomorphic to $\mathbb{R}^n$. This proves the first assertion. For every $n \geq 4$, the uniform-decay conclusion of Theorem 6.1, together with $\mathrm{Sc}_{g^{(t)}} = t^{-4/(n-2)}\mathrm{Sc}_{\bar{g}^{(t)}}$, shows that $\mathrm{Sc}_{g^{(t)}}(x) \to 0$ as $d_{g_{st}}(x, K) \to 0$. By Proposition 6.7, $\dim_{\mathcal{H}} K = 1 < n$, so K has empty interior in S^n. Fix $p \in K$. Since K has empty interior, for every $i \geq 1$ one may choose $x_i \in B_{g_{st}}(p, 1/i) \setminus K \subset M$. Then $d_{g_{st}}(x_i, K) < 1/i$, and hence $\mathrm{Sc}_{g^{(t)}}(x_i) \to 0$. Since $\mathrm{Sc}_{g^{(t)}} > 0$ on M, it follows that $\inf_M \mathrm{Sc}_{g^{(t)}} = 0$. Thus the shifted metrics of Theorem 6.1 are not uniformly PSC in any dimension. For $n \geq 6$, however, Theorem 6.21 supplies a manifold of the same kind carrying a complete locally conformally flat metric whose scalar curvature has a positive lower bound, proving the second assertion. □

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