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Mini-Workshop: Higher Category Theory in Geometry and Representation Theory

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Abstract. The mini-workshop brought together researchers with the aim of understanding recent uses of higher category theory in geometry and representation theory. It focused on the very recent theory of Gestalten due to Stefanich and Scholze and on the notion of an extended topological quantum field theory (TQFT) and its applications in (classical and categorical) representation theory. The program combined survey lectures, research presentations, and discussion sessions.

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Introduction by the Organizers

In various parts of modern mathematics: homotopy theory, K-theory, geometric representation theory, algebraic geometry, etc., the theory of $(\infty, 1)$-categories as developed by Lurie, building on earlier work of Boardman–Vogt and Joyal, now plays a central role. In an $(\infty, 1)$-category, contrary to what happens in an ordinary 1-category, one does not only have objects and morphisms, but also higher morphisms (morphisms between morphisms, etc.); consequently, morphisms between objects of an $(\infty, 1)$-category are not just sets, but topological spaces up to homotopy (anima). As a matter of fact, the theory of $(\infty, 1)$-categories is a combination of ordinary category theory and homotopy theory and provides a very powerful language for understanding homotopy-theoretic counterparts of familiar algebraic structures such as rings or modules.

However, in the foundations of the theory, it is required that all n-morphisms are invertible for $n > 1$ (whence the "1" in "$(\infty, 1)$"). If one drops this condition, or rather relax it to only require invertibility of n-morphisms for $n > k$ for a fixed k, the theory has been less developed. But there are very good reasons to try to do it, not just for the sake of generality. Let us mention two:

(1) The notion of *topological quantum field theory* (*TQFT* for short), or more precisely the notion of *extended TQFT*, uses critically the notion of (∞, n)-categories for arbitrary (finite) values of n. The cobordism hypothesis, as first formulated by Baez-Dolan and announced by Lurie, gives a very clean description of what extended TQFT's are, which one cannot formulate, prove and use without a robust theory of (∞, n)-categories. If the notion of TQFT originates from physics, it gave rise to an enormous amount of mathematics, most notably in the work of Witten. For example, Kapustin and Witten showed how the Langlands program can be, at least heuristically (for now!), thought as an equivalence of extended TQFT's of dimension 4.
(2) A central idea of algebraic geometry is that algebra and geometry should in some sense be "dual" to each other. Making this idea precise requires a flexible enough notion of *geometry*. This was made possible in the hands of Zariski, Weil, Grothendieck in the course of the XXth century. Their work was confined to the algebraic setting, but the recent developement of *analytic geometry* by Clausen-Scholze greatly expanded its scope, to include all sort of more analytic examples. Nevertheless, in this formulation, the duality between algebra and geometry is not perfect: e.g. non-affine schemes are not described by their global sections and one categorical level higher certain stacks are not described by their $(\infty, 1)$-category of quasi-coherent sheaves. To restore the duality, one has to be able to arbitrarily increase the categorical level. Ongoing work of Stefanich-Scholze shows that this can be turned into a satisfying theory, the theory of *Gestalten*. This new language has already found remarkable applications in the PhD thesis of Aoki.

The goal of this mini-workshop was to learn about the use of higher category theory in geometry and representation theory, with special focus on the two lines of research just mentioned. In particular, even though these two themes may seem a bit disjoint at first sight, one of our objectives was also to try to understand better how they are connected to each other.

From Monday to Thursday, the workshop consisted in lectures of 120mn or 90mn, discussing various aspects of the theory of ∞-categories (1 talk), of Gestalten (4 talks) and the theory of TQFT's (3 talks). The talks on Gestalten and TQFT's were intertwinned with each other, so as to highlight the relations between the two topics. Each talk was followed by a discussion session of roughly 30-45mn. This format proved highly effective in fostering active engagement from the participants and allowing them to go in depth in the details of the theory.

As an illustration, here are some points of contact between topics (1) and (2) that were discussed during the week:

- Stefanich recently constructed TQFT's outputting higher categorical versions of the categories of quasi-coherent or ind-coherent sheaves on stacks. Even for quasi-coherent sheaves on arbitrary algebraic stacks, this was an open problem: indeed, factorization homology techniques do not apply, due to the lack of a categorical Künneth formula without some assumptions on the stacks. Applied to ind-coherent sheaves, it provides a rigorous construction of what should be the B-model in Kapustin-Witten's proposal. In fact, this construction is a particular case of a much more general construction, providing a categorification of six-functor formalisms. This idea has been recasted in the language of Gestalten, where it appears as a general form of the idea of *transmutation* (an idea going back to Simpson for de Rham cohomology). This construction in turn promises to be useful in arithmetic geometry: the theory of étale ℓ-adic sheaves and more generally motives do not satisfy categorical Künneth, so realizing them "geometrically" requires thinking higher categorically.
- As it turns out, the notion of TQFT can be naturally reinterpreted in terms of the theory of Gestalten. More precisely, given a Gestalt X, one can associate for all $n \in \mathbb{N}$ a presentable (∞, n)-category of quasi-coherent sheaves $\mathcal{O}(X)_n$. The functor sending a Gestalt X to the underlying anima $\mathcal{O}(X)_n^{\simeq}$ is represented by a Gestalt that one can denote as $\mathbb{A}^1_{[n]}$, whose notation is motivated from the fact that when $n = 0$ this is (after base change to $\mathbb{Q}$) the Gestalt represented by the classical affine line. Thus, the moduli of n-dimensional TQFT's is nothing but the subobject of $\mathbb{A}^1_{[n]}$ corepresenting fully dualizable objects in $\mathcal{O}(X)_n$; this subobject is also corepresented by a Gestalt, say $\mathbb{A}^{1,\mathrm{dual}}_{[n]}$. On the other hand, a presentable version of the n-bordism category gives rise to a Gestalt $\mathrm{Gest}(\mathbf{Bord}_n)$ and one has a natural map $\mathrm{Gest}(\mathbf{Bord}_n) \to \mathbb{A}^1_{[n]}$ of Gestalten that factors through $\mathbb{A}^{1,\mathrm{dual}}_{[n]}$. The cobordism hypothesis then states that $\mathrm{Gest}(\mathbf{Bord}_n) = \mathbb{A}^{1,\mathrm{dual}}_{[n]}$.
- A special case of TQFT's are the invertible ones. In the language of Gestalten, invertible TQFT's organize in a spectrum object in Gestalten denoted GL_1, corepresenting invertible objects in all (∞, n)-categories of quasi-coherent sheaves $\mathcal{O}(X)_n^{\times}$. The previous discussion leads to a general theory of Cartier duality in Gestalten where all known geometric incarnations of Cartier duality live (it even includes a Fourier transform over the real numbers). Furthermore, using this theory of Cartier duality, Scholze and Stefanich have computed GL_1 for $\mathbb{C}$-linear categories obtaining the dual homotopy groups of the sphere spectrum. The previous points explain that the conjectural relation (Kitaev, Freed-Hopkins, ...) of invertible $\mathbb{C}$-linear (∞, n)-categories with the duals of the homotopy groups of the sphere holds internally in the topos of Gestalten.

In addition, on Friday morning, 1h research talks were given, discussing even more recent developments and proposing promising directions for future research.

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Mini-Workshop: Higher Category Theory in Geometry and Representation Theory

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Abstracts

Presentable and accessible ∞-categories

Akhil Mathew

The goal of this lecture is to review the theory of presentable and accessible ∞-categories from [4]. This extends the theory for 1-categories, cf. [3, 5, 1].

Throughout, we fix a regular cardinal κ.

Definition. Let $\mathcal{C}$ be an ∞-category with κ-filtered colimits. We say that $x \in \mathcal{C}$ is *κ-compact* if $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}$ (the ∞-category of spaces) commutes with κ-filtered colimits.

Definition. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *κ-accessible* if:

(1) $\mathcal{C}$ admits κ-filtered colimits.
(2) Every object of $\mathcal{C}$ is a κ-filtered colimit of κ-compact objects, and the κ-compact objects form an essentially small ∞-category.

Given a κ-accessible ∞-category $\mathcal{C}$, let $\mathcal{C}^{\kappa}$ be the full subcategory of κ-compact objects.

Definition. We say that $\mathcal{C}$ is *κ-presentable* if $\mathcal{C}$ is κ-accessible and admits all small colimits, and *presentable* if it is κ-presentable for some regular cardinal κ.

Let Pr^L_{κ} be the ∞-category where:

(1) The objects are κ-presentable ∞-categories $\mathcal{C}$.
(2) Given $\mathcal{C}, \mathcal{D}$, the morphisms $\mathcal{C} \to \mathcal{D}$ form the ∞-category $\mathrm{Fun}^{L,\kappa}(\mathcal{C}, \mathcal{D})$ of continuous functors which preserve κ-compact objects.

We let $\mathrm{Pr}^L = \varinjlim_{\kappa} \mathrm{Pr}^L_{\kappa}$ be the ∞-category of presentable ∞-categories and continuous functors.

Remark. The adjoint functor theorem states that any continuous functor between presentable ∞-categories admits a right adjoint. A continuous functor between $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L_{\kappa}$ defines a morphism in Pr^L_{κ} (rather than simply in Pr^L) if and only if the right adjoint preserves κ-filtered colimits.

Proposition (Gabriel–Ulmer duality)**.** *Via $\mathcal{C} \mapsto \mathcal{C}^{\kappa}$, Pr^L_{κ} is equivalent to the ∞-category Rex_{κ} of small, idempotent complete ∞-categories that admit κ-small colimits, and functors between them that preserve κ-small colimits.*

We use the Lurie tensor product $\otimes : \mathrm{Pr}^L_{\kappa} \times \mathrm{Pr}^L_{\kappa} \to \mathrm{Pr}^L_{\kappa}$ (more generally, this works on Pr^L).

Definition. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L_{\kappa}$. Then $\mathcal{C} \otimes \mathcal{D} \in \mathrm{Pr}^L_{\kappa}$ is characterized by the following universal property: given $\mathcal{E} \in \mathrm{Pr}^L_{\kappa}$, $\mathrm{Fun}^{L,\kappa}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ is identified with the ∞-category of functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in each variable as well as κ-compact objects.

Via Gabriel–Ulmer duality, this translates to a tensor product on Rex_κ. In this way, Pr^L_κ (or equivalently Rex_κ) becomes a symmetric monoidal ∞-category.

The ∞-category Pr^L_κ has all small limits and colimits. Colimits (which are compatible with change of κ, and thus work in Pr^L) are constructed as follows: given a diagram $I \to \mathrm{Pr}^L_\kappa$, $i \mapsto \mathcal{C}_i$, the colimit $\varinjlim_{i\in I} \mathcal{C}_i$ is obtained by passing to the right adjoints and forming the limit of the $\mathcal{C}_i$ (along the right adjoint functors) at the level of underlying ∞-categories. If κ is uncountable, then κ-small limits in Pr^L_κ are computed at the level of underlying ∞-categories [2, Propositions 3.1 and 3.2].

The Lurie tensor product on Pr^L_κ is closed; the internal hom is given by

$$\mathrm{Hom}_{\mathrm{Pr}^L_\kappa}(\mathcal{C},\mathcal{D}) = \mathrm{Ind}_\kappa(\mathrm{Fun}^\kappa(\mathcal{C}^\kappa,\mathcal{D}^\kappa)),$$

where the superscript refers to functors that preserve κ-small colimits.

Theorem (Aoki [2])**.** $\mathrm{Pr}^L_\kappa \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$.

Note that in particular, $\mathrm{Pr}^L_\kappa \in \mathrm{Pr}^L_\kappa$; this does not contradict Russell's paradox, since Pr^L_κ is not literally a set-theoretic element of itself; rather, it is *equivalent* to an object of Pr^L_κ, namely Rex_κ.

Given $A \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$, we can consider $\mathrm{Mod}_A(\mathrm{Pr}^L_\kappa) \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$. This defines a functor

$$\mathrm{Mod} : \mathrm{CAlg}(A) \to \mathrm{CAlg}(\mathrm{Mod}_A(\mathrm{Pr}^L_\kappa)),$$

which preserves colimits. This functor is fully faithful and its right adjoint carries $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Mod}_A(\mathrm{Pr}^L_\kappa))$ to $\mathrm{End}_{\mathcal{D}}(\mathbf{1})$.

Definition. Inductively, we define $n\mathrm{Pr}^L_\kappa \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$ via:

(1) $0\mathrm{Pr}^L_\kappa = \mathcal{S}$ (under $\times$).
(2) $(n+1)\mathrm{Pr}^L_\kappa = \mathrm{Mod}_{n\mathrm{Pr}^L_\kappa}(\mathrm{Pr}^L_\kappa)$.

If A is an E_∞-ring spectrum, then we can define $n\mathrm{Pr}^L_{A,\kappa}$ inductively via $0\mathrm{Pr}^L_{A,\kappa} = \mathrm{Mod}_A$, continuing as above.

Let κ be an uncountable regular cardinal (usually, $\kappa = \aleph_1$).

Definition. A *Stefanich ring* consists of objects $\mathcal{C}_n \in \mathrm{CAlg}(n\mathrm{Pr}^L_\kappa)$ for $n \geq 1$, together with equivalences $\mathcal{C}_n \simeq \mathrm{End}_{\mathcal{C}_{n+1}}(\mathbf{1})$. Equivalently, we define the ∞-category StRing_κ as the colimit (in Pr^L) of the sequence of fully faithful functors Mod,

$$\mathrm{StRing}_\kappa = \varinjlim \left(\mathrm{CAlg}(\mathrm{Pr}^L_\kappa) \to \mathrm{CAlg}(2\mathrm{Pr}^L_\kappa) \to \mathrm{CAlg}(3\mathrm{Pr}^L_\kappa) \to \dots\right).$$

Note that any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$ defines an object of StRing_κ, given by the sequence $(\mathcal{C}, \mathrm{Mod}_{\mathcal{C}}, \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}}, \dots)$. Alternatively, this comes from the inclusion of the first stage $\mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$ into the colimit.

AI disclosure: Claude was used to help with proofreading and editing this abstract.

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Quasi-coherent sheaves of higher categories

Rok Gregoric

Following [4, Lectures IV-V], we build upon the foundations of the theory of presentable (∞, n)-categories, carried out in the previous talk, to explain how to embed the fpqc stacks of derived algebraic geometry sheaves into Gestalten.

We begin with the affine setting. The ∞-category of affines C is taken to be the opposite of any of the myriad possible choices for "derived affines", here for concreteness taken to be $\mathbb{E}_\infty$-ring spectra $\mathrm{CAlg}(\mathcal{D}(\mathbb{S}))$. For any $\mathrm{Spec}(A) \in C$, we take $0\mathrm{Pr}_A := \mathcal{D}(A)$ to be the derived ∞-category of A-modules, and inductively define

$$(n+1)\mathrm{Pr}_A := \mathrm{Mod}_{n\mathrm{Pr}_A}((n+1)\mathrm{Pr},$$

equipped with the relative Lurie tensor product.

A map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ in C, corresponding to a map of $\mathbb{E}_\infty$-rings $A \to B$, induces for any fixed $n \geq 0$ an adjunction

$$f_n^* : n\mathrm{Pr}_A \rightleftarrows n\mathrm{Pr}_B : f_{n,*},$$

with a further left adjoint $f_{n,\sharp} : n\mathrm{Pr}_B \to n\mathrm{Pr}_A$. The association $\mathrm{Spec}(A) \mapsto n\mathrm{Pr}_A$ gives a presentable six-functor formalism in the sense of [3]

$$\mathrm{Corr}(C, E) \to \mathrm{Pr}^L,$$

where E are all maps in C. If we restrict the class E to the *countably presented* maps $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ in C, i.e. those for which $B \in \mathrm{CAlg}(\mathcal{D}(A))^{\aleph_1}$, this six-functor formalism even takes values in $(n+1)\mathrm{Pr}_A$. Any countably presented f is also *étale*, in the sense that f_n^* admits a further left adjoint $f_{n\sharp}$ for all $n \geq 1$, and an ambidexterity result $f_{n\sharp} \simeq f_{n*}$ canonically identifies the left and right adjoints. As a formal consequence, $n\mathrm{Pr}_B$ is a duablizable object in $(n+1)\mathrm{Pr}_A$ for all $n \geq 1$.

To extend to non-affine stacks, we verify that the functor $C^{\mathrm{op}} \to 1\mathrm{Pr}$, given by $\mathrm{Spec}(A) \mapsto n\mathrm{Pr}_A$ with its pullback functoriality, satisfies fpqc hyperdescent, with respect to the usual notion of flatness for $\mathbb{E}_\infty$-rings due to Lurie, see e.g.

[2]. This shows that the naive notion of *quasi-coherent sheaves of (presentable)* (∞, n)*-categories* on a prestack $X : C^{\mathrm{op}} \to \mathrm{Ani}$, given by

$$n\mathrm{Pr}_X \simeq \lim_{\mathrm{Spec}(A)\to X} n\mathrm{Pr}_A,$$

is invariant under sheafification. For $n = 0$ this recovers (at least for countable colimits of affines) the usual derived ∞-category $\mathcal{D}(X)$, and likewise for $n = 1$ it is the sheaves of categories of Gaitsgory [1], also called quasi-coherent stacks in [2]. The canonical tensor products make each $n\mathrm{Pr}_X$ into a symmetric monoidal ∞-category, with respect to which we have

$$\mathrm{End}_{(n+1)\mathrm{Pr}_X}(\mathbf{1}) \simeq n\mathrm{Pr}_X.$$

This allows us to identify $n\mathrm{Pr}_X \simeq \mathbf{1} \in \mathrm{CAlg}((n+1)\mathrm{Pr}_X)$ for all $n \geq 1$, and shows that the collection of $n\mathrm{Pr}_X$ for all $n \geq 0$ naturally constitutes a Stefanich ring.

For any map of fpqc stacks $f : Y \to X$, we automatically obtain the adjoint pullback and pushforward functorialities

$$f_n^* : n\mathrm{Pr}_X \rightleftarrows n\mathrm{Pr}_Y : f_{n,*},$$

compatible in the expected way with the symmetric monoidal structures. In order to extend this to a 6-functor formalism, it is convenient to impose the following mild requirement: a map of fpqc stacks $f : Y \to X$ is *bi-countably presented* if for any map from an affine $\mathrm{Spec}(A) \to X$, the pullback $Y \times_X \mathrm{Spec}(A)$ is a countable colimit of affines $\mathrm{Spec}(B_i)$, with each B_i countably presented over A. With E taken to be the bi-countably presented maps, we then have a 6-functor formalism

$$\mathrm{Corr}(\mathrm{Shv}(C, E)^{\mathrm{bi}\aleph_1}) \to (n+1)\mathrm{Pr};$$

given by $X \mapsto n\mathrm{Pr}_X$. In particular, for any bi-countably presented $f : Y \to X$, there exists for all $n \geq 1$ an internal left adjoint $f_{n,\sharp}$ to f_n^*, satisfying for all $n \geq 2$ ambidexterity in the form $f_{n,\sharp} \simeq f_{n,*}$. For such f, the object $n\mathrm{Pr}_Y$ is dualizable (in fact self-dual) in $(n+1)\mathrm{Pr}_X$ for all $n \geq 1$, and for another bi-countably presentable $Y' \to X$ we have for all $n \geq 1$ a a Künneth formula

$$n\mathrm{Pr}_Y \otimes_{(n+1)\mathrm{Pr}_X} \mathrm{Pr}'_Y \simeq n\mathrm{Pr}_{Y\times_X Y'}.$$

The tensor product on the left-hand side must be taken inside the ∞-category $(n+1)\mathrm{Pr}_X$, otherwise some geometricity assumptions might be required on the stacks.

Much of the discussion of the previous paragraph can be carried out in the slightly greater generality of those stacks which may be written as countable colimits of affines. That is to say, the functor to stable Stefanich rings

$$\mathrm{Shv}(C)^{\aleph_1} \to \mathrm{StRing}^{\mathrm{op}}_{/\mathbb{S}},$$

given by $X \mapsto (\mathcal{O}(X), \mathcal{D}(X), 1\mathrm{Pr}_X, 2\mathrm{Pr}_X, \ldots)$, commutes with countable colimits and finite limits, under the image of which all maps are 1-étale and 2-proper. Up to the issues of countability, this means that we have constructed a geometric morphism from Gestalten over the sphere spectrum to fpqc stacks.

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Topological Quantum Field Theories

Adam Dauser

We present an account of fully extended topological quantum field theories in terms of symmetric monoidal presentable (∞, n)-categories.

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. An object X is called *dualisable* if it admits a dual $X^{\vee}$, a coevaluation and an evaluation

$$\mathrm{coev}_X : 1_{\mathcal{C}} \longrightarrow X \otimes X^{\vee}, \qquad \mathrm{ev}_X : X^{\vee} \otimes X \longrightarrow 1_{\mathcal{C}}$$

satisfying the triangle identities

$$(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \simeq \mathrm{id}_X,$$
$$(\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{X^\vee} \otimes \mathrm{coev}_X) \simeq \mathrm{id}_{X^\vee}.$$

The starting point of our story is a graphical calculus representing dualisable objects. The objects themselves are oriented points, positive orientation signifying X and negative orientation $X^{\vee}$. The identity map id_X is drawn as a straight line. Coevaluation is a cup and evaluation is a cap and the empty diagram represents the unit. To compose maps, glue their representing diagrams together, and to take their tensor product, stack them on top of each other. The two triangle identities say that we can straighten the wobbly lines, which come from the graphical representations of the maps featured in the triangle identities, into a straight line. Geometrically, this straightening move is an isotopy. Such graphical calculi were first formalised by Joyal and Street [1].

We can also close up a line to obtain a circle. Algebraically, this corresponds to composing coevaluation, the symmetry and evaluation, yielding $\dim(X) := \mathrm{Tr}(\mathrm{id}_X | X)$, an endomorphism of $1_{\mathcal{C}}$. This yields another test of our graphical calculus: It predicts $\dim(X^{\vee}) = \dim(X)$, which turns out to be true.

This graphical calculus has a purely geometric interpretation. Denote by $\mathrm{Bord}_{0,1}^{\text{1-fr}}$ the symmetric monoidal ∞-category whose objects are finite signed sets and whose morphisms are compact oriented 1-manifolds with collared boundary. We make this into an ∞-category by letting spaces of diffeomorphisms between them carry their C^{∞}-topology, and a symmetric monoidal one via disjoint union. Harpaz [2] proves that evaluation at the positive point gives an equivalence between the anima of symmetric monoidal functors $\mathrm{Bord}_{0,1}^{\text{1-fr}} \to \mathcal{C}$ and the full sub-anima on dualisable objects of $\mathcal{C}$.

The same statement can be expressed in terms of κ-presentable categories. As $(\mathrm{Bord}_{0,1}^{1\text{-fr}})^{\mathrm{op}} \simeq \mathrm{Bord}_{0,1}^{1\text{-fr}}$ by duality, its presheaf category is equivalent to

$$\mathrm{Fun}(\mathrm{Bord}_{0,1}^{1\text{-fr}}, \mathrm{Ani}),$$

with tensor product given by Day convolution. For $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}_\kappa^L)$, denote its full sub-anima on dualisable objects by $\mathcal{C}^{\mathrm{dual}}$. We obtain an equivalence

$$\mathrm{ev}_{\mathrm{pt}^+} : \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}_\kappa^L)}(\mathrm{Fun}(\mathrm{Bord}_{0,1}^{1\text{-fr}}, \mathrm{Ani}), \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{\mathrm{dual}}.$$

Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. An object X is 2-*dualisable* if it is dualisable and both ev_X and coev_X admit left and right adjoints. We denote the right adjoint of ev_X by ev_X^R. Extending the 1-dimensional case, we might hope for a graphical calculus where we additionally have surfaces depicting e.g. the unit of $\mathrm{ev}_X \dashv \mathrm{ev}_X^R$. However, all of a sudden, we can define non-trivial endomorphisms purely from this data, so the straight line we used to represent id_X in the old graphical calculus can't be good enough anymore.

More precisely, let the *Serre twist* be the automorphism $S_X : X \to X$ given by

$$X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X^R} X \otimes X^\vee \otimes X \xrightarrow{\beta_{X,X^\vee} \otimes \mathrm{id}_X} X^\vee \otimes X \otimes X \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_X} X,$$

where β is the symmetry. The geometry has to remember this. The trick turns out to be to equip our manifolds with framings.

Let M be an m-dimensional manifold, $m \leq n$. An n-*framing* is a trivialisation of

$$TM \oplus \mathbb{R}^{n-m}_M.$$

On a collared bordism, the framing must agree with the given boundary framings under the collar identifications, so that framed bordisms can be glued. In the resulting graphical calculus, the Serre twist corresponds to turning a basis of a 2-dimensional tangent space once around; see [3, §4.2].

To formally define a symmetric monoidal κ-presentable (∞, n)-category of n-framed bordisms, we proceed inductively as follows.

Recall the inductive definition [5, Lecture III]

$$0\mathrm{Pr}_\kappa^L := \mathrm{Ani}, \qquad n\mathrm{Pr}_\kappa^L := \mathrm{Mod}_{(n-1)\mathrm{Pr}_\kappa^L}\mathrm{Pr}_\kappa^L.$$

Denote by $\mathrm{Bord}_{k,k+1}^{n\text{-fr}}$ the symmetric monoidal ∞-category of closed n-framed k-manifolds and n-framed bordisms between them, defined analogously to $\mathrm{Bord}_{0,1}^{1\text{-fr}}$. For $k \leq n-2$, we denote by

$$B_k^{n\text{-fr}} : \mathrm{Bord}_{k,k+1}^{n\text{-fr}} \longrightarrow \mathrm{Cat}_\infty$$

the lax symmetric monoidal functor sending X to the ∞-category $B_k^{n\text{-fr}}(X)$ of n-framed $(k+1)$-manifolds with boundary X and bordisms relative to X. Formally, this can be defined by representing its symmetric monoidal unstraightening in a way analogous to our definition of $\mathrm{Bord}_{0,1}^{1\text{-fr}}$. Each of these is automatically a module over $B_k^{n\text{-fr}}(\emptyset) = \mathrm{Bord}_{k+1,k+2}^{n\text{-fr}}$.

For $n \geq 2$, start with the lax symmetric monoidal composite

$$\mathcal{B}^{n\text{-fr}}_{n-2} : \mathrm{Bord}^{n\text{-fr}}_{n-2,n-1} \xrightarrow{B^{n\text{-fr}}_{n-2}} \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,\mathrm{Ani})} \mathrm{Pr}^L_\kappa.$$

For $k = n-3, \dots, 0$, inductively define lax symmetric monoidal functors

$$\mathcal{B}^{n\text{-fr}}_k : \mathrm{Bord}^{n\text{-fr}}_{k,k+1} \longrightarrow (n-k-1)\mathrm{Pr}^L_\kappa$$

via the composite

$$\begin{aligned}\mathrm{Bord}^{n\text{-fr}}_{k,k+1} &\xrightarrow{B^{n\text{-fr}}_k} \mathrm{Mod}_{\mathrm{Bord}^{n\text{-fr}}_{k+1,k+2}}\mathrm{Cat}_\infty \\ &\xrightarrow{\mathrm{Fun}(-,(n-k-2)\mathrm{Pr}^L_\kappa)} \mathrm{Mod}_{\mathrm{Fun}(\mathrm{Bord}^{n\text{-fr}}_{k+1,k+2},(n-k-2)\mathrm{Pr}^L_\kappa)}(n-k-1)\mathrm{Pr}^L_\kappa \\ &\xrightarrow{\mathrm{Mod}_{\mathcal{B}^{n\text{-fr}}_{k+1}}} (n-k-1)\mathrm{Pr}^L_\kappa.\end{aligned}$$

Finally, set

$$\mathcal{B}\mathrm{ord}^{n\text{-fr}}_n := \mathrm{Mod}_{\mathcal{B}^{n\text{-fr}}_0}\mathrm{Fun}\big(\mathrm{Bord}^{n\text{-fr}}_{0,1}, (n-1)\mathrm{Pr}^L_\kappa\big) \in \mathrm{CAlg}(n\mathrm{Pr}^L_\kappa).$$

A morphism $\mathcal{B}\mathrm{ord}^{n\text{-fr}}_n \to \mathcal{C}$ in $\mathrm{CAlg}(n\mathrm{Pr}^L_\kappa)$ is called a *fully extended framed topological quantum field theory*.

In higher dimensions, we need duals and adjoints at each level. A symmetric monoidal (∞,n)-category $\mathfrak{C}$ *admits duals* if every object has a dual and every k-morphism admits left and right adjoints for $1 \leq k < n$. An object is called *fully dualisable* if it is contained in a symmetric monoidal subcategory admitting duals.

The cobordism hypothesis [3] can now be reformulated as follows.

Conjecture. Let $\mathcal{C} \in \mathrm{CAlg}(n\mathrm{Pr}^L_\kappa)$, and denote by $\mathcal{C}^{\mathrm{dual}}$ the full sub-anima on fully dualisable objects of its underlying (∞,n)-category. Evaluation at the positively framed point yields an equivalence of anima

$$\mathrm{ev}_{\mathrm{pt}^+} : \mathrm{Hom}_{\mathrm{CAlg}(n\mathrm{Pr}^L_\kappa)}(\mathcal{B}\mathrm{ord}^{n\text{-fr}}_n, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{\mathrm{dual}}.$$

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TQFTs and Representation Theory

Jackson Van Dyke

Many examples of TQFTs can be described via the cobordism hypothesis using categories of sheaves (of higher categories) on stacks of various kinds. For instance, the category of quasicoherent sheaves on the classifying stack $*/G$ of an algebraic group is equivalent to the category of algebraic representations of G. This defines an object of the symmetric-monoidal $(\infty, 2)$-category of $\mathbb{C}$-linear presentable categories:

$$\mathbf{QC}(*/G) = \mathbf{Rep}(G) \in \mathbf{Pr}_{\mathbb{C}} = \mathbf{Mod}_{D(\mathbb{C})}\left(1\mathbf{Pr}^{L}_{\mathrm{st}}\right).$$

As is necessary to define a TQFT via the cobordism hypothesis, categories of this form are often dualizable. For instance, it was shown in [1] that $\mathbf{QC}(X)$ is dualizable in $\mathbf{Pr}_{\mathbb{C}}$ when X is perfect. The dualizability data itself is given by pulling and pushing through the diagonal correspondence:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow^{\Delta} & \\ * & & & & X \times X \end{array}$$

This is a part of the more general discussion of the categorical Künneth formula: one can ask, given a pair of maps of algebraic stacks $X \to S \leftarrow Y$, is the natural comparison map

$$\mathbf{QC}(X) \otimes_{\mathbf{QC}(S)} \mathbf{QC}(Y) \to \mathbf{QC}(X \times_S Y) \tag{1}$$

an equivalence? As is shown in [1], this is an equivalence in the setting of perfect stacks, which includes schemes, and many algebraic stacks in characteristic zero. In fact, it also shown that the natural map

$$\mathbf{QC}(X \times_S Y) \to \mathrm{Hom}_{\mathbf{QC}(S)}(\mathbf{QC}(X), \mathbf{QC}(Y)) \tag{2}$$

is an equivalence. As is discussed in [5], though this often fails for more exotic sheaf theories, it can be remedied using categorification.

Dualizability alone merely defines a framed theory, whereas an oriented theory requires the extra data of a homotopy-fixed point structure for the appropriate special-orthogonal group. We return to our running example $X = */G$ to illustrate this point. Now additionally assume that $G = \Gamma$ is a finite group. The category in question, finite-dimensional modules over the group algebra $\mathbb{C}[\Gamma]$, is even a 2-dualizable object of $\mathbf{Pr}_{\mathbb{C}}$ (since the group algebra is separable). By the framed cobordism hypothesis, this classifies a representation of $\mathbf{Bord}^{\mathrm{fr}}_2$. There is a natural functor which takes a framed bordism to the underlying oriented one, however not every framed theory factors through this. The oriented cobordism hypothesis says that, given a framed theory, the data of a factorization:

$$\begin{array}{ccc} & & \mathbf{Bord}^{\mathrm{or}}_2 \\ & \nearrow & \uparrow \\ \mathbf{Bord}^{\mathrm{fr}}_2 & \longrightarrow & \{\text{target category}\} \end{array} \tag{3}$$

is equivalent to the data of an $\mathrm{SO}(2)$-fixed point structure on the assignment to the point of the original framed theory. In the case of the category of modules over the group algebra, this is equivalent to the data of a symmetric Frobenius structure on $\mathbb{C}[\Gamma]$. Recall that this is a map τ from $\mathbb{C}[\Gamma]$ to $\mathbb{C}$ such that the pairing $(f,g) \mapsto \tau(fg)$ is a nondegenerate symmetric form. In the case of a finite group algebra, such a map is defined by sending f to $f(e)/|\Gamma|$. The associated oriented TQFT sends the circle to the vector space of functions on the stack of Γ-bundles on the circle, i.e. the class functions $\mathbb{C}[\Gamma]^{\Gamma} = \mathbb{C}[\Gamma/_{\mathrm{ad}}\Gamma]$. Note that this is the center of $\mathbb{C}[\Gamma]$, and that the trace map being symmetric is equivalent to factoring through a trace $\mathbb{C}[\Gamma]^{\Gamma} \to \mathbb{C}$. An oriented surface Σ is sent to the volume of the stack of Γ-bundles, i.e. the cardinality of $\mathrm{Hom}(\pi_1(\Sigma), \Gamma)/\Gamma$.

For a general object of $\mathbf{Pr}_{\mathbb{C}}$ of the form $\mathbf{QC}(X)$, an $\mathrm{SO}(2)$-fixed point structure is not guaranteed to exist: This requires a Calabi-Yau structure on X in the usual sense (a trivialization of the canonical bundle).

Thanks to the symmetric-monoidal structure on $\mathbf{QC}(X)$, it can be regarded as an object of the symmetric-monoidal $(\infty, n+2)$-category $\mathbf{Mor}_n(\mathbf{Pr}_{\mathbb{C}})$ of $\mathbb{E}_n$-algebras in $\mathbf{Pr}_{\mathbb{C}}$ (for any positive integer n). At an informal level, this category can be defined inductively: the morphisms between two $\mathbb{E}_n$-algebras $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{E}_{n-1}$-algebras among $(\mathcal{A}, \mathcal{B})$-bimodules. When the stack X is n-affine, then the object $\mathbf{QC}(X)$ of $\mathbf{Mor}_n(\mathbf{Pr}_{\mathbb{C}})$ can be thought of as an avatar of $n\mathbf{Pr}_X \in (n+1)\mathbf{Pr}_{\mathbb{C}}$.

As is discussed in [1], the object $\mathbf{QC}(X) \in \mathbf{Mor}_1(\mathbf{Pr}_{\mathbb{C}})$ is 2-dualizable if X is perfect. Again, the dualizability data is defined by the diagonal correspondence, only now the 1-morphisms are bimodule categories, and the perfect condition enters at the level of 2-morphisms, where again diagonal correspondences (between correspondences) will induce the 2-morphisms furnishing the 2-dualizability data.

For example, the classifying stack of a reductive algebraic group is perfect and 1-affine, so the 2-category of sheaves of categories on $*/G$ is 2-dualizable in the $(\infty, 3)$-category $2\mathbf{Pr}_{\mathbb{C}}$.

Again, this dualizability furnishes a framed theory, but one might ask if this factors through an oriented theory as in (3). In $\mathbf{Mor}_1(\mathbf{Pr}_{\mathbb{C}})$ there is a canonical homotopy-fixed point structure of $\mathrm{SO}(2)$ induced by the correspondence given by the inclusion of constant loops:

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ * & & X^{S^1} \end{array}$$

For a subgroup $H \subset G$, descent along the map $BH \to BG$ provides an equivalence:

$$\mathbf{Mod}_{\mathbf{Rep}(G),\otimes}(\mathbf{Pr}_{\mathbb{C}}) \xrightarrow[\otimes_{\mathbf{Rep}(H)}]{\sim} \mathbf{Mod}_{\mathbf{QC}(H\backslash G/H),*}(\mathbf{Pr}_{\mathbb{C}}) \tag{4}$$

Note that in the target, we are considering the category of sheaves on $BH \times_{BG} BH \cong H\backslash G/H$ with convolution. Therefore one can think of $(\mathbf{Rep}(G), \otimes)$ and $(\mathbf{QC}(G), *)$ as two different representatives for the same object $\mathbf{Pr}_{*/G} \in 2\mathbf{Pr}_{\mathbb{C}}$. When H is the trivial group, this provides a Morita equivalence between $\mathbf{Rep}(G)$

(with the usual symmetric-monoidal tensor product) and $\mathbf{QC}(G)$ (with convolution), as objects of $\mathbf{Mor}_1(\mathbf{Pr}_{\mathbb{C}})$.

Rather than being presented directly using quasi-coherent sheaves, many TQFTs are presented via more exotic sheaf theories such as $\mathcal{D}$-modules and ind-coherent sheaves. Though this obscures some properties of the theory (such as the categorical Künneth formula and the existence of orientation data), similar properties can typically be restored, e.g. by rewriting the given sheaf theory in terms of $\mathbf{QC}$ via transmutation. For instance, the character field theory studied in [4, 2] employs the theory of $\mathcal{D}$-modules, which can be defined as quasicoherent sheaves on the de Rham stack. In particular, the theory sends the point to:

$$\mathbf{D\text{-}Mod}(G) = \mathbf{QC}(G_{\mathrm{dR}}) \in \mathbf{Mor}_1(\mathbf{Pr}_{\mathbb{C}})$$

with monoidal structure coming from convolution.

The theory of $\mathcal{D}$-modules does define a six-functor formalism, however one cannot automatically conclude analogues of results such as (1), (2), and (4). When the base stack is the classifying stack of a reductive group, the category of $\mathcal{D}$-modules is shown to satisfy an analogue of the Künneth and integral transforms formulas in [4, 2]. In addition, they show that $\mathbf{D\text{-}Mod}(G)$ is in fact a 2-dualizable object with an $\mathrm{SO}(2)$-fixed point structure, in analogy with the theory based on $\mathbf{QC}$.

Explicitly, the theory sending the point to $\mathbf{D\text{-}Mod}(G)$ sends the circle to the category of G-equivariant $\mathcal{D}$-modules on G, i.e. $\mathcal{D}$-modules on the quotient stack $G/_{\mathrm{ad}}G$. This theory assigns a closed oriented surface Σ to the Borel-Moore chains on the mapping stack from S to BG.

There is also a Morita equivalence of categories of $\mathcal{D}$-modules, which is an analogue of (4). It is shown in [3] that there is a Morita equivalence between $\mathbf{D\text{-}Mod}(G)$ and $\mathbf{D\text{-}Mod}(N\backslash G/N)$, both with convolution, in $\mathbf{Mor}_1(\mathbf{Pr}_{\mathbb{C}})$. Here N is the unipotent radical of a Borel subgroup. Therefore the above TQFT is equivalently described by either $\mathbf{D\text{-}Mod}(G)$, or by this universal Hecke category.

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Gestalten

Ko Aoki

This report summarizes my lecture, which followed Lectures VI–VII of Scholze's notes [2]. The aim was to explain in what sense the category of *Gestalten*, introduced by Scholze and Stefanich, is a good category of geometric objects. The

category Gest is basically a topos, but this statement requires two qualifications. One must restrict to n-étale gestalten for a fixed $n \geq 0$, since Gest is not locally cartesian closed. Moreover, after fixing the cardinal $\aleph_1$ in the foundations, $\mathsf{Gest}^{n\text{-ét}}$ has only countable colimits, not arbitrary colimits.

We write $n\mathsf{Pr}$ for the symmetric monoidal category of $\aleph_1$-compactly generated n-categories. A *Stefanich ring* is a sequence

$$A = (A_0, A_1, \dots), \qquad A_m \in \mathrm{CAlg}(n\mathsf{Pr}), \qquad A_m \simeq \mathrm{End}_{A_{n+1}}(\mathbf{1}).$$

We write StRing for the category of Stefanich rings and set

$$\mathsf{Gest} = \mathsf{StRing}^{\mathrm{op}}.$$

Thus a gestalt is, by definition, the geometric object dual to a Stefanich ring.

The relative description is essential. A map $f\colon A \to B$ of Stefanich rings is a compatible sequence of maps

$$f_n^*\colon A_{n+1} \to B_{n+1}.$$

Its right adjoint $f_{n,*}$ is a lax symmetric monoidal functor and sends the unit to a commutative algebra

$$(B/A)_n = f_{n,*}\mathbf{1} \in \mathrm{CAlg}(A_{n+1}).$$

These objects identify the category of relative Stefanich rings as

$$\mathsf{StRing}_{A/} \simeq \operatorname*{colim}_{n\geq 0} \mathrm{CAlg}(A_{n+1}),$$

where the transition functors are fully faithful. A map $f\colon A \to B$ is called *n-affine* if it already occurs in $\mathrm{CAlg}(A_{n+1})$.

There are many interesting geometric objects that are not n-affine for any n. Two other useful conditions are *properness* and *étaleness*.

We say that a map $f\colon A \to B$ of Stefanich rings is *n-prim* if

$$f_m^*\colon \mathbf{1} \to (B/A)_{m+1}$$

admits a right adjoint in A_{m+2} for all $m \geq n$. It is *n-proper* if it is n-prim and all the iterated diagonals are n-prim.

We call a map $f\colon A \to B$ of Stefanich rings *n-suave* if the following two conditions hold. First, for every $m \geq n$,

$$f_m^*\colon \mathbf{1} \to (B/A)_{m+1}$$

admits a left adjoint in A_{m+2}, which we call $f_{m,\natural}$. In this situation, $f_{m,\natural}$ is oplax symmetric monoidal, and hence sends the unit to a commutative coalgebra

$$(B/A)^!_m = f_{m,\natural}(\mathbf{1}) \in \mathrm{cCAlg}(A_{m+2}).$$

Second, we require that the counit map

$$f_{m,!}\colon (B/A)^!_{m+1} \to \mathbf{1}$$

admits a right adjoint in A_{m+2} for every $m \geq n$. It is *n-étale* if it is n-suave and all the iterated diagonals are n-suave.

By ambidexterity, n-proper maps are $(n+1)$-étale, and n-étale maps are $(n+1)$-proper. It follows that many gestalten are n-étale for some n.

For a gestalt $X = \mathrm{Gest}(A)$, we write

$$\mathsf{Gest}^{n\text{-ét}}_{/X}$$

for the category of n-étale gestalten over X.

Theorem (Scholze–Stefanich)**.** For every gestalt X and every $n \geq 0$,

$$\mathrm{Ind}_{\aleph_1}\left(\mathsf{Gest}^{n\text{-ét}}_{/X}\right)$$

is a topos.

The key input is that countable colimits of n-étale gestalten can be computed in a way that is compatible with tensor products. These colimits correspond to countable limits of Stefanich rings. The limit of a countable diagram of n-suave Stefanich rings can be computed by taking the corresponding countable colimit of commutative coalgebras on the shriek side.

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Factorization homology and TQFTs

Viktoriya Ozornova

Factorization homology is a vibrant and by now quite broad topic, so we can only scratch the surface of the subject here. For many of the precise proofs, the thesis of Claudia Scheimbauer [Sch14] is still the main source; however, since one is rarely precise in a talk, more survey-like articles by Lurie [Lur09] and Ayala–Francis [AF20] provide a good intuition of what is going on.

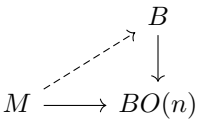
Factorization homology takes two inputs, an algebraic one and a topological one, and produces a new (maybe slightly different) algebraic output. The topological input is mainly some version of a manifold, and the procedure can be informally described as 'integrating' the algebraic input over the manifold; this is also recorded by the integral sign denoting factorization homology.

We will be mainly concerned with a variant where the algebraic input is an E_n-algebra A in some symmetric monoidal ∞-category $\mathcal{C}$. Mostly, it will be useful to assume that $\mathcal{C}$ is presentable and that the monoidal product in $\mathcal{C}$ is compatible with certain colimits (for many things, sifted colimits would be enough; sometimes it is useful to assume compatibility with all colimits.)

One of the main sources of E_n-algebras (and the original motivation for considering them) is given by n-fold loop space of a topological space.

On the topological side, our input will typically be an n-dimensional smooth manifold (technically, satisfying some finiteness assumptions). Moreover, these manifolds will be often equipped with a tangential structure (mainly with a 'framing', but more general definitions are of the same complexity). Fix a continuous

map $B \to BO(n)$. A *B-framing* on an n-dimensional smooth manifold M is a choice of a lift (up to a chosen homotopy!) of the tangent bundle in the following diagram:



Our main examples come from $\mathrm{id}\colon BO(n) \to BO(n)$ (corresponding to no structure) and any point $\mathrm{pt} \to BO(n)$ (corresponding to framings).

For factorization homology, we discussed first the ∞-category of n-dimensional manifolds $\mathcal{M}\mathrm{fld}_n$ and embeddings between them as well as a variant $\mathcal{M}\mathrm{fld}_n^B$ incorporating the B-framing. Amongst general manifolds, we singled out specific ones which are finite disjoint unions of manifolds diffeomorphic to $\mathbb{R}^n$, and the full sub-∞-category of those is then written as $\mathcal{D}\mathrm{isk}_n$ (or $\mathcal{D}\mathrm{isk}_n^B$ if we also want a B-framing.) All of these ∞-categories are symmetric monoidal with respect to disjoint union of manifolds. Note though that it is not the categorical coproduct.

A reformulation of having an E_n-algebra A in a symmetric monoidal ∞-category $\mathcal{C}$ amounts to having a symmetric monoidal functor $A\colon \mathcal{D}\mathrm{isk}_n^{\mathrm{fr}} \to \mathcal{C}$ (where the tangential structure on the source denotes the framing; getting back a single object of $\mathcal{C}$ is given by evaluating at $\mathbb{R}^n$ with standard framing.) As an aside on the notation, having no tangential structure is instead referred to as 'framed E_n-algebra'.

This allows us to make the following definition.

Definition. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Let $A\colon \mathcal{D}\mathrm{isk}_n^{\mathrm{fr}} \to \mathcal{C}$ be an E_n-algebra in $\mathcal{C}$ and let M be a B-framed manifold. The *factorization homology* of M with coefficients in A is given by the following formula:

$$\int_M A := \mathrm{colim}\left(\mathcal{D}\mathrm{isk}_n^{\mathrm{fr}}/M \to \mathcal{D}\mathrm{isk}_n^{\mathrm{fr}} \xrightarrow{A} \mathcal{C}\right)$$

The motivation for calling it homology comes from two key examples:

Example. If we look at the case $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ with *direct sum* as a monoidal structure (as was pointed out, this example is not completely well-behaved with respect to colimits, but it is sufficient for our purposes) and any abelian group A (viewed as a chain complex concentrated in degree 0), then the factorization homology is computing the singular chain complex with coefficients in A.

Example. If A is an associative algebra (so an E_1-algebra in the ∞-category $\mathcal{C} = \mathcal{D}(\mathbb{Z})$, but this time with tensor product as a monoidal structure) and $M = S^1$, the factorization homology computes the Hochschild chain complex of A.

A central tool used for proving the statement of the latter example is a variant of excision satisfied by factorization homology, further motivating the name 'homology'.

As already sketched by Lurie [Lur09], the factorization homology can be extended to give a TQFT, meaning a symmetric monoidal (∞, n)-functor

$$\mathrm{Bord}_n^{\mathrm{fr}} \to \mathrm{Morita}_n.$$

The source has already been described in Jackson Van Dyke's talk. Constructing the target in a rigorous and suitable manner is highly non-trivial. Previous work by Haugseng is probably not giving the 'right' notion of dualizability. There is a new and very recent meaning given to the target by [SSS26]. But informally, we can roughly describe the Morita $(\infty, n+1)$-category here following Scheimbauer–Steffens–Stewart. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then E_k-algebras in $\mathcal{C}$ inherit a monoidal structure which we will need momentarily. Objects of Morita_n are E_n-algebras in some nice symmetric monoidal ∞-category $\mathcal{C}$, 1-morphisms are bimodules in E_{n-1}-algebras, 2-morphisms are bimodules of bimodules in E_{n-2}-algebras, and so on. The last potentially non-invertible level is different and given by homomorphisms.

Now we proceed with the description of a TFT, i.e. a symmetric monoidal functor as above. First, given an E_n-algebra A and a framed k-dimensional manifold M, we need to assign to it an object of $\mathcal{C}$ (carrying further structure which we will not make explicit in this sketch.) We can look at the n-dimensional (framed) manifold $M \times \mathbb{R}^{n-k}$ and assign to M the value

$$\int_{M \times \mathbb{R}^{n-k}} A.$$

To give a taste for the structure appearing, we want to understand how cobordisms give rise to bimodules, and we will concentrate on codimension 1 and consider an n-dimensional framed cobordism W between $(n-1)$-dimensional manifolds M_1 and M_2. Observe that finding a collar of the boundary yields an embedding

$$(M_1 \times \mathbb{R}) \amalg \mathring{W} \amalg (M_2 \times \mathbb{R}) \hookrightarrow \mathring{W}.$$

The compatibility with symmetric monoidal structures makes then the value at $\mathring{W}$ into a bimodule over values for M_1 and M_2.

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Examples and 6 functor formalisms

Ishan Levy

This talk first discussed several examples of Gestalten following lecture 8 of Scholze's notes. We started with Gestalten over a base coming from countable anima and discussed their properties (affineness, properness, etaleness). In particular, it was shown that $B^n\mathbb{Z}/2\mathbb{Z}$ is n-affine but not $n-1$-affine, using a criterion of Stefanich that reduced to the case $n = 1$ using the diagonal. We then discussed variants of the affine line and multiplicative group, and showed that the sheaf Pic on Gestalten can be described as an extension of $\mathbb{Z}/2\mathbb{Z}$ and BGL_1.

The talk then turned to a construction of the Gestalt of higher sheaves from a (ω_1-presentable) six functor formalism due to Stefanich. It began with explaining the general definition of 6-functor formalisms as a lax symmetric monoidal functor out of a category of spans into Cat with the condition that all maps get sent to left adjoints. In the case the case that the target category is replaced with Pr^L or $\mathrm{Pr}^L_{\omega_1}$, this condition is automatic, and we call it a (ω_1-)presentable six functor formalism. We discussed the 2-category of kernels of a six functor formalism, and an alternative construction as a marked atomically generated 1Pr-linear presentable category due to Dauser. The original six functor formalism refines to a *strong* symmetric monoidal functor into this presentable 2-category. The naturality of this presentable version of the 2-category of kernels gives a six functor formalism with values in 2Pr. We can then keep iterating this process of categorifying the six functor formalism to obtain a functor from the original category to Gestalten that at the level of 1-categories recovers the six functor formalism. This functor preserves finite limits, and can be thought of as a generalized kind of transmutation factoring the original six functor formalism. The reason it preserves finite limits is because the Kunneth formula at the n-categorical level is fixed by categorifying to the $n+1$st categorical level.

The talk ended with a discussion of the construction of unoriented extended field theories from Gestalten, due to Stefanich. A particular example coming from the IndCoh six functor formalism gives the spectral side of the fully extended version of the Betti geometric Langlands correspondence up to normalization issues. There is a category nCorr of higher correspondences that can be used to encode the higher categorical six functor formalism that exists at the n-categorical level of the Gestalt associated to a six functor formalism.

The first step of the construction of the field theory is to note that there is a functor from the unoriented cobordims category to $n\mathrm{Corr}(\mathrm{Ani}^{op})$ taking a cobordism to the underlying iterated cospans of anima. This can be composed with the functor Ani^{op} into stacks taking the mapping stack to a fixed stack X. The last step is to apply the symmetric monoidal functor from nCorr coming from the $n+1$-categorical level of the Gestalt of a six functor formalism on stacks.

Cartier Duality in Gestalten

Lucas Mann

In this lecture we discussed Cartier duality in the setting of Gestalten.

1. Motivation

Consider a commutative group scheme G over some base ring R. How do you understand the category $\mathrm{QCoh}(*/G)$ of algebraic G-representations?

Example.

(1) The category $\mathrm{QCoh}(*/\mathbb{Z})$ is the category of $\mathbb{Z}$-representations, i.e. $R[t^{\pm}]$-modules, i.e. $\mathrm{QCoh}(*/\mathbb{Z}) = \mathrm{QCoh}(\mathbb{G}_m)$.
(2) The category $\mathrm{QCoh}(*/\mathbb{G}_m)$ is the category of algebraic $\mathbb{G}_m$-representations, i.e. $\mathbb{Z}$-graded R-modules, i.e. $\mathrm{QCoh}(*/\mathbb{G}_m) = \mathrm{QCoh}(\mathbb{Z})$.
(3) Suppose R is a field of characteristic 0. Then $\mathrm{QCoh}(*/\mathbb{G}_a)$ is the category of R-modules equipped with a locally nilpotent endomorphism, i.e. $\mathrm{QCoh}(*/\mathbb{G}_a) = \mathrm{QCoh}(\hat{\mathbb{G}}_a)$.

In the first example, the equivalence can be constructed as follows: In the derived category of sheaves of abelian groups on rings we have

$$\underline{\mathrm{Hom}}(*/\mathbb{Z}, */\mathbb{G}_m) = \underline{\mathrm{Hom}}(\mathbb{Z}[1], \mathbb{G}_m[1]) = \underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m,$$

which induces a natural pairing $*/\mathbb{Z} \times \mathbb{G}_m \to */\mathbb{G}_m$ and thus a canonical line bundle $\mathcal{L} \in \mathrm{QCoh}(*/\mathbb{Z} \times \mathbb{G}_m)$. Now the equivalence is given by the functor

$$\mathrm{QCoh}(*/\mathbb{Z}) \to \mathrm{QCoh}(\mathbb{G}_m), \qquad M \mapsto q_!(p^*M \otimes \mathcal{L}),$$

where p and q are the two projections from $*/\mathbb{Z} \times \mathbb{G}_m$. In the other two examples, the equivalence can be constructed similarly.

Example.

(4) One categorical level down, one gets an equivalence $H_*(\mathbb{Z}) = \bigoplus_{\mathbb{Z}} R = R[t^{\pm}] = H^*(\mathbb{G}_m)$. This is a decompleted version of the Fourier transform $L^2(S^1) = L^2(\mathbb{Z})$.
(5) A similar duality as above exists also for étale sheaves instead of quasicoherent sheaves, observed by Deligne.

The goal of this talk is to explain all of the above examples through one general duality.

2. Statement

Fix some base Gestalt X. We work with spectrum objects in Gestalten over X:

$$\begin{aligned}\mathrm{Sp}(\mathrm{Gest}_{/X}) &= \varprojlim_{\Omega} \mathrm{CGrp}(\mathrm{Gest}_{/X}) \\ &= \{(M_0, M_1, \dots) \mid M_n \in \mathrm{CGrp}(\mathrm{Gest}_{/X}), M_n = \Omega M_{n+1}\}.\end{aligned}$$

There is an object $\mathrm{GL}_1 = ((\mathrm{GL}_1)_0, (\mathrm{GL}_1)_1, \dots) \in \mathrm{Sp}(\mathrm{Gest}_{/X})$ such that $(\mathrm{GL}_1)_n$ represents the functor $Y \mapsto \mathcal{O}(Y)_n^{\times}$. Explicitly, in the case $X = *$ the Gestalt $(\mathrm{GL}_1)_n$ is n-affine and corresponds to $\mathbb{S} \otimes (n-1)\mathrm{Pr}$.

Theorem (Cartier Duality, [1, §10–12])**.** *For all $n \geq 0$ the assignment $M \mapsto M^* := \underline{\mathrm{Hom}}(M, \mathrm{GL}_1)$ defines an equivalence*

$$\mathrm{Sp}(\mathrm{Gest}_{/X}^{n\text{-et}}) \xrightarrow{\sim} \mathrm{Sp}(\mathrm{Gest}_{/X}^{n\text{-prop}}).$$

Moreover, for a fixed M in the source we have:

1. *The map $M \xrightarrow{\sim} M^{**}$ is an isomorphism (note that $M^* \in \mathrm{Sp}(\mathrm{Gest}_{/X}^{(n+1)\text{-et}})$, so we can apply $(-)^*$ to it).*
2. *For all $m \geq n$ and $k = 0, \ldots, m$ there is a canonical equivalence*
$$\mathcal{O}(M_n/X)^!_m \cong \mathcal{O}(M^*_{m-k}/X)_m$$
induced by an explicit Fourier transform.
3. *M^* is connective if and only if $M_{n+2} \to *$ is $(n+2)$-affine.*

Remark. In a sense, the duality "swaps" the two spectrum structures on a spectrum Gestalt:

$$\begin{array}{ccccc} & & \mathcal{O}(M_2)_2 & & \\ & \mathcal{O}(M_1)_1 & \mathcal{O}(M_1)_2 & & \\ \mathcal{O}(M_0)_0 & \mathcal{O}(M_0)_1 & \mathcal{O}(M_0)_2 & \cdots & \end{array} \quad \rightsquigarrow \quad \begin{array}{cccc} & & \mathcal{O}(M_0^*)^!_2 & \\ & \mathcal{O}(M_0^*)^!_1 & \mathcal{O}(M_1^*)^!_2 & \\ \mathcal{O}(M_0^*)^!_0 & \mathcal{O}(M_1^*)^!_1 & \mathcal{O}(M_2^*)^!_2 & \cdots \end{array}$$

3. Proof

We fix some $M \in \mathrm{Sp}(\mathrm{Gest}_{/X}^{n\text{-et}})$ and note that

$$M_0^* = \tau_{\geq 0}\, \underline{\mathrm{Hom}}(M, \mathrm{GL}_1) = \varprojlim_m \underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(M_m, (\mathrm{GL}_1)_m).$$

The proof of the theorem proceeds in five steps:

1. We first observe that for all $m \geq n$, $\underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(M_m, (\mathrm{GL}_1)_m)$ is an m-affine Gestalt represented by $\mathcal{O}(M_m/X)^!_m$. This follows formally by playing around with adjunctions, see [1, Theorem 11.1].
2. By induction on m we show that for all $m \geq n$, $\underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(M_m, (\mathrm{GL}_1)_m)$ is n-prim and its n-affine truncation is $\underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(M_n, (\mathrm{GL}_1)_n)$, see [1, Theorem 11.4].

From the above two steps it follows that M_0^* is n-prim and its n-affine truncation is given by $\mathcal{O}(M_n/X)^!_n$.

3. One checks that $* \to M_0^*$ is n-prim, see [1, Proposition 12.1]. Combining with step (2) and shifting implies that both $M_k^* \to *$ and $* \to M_k^*$ are n-prim for all $k \in \mathbb{Z}$. The diagonal of $* \to M_k^*$ is $* \to M_{k-1}^*$, so we deduce that $* \to M_k^*$ is n-proper. The diagonal of $M_k^* \to *$ is a pullback of $* \to M_k^*$ so we conclude that all M_k^* are n-proper.
4. We now show that the natural map $M \to M^{**}$ is an isomorphism, see [1, Corollary 12.2]. The key observation is that by ambidexterity, $\mathcal{O}(M_k^*/X)_m$ is dualizable for $m \geq n+1$, and so the claimed reflexivity follows from the automatic reflexivity of dualizable objects (a priori on categorical levels $\geq n+2$, but a posteriori on all categorical levels). Here we see the magic of

Gestalten: By allowing us to go to a high enough categorical level, Cartier duality reduces to dualizable duality!

(5) At last one checks that M^* is connective iff $M_{n+2} \to *$ is $(n+2)$-affine, see [1, Theorem 12.3(iii)]. This follows quite directly from the fact that we already know the m-affine truncations of M_k^* by steps (1) and (2).

4. Examples

Fix $X = \mathrm{Gest}(\mathcal{D}(R))$ for a ring R. Here are some applications of the theorem:

Example.

(1) Let $G = \mathbb{Z} = (\mathbb{Z}, B\mathbb{Z}, B^2\mathbb{Z}, \dots) \in \mathrm{Sp}(\mathrm{Gest}_{/X})$. By step (1) in the proof we have

$$\underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(\mathbb{Z}, (\mathrm{GL}_1)_0) = \mathrm{Gest}_X(\mathcal{O}(\mathbb{Z}/X)_0^!) = \mathrm{Gest}_X(R[t^{\pm}]) = \mathbb{G}_m.$$

Thus $\tau_{\geq 0}\mathbb{Z}^* = \mathbb{G}_m$. On the other hand, $B\mathbb{Z}$ is 1-affine, hence $B^2\mathbb{Z}$ is 2-affine and we conclude that $\mathbb{Z}^*$ is connective. Therefore $\mathbb{Z}^* = \mathbb{G}_m$ and $\mathbb{G}_m^* = \mathbb{Z}$. The identities $\mathcal{O}(M_k/X)_m^! \cong \mathcal{O}(M_{m-k}^*/X)_m$ evaluate to:

$$\begin{aligned}
m = k = 0\colon &\qquad H_*(\mathbb{Z}) = H^*(\mathbb{G}_m)\\
m = 1, k = 0\colon &\qquad \mathrm{QCoh}(\mathbb{Z}) = \mathrm{QCoh}(*/\mathbb{G}_m)\\
m = 1, k = 1\colon &\qquad \mathrm{QCoh}(*/\mathbb{Z}) = \mathrm{QCoh}(\mathbb{G}_m)\\
m = 2, k = 0\colon &\qquad 1\mathrm{Pr}_{\mathbb{Z}} = 1\mathrm{Pr}_{B^2\mathbb{G}_m}\\
m = 2, k = 1\colon &\qquad 1\mathrm{Pr}_{B\mathbb{Z}} = 1\mathrm{Pr}_{B\mathbb{G}_m}\\
m = 2, k = 2\colon &\qquad 1\mathrm{Pr}_{B^2\mathbb{Z}} = 1\mathrm{Pr}_{\mathbb{G}_m}\\
\vdots &\qquad \vdots
\end{aligned}$$

(2) Let G be a finite flat group scheme over R, viewed implicitly as Gestalt. Then G is lci and hence 0-suave, therefore

$$\underline{\mathrm{Hom}}_{\mathrm{CGrp}_{/X}}(G, (\mathrm{GL}_1)_0) = \mathrm{Gest}_X(\mathcal{O}(G/X)_0^!) = \mathrm{Gest}_X(\mathcal{O}(G)^*).$$

Since BG is 1-affine, G^* is connective and thus given by $\mathrm{Gest}_X(\mathcal{O}(G)^*)$. This is the classical Cartier dual of G. Note that the dualizability from step (4) in the proof already holds on categorical level 0.

(3) Fix $R = \mathbb{C}$. By (1), the spectrum object GL_1 in Gestalten over $\mathbb{C}$ is an internal Brown–Comenetz dual of $\mathbb{G}_m$ in the sense that $\underline{\mathrm{Hom}}(\mathbb{Z}, \mathrm{GL}_1) = \mathbb{G}_m$. From this one can deduce that for $n > 0$ one has

$$\pi_{-n}\,\mathrm{GL}_1 = \mathrm{Hom}(\pi_n\mathbb{S}, \pi_0\mathbb{G}_m) = \mathrm{Hom}(\pi_n\mathbb{S}, \mathbb{C}^\times),$$

see [1, Corollary 10.12].

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Equivariant de Rham spaces

Germán Stefanich
(joint work with David Ben-Zvi and David Nadler)

The goal of this talk was to describe a theory of de Rham spaces in the setting of equivariant algebraic geometry. The quest for this theory was originally motivated by considerations in local geometric Langlands and three dimensional topological field theory [1], aspects which were discussed in a subsequent talk by David Nadler [3]. For the present talk, we instead took as a starting point the problem of geometrizing equivariant de Rham cohomology.

Suppose given a smooth variety X over an algebraically closed field k of characteristic zero. Taking the quotient of X by the equivalence relation which identifies infinitesimally close points gives rise to the so-called de Rham space [4], which geometrizes de Rham cohomology in the sense that one has an isomorphism

$$H^{\bullet}_{\mathrm{dR}}(x) = \mathcal{O}(X_{\mathrm{dR}})$$

between the de Rham cohomology of X and the coherent cohomology of X_{dR}.

Assume now that we are given an action of a reductive algebraic group G on X. There are several different meanings one can give to the G-equivariant de Rham cohomology of X, which depend on a choice of a one dimensional algebraic group A with trivialized Lie algebra: in addition to the "naive" Borel equivariant cohomology (corresponding to $A = \mathbb{G}_a$), one also has equivariant K-theory (when $A = \mathbb{G}_m$) and equivariant elliptic cohomology (when A is an elliptic curve). The main output of the talk was the construction, for each such choice of A, of a Gestalt $(X/G)_{\mathrm{dR},A}$ which geometrizes the corresponding equivariant cohomology theory, in the sense that one has an isomorphism

$$H^{\bullet}_{\mathrm{dR},A}(X/G) = \mathcal{O}((X/G)_{\mathrm{dR},A}).$$

To approach this construction it turns out to be convenient to restrict attention to stacks of the form X/G where X is a finite type affine scheme and G is reductive. These form a $(2,1)$-category which we denote Glo_k. Once one knows what the de Rham space is for stacks of this form, more general stacks can be accessed by Kan extension.

A basic desideratum for a theory of equivariant de Rham spaces is then that it gives rise to a left exact functor from Glo_k into Gestalten (over some yet to be specified base). Abstracting the target for a moment, it would be desirable to have a general understanding of how to construct interesting left exact functors $\mathrm{Glo}_k \to \mathcal{T}$ where $\mathcal{T}$ is some ∞-topos.

In the non-equivariant setting, where Glo_k is replaced by the category Aff_k of finite type affine schemes, one has the following wonderful observation: left exact functors $\mathrm{Aff}_k \to \mathcal{T}$ are in correspondence with k-algebra objects in $\mathcal{T}$, with the correspondence sending a left exact functor F to $F(\mathbb{A}^1)$. As it turns out, this observation can be extended to the equivariant setting, with the relevant algebraic structure being given by the following:

Definition. A 2-algebra is a k-linear, additive, idempotent complete, rigid symmetric monoidal $(1,1)$-category.

For any object X/G of Glo_k the category $\mathrm{Vec}(X/G)$ of vector bundles is an example of a 2-algebra. As we vary X/G, the assignment $X/G \mapsto \mathrm{Vec}(X/G)$ gives rise to a 2-algebra object Vec internal to the presheaf ∞-topos $\mathcal{P}(\mathrm{Glo}_k)$. Consequently, given a left exact functor $F : \mathrm{Glo}_k \to \mathcal{T}$, one may obtain a 2-algebra object in $\mathcal{T}$ by evaluating the left Kan extension of F on Vec. As it turns out, this assignment provides an embedding

$$\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Glo}_k, \mathcal{T}) \hookrightarrow 2\mathrm{Alg}(\mathcal{T}).$$

Unlike the non-equivariant setting, the above embedding is not an equivalence in general (though its image may be given a concrete description).

The above considerations replace the issue of constructing interesting left exact functors to that of constructing interesting 2-algebras. The simplest class of 2-algebras one can conceive are the ones that we call *diagonalizable*: these are those 2-algebras which are Zariski local, generated by invertible objects of dimension 1, and with the feature that every morphism between invertible objects is either zero or an endomorphism. The data of a diagonalizable 2-algebra $\mathcal{R}$ turns out to be equivalent to the following pieces of information:

- An algebra R (the underlying algebra to $\mathcal{R}$).
- An abelian 2-group P (the Picard group of $\mathcal{R}$).
- A monomorphism $\mathrm{B}R^\times \hookrightarrow P$ (the orientation data).

The role played by oriented abelian groups in this story parallels the role they play in the construction of equivariant cohomology theories [5, 6]: traditionally, one works not with Glo_k but with a topological version thereof, and then an oriented group is precisely the input needed to construct an equivariant theory.

For our desired application, we let $R = \mathbb{A}^1_{\mathrm{dR}}$, the algebra encoding non-equivariant de Rham spaces. One is then looking for an enlargement P of $\mathrm{B}\mathbb{G}_{m,\mathrm{dR}}$ with the feature that $\mathcal{O}(P) = H^\bullet_{\mathrm{dR},A}(\mathrm{B}\mathbb{G}_m) = \mathcal{O}(A)$, where we work implicitly over the 2-periodic base $k(\beta)$ with β a variable of homological degree -2 (this 2-peridicity can be avoided in the case $A = \mathbb{G}_a$, but it is necessary in general). We observe that $\mathcal{O}(\mathrm{B}\mathbb{G}_{m,\mathrm{dR}})$ may be identified with functions on the formal completion $A^\wedge$ of A at the unit. This leads to the following:

Ansatz 1. The 2-group P should sit in an exact sequence

$$\mathrm{B}\mathbb{G}_{m,\mathrm{dR}} \hookrightarrow P \to A/A^\wedge = A_{\mathrm{dR}}$$

In order to find a 2-group P with the desired property, it is helpful to rewrite $\mathrm{B}\mathbb{G}_{m,\mathrm{dR}}$ as the fiber of the canonical map $\mathrm{B}^2\mathbb{G}_m^\wedge \to \mathrm{B}^2\mathbb{G}_m$. Now the fundamental observation is the following: when working 2-periodically one has an isomorphism between $\mathrm{B}^2\mathbb{G}_m^\wedge$ and the Gestalt of $2\mathrm{IndCoh}(A^\wedge)$. Here 2IndCoh refers to the categorification, in the sense of [2], of the six-functor formalism IndCoh, where we implicitly work over the base $3\mathrm{IndCoh}(\mathrm{Spec}(k))$. This observation may be thought of as a variant of the (somewhat more classical) fact that one has an isomorphism $\mathrm{B}^2\mathbb{G}_a = \mathbb{G}_a^\wedge$ when working 2-periodically in the descendable topology.

The above ideas lead one to the following improved version of Ansatz 1:

Ansatz 2. The 2-group P should arise as the fiber of a character

$$\chi : A^! \to \mathrm{B}^2\mathbb{G}_m$$

where $A^!$ denotes the Gestalt of $2\mathrm{IndCoh}(A)$.

There is a universal base where a character as in the above Ansatz exists, namely the 2-shifted Cartier dual to $A^!$. We call this base $k[\beta]^!_A$; it turns out to be 2-affine relative to $3\mathrm{IndCoh}(\mathrm{Spec}(k))$, and its 0-affinization is the algebra $k[\beta]$. Inverting β gives rise to a 2-periodic base $k(\beta)^!_A$. One may then define χ to be canonical map

$$\chi : A^! \to \mathrm{B}^2\mathbb{G}_m$$

defined over $k(\beta)^!_A$. The fiber of χ supplies an oriented abelian 2-group P which gives rise to a diagonalizable 2-algebra in Gestalten over $k(\beta)^!_A$ encoding the desired theory of equivariant de Rham spaces.

Remarkably, while the 2-group P described above satisfies Ansatz 1 and is therefore an extension between two rather classical objects, in order to construct this extension we needed the object $A^!$, which forced us outside of classical geometry, into the world of Gestalten.

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Three-dimensional mirror symmetry

David Nadler
(joint work with David Ben-Zvi, Germán Stefanich)

This talk presented work in progress, joint with David Ben-Zvi and Germán Stefanich, applying new notions of genuine equivariance in algebraic geometry to three-dimensional gauge theory. (See [1] for an overview of the project.)

Throughout the discussion, we fixed k an algebraically closed field of characteristic zero, which in practice we took to be the complex numbers $\mathbb{C}$. All constructions are over k without further comment, and by a point we mean $pt = \mathrm{Spec}(k)$.

In a prior talk at the workshop [10], Stefanich introduced the main character of our story, an enhancement in the setting of Gestalten [9] of the de Rham stack of a group quotient. Let X be a scheme and G a reductive group. Then we have the quotient stack X/G and further its de Rham stack $(X/G)_{\mathrm{dR}} = X_{\mathrm{dR}}/G_{\mathrm{dR}}$. Given

additionally an abelian group A, Stefanich defined a Gestalt $(X/G)_{\mathrm{dR},A}$ which we will refer to here as the A-expanded de Rham stack. Crucially, the construction requires we work within the Getsalt given by $3\mathrm{IndCoh}(pt)$, and over the two-periodic base Gestalt $k(u)^!_A$ given by $2\mathrm{IndCoh}(A)/2\mathrm{QCoh}(A)$. All further constructions are over $k(u)^!_A$ without further comment. (See [11] for the construction of nIndCoh and other notions of categorical sheaf theory.)

After recalling this, we focused on the case where X is a point, so the A-expanded de Rham classifying stack $(BG)_{\mathrm{dR},A}$. We began by stating a Langlands duality conjecture that serves as a lodestar for much of our efforts.

Conjecture. Set $A = \mathbb{G}_a/\mathbb{Z}$. Then there is an equivalence of 2-categories

$$2\mathrm{Shv}((BG)_{\mathrm{dR},A}) \simeq 2\mathrm{Shv}((B\check{G})_{\mathrm{dR},A})$$

where $\check{G}$ is the Langlands dual of G.

From here, the talk focused on the study of $(BG)_{\mathrm{dR},A}$ in the "linear" case where $A = \mathbb{G}_a$. To motivate its study, we first explained how $(BG)_{\mathrm{dR},A}$ provides a realization in the realm of algebraic geometry of pure gauge theory in three dimensions. To approach this, we first recalled fully extended topological field theory as formalized by the Cobordism Hypothesis [6]. We then explained how $(n+1)$-dimensional gauge theory is a powerful tool for studying n-dimensional field theories with symmetry. We detailed how 2-dimensional gauge theory of a finite group Γ organizes the representation theory of Γ. Specifically, we explained how the 2-dimensional B-model of coherent sheaves on the spectrum of class functions provides a mirror description of the representation theory. Then following Teleman [12], we outlined a parallel story of how 3-dimensional gauge theory of a compact group G_c organizes the categorical representation theory of G_c. Specifically, we explained how Rozansky-Witten theory [8, 4, 5] with target the Coulomb branch [3], as realized by the group-scheme of regular centralizers $J_{\check{G}} \to \mathfrak{c}_{\mathfrak{g}}$ over the Chevalley base [2], provides a mirror description.

Now returning to $(BG)_{\mathrm{dR},A}$, so the realm of algebraic geometry and de Rham stacks rather than homotopy theory and topological stacks, we aimed to state a similar mirror description, at least for the "semisimplification" of $(BG)_{\mathrm{dR},A}$ as introduced below. In comparison with the above topological theory, the de Rham theory is less "affine," and the relevant mirror target for its semisimplification is not an affine scheme but rather a variant of the group-scheme of regular centralizers

$$J^{\nabla}_{\check{G}} := \mathrm{Hom}_{\mathfrak{c}_{\mathfrak{g}}}((\mathbb{G}_m)_{\mathrm{dR}}, BJ_{\check{G}}) \to \mathfrak{c}_{\mathfrak{g}}$$

Note we have $J_{\check{G}} \simeq \mathrm{Hom}_{\mathfrak{c}_{\mathfrak{g}}}(B\mathbb{Z}, BJ_{\check{G}})$, and $J^{\nabla}_{\check{G}} := \mathrm{Hom}_{\mathfrak{c}_{\mathfrak{g}}}((\mathbb{G}_m)_{\mathrm{dR}}, BJ_{\check{G}})$ results from replacing the topological circle $S^1 \simeq B\mathbb{Z}$ with the "de Rham circle" $(\mathbb{G}_m)_{\mathrm{dR}}$.

Now some bad news: unlike the original group-scheme $J_{\check{G}} \to \mathfrak{c}_{\mathfrak{g}}$, its de Rham variant $J^{\nabla}_{\check{G}} \to \mathfrak{c}_{\mathfrak{g}}$ is not affine. But in contrast, some good news: assuming $\check{G}$ has connected center (equivalently, the derived group of G is simply-connected), the symplectic stack $J^{\nabla}_{\check{G}}$ admits a presentation

$$J^{\nabla}_{\check{G}} = T^*(\mathfrak{c}_{\mathfrak{g}})/\mathcal{L}$$

as a quotient of the cotangent bundle of the base by the coweight lattice

$$\mathcal{L} = \mathrm{Hom}_{\mathfrak{c}_{\mathfrak{g}}}(G_m, J_{\check{G}}) \to \mathfrak{c}_{\mathfrak{g}}$$

This presentation provides access to the Rozansky-Witten theory of $J_{\check{G}}^{\nabla}$ via higher sheaf theory, as formalized by the following Ansatz.

Ansatz. Assume that $\check{G}$ has connected center. Then Rozansky-Witten theory of $J_{\check{G}}^{\nabla}$ is given by the 2-category of $\mathcal{L}$-invariants $2\mathrm{IndCoh}(\mathfrak{c}_{\mathfrak{g}})^{\mathcal{L}}$.

Here we took a short detour and discussed how we expect the Ansatz to be a special case of a broader approach to Rozansky-Witten theory via microlocal higher sheaf theory.

Then we returned to our goal of a mirror description of the semisimplification of $(BG)_{\mathrm{dR},A}$. Namely, we introduced the enhanced Chevalley base $\mathfrak{c}_{\mathfrak{g}}^!$ as the Gestalt whose 2-sheaves are $2\mathrm{IndCoh}(\mathfrak{c}_{\mathfrak{g}})$. We have the evident map of Gestalten $\mathfrak{c}_{\mathfrak{g}}^! \to \mathfrak{c}_{\mathfrak{g}}$ induced by the map of 2-sheaves $2\mathrm{IndCoh}(\mathfrak{c}_{\mathfrak{g}}) \leftarrow 2\mathrm{QCoh}(\mathfrak{c}_{\mathfrak{g}})$. We then stated the following observation (a natural generalization holds for any A where the linear Chevalley base $\mathfrak{c}_{\mathfrak{g}}$ relevant to the case $A = \mathbb{G}_a$ is replaced by a general A-variant).

Lemma. For $A = \mathbb{G}_a$, equivariant cohomology induces a natural map

$$\chi : (BG)_{\mathrm{dR},A} \to \mathfrak{c}_{\mathfrak{g}}^!$$

Now we defined the semisimplification

$$\chi^{ss} : (BG)_{\mathrm{dR},A}^{ss} \to \mathfrak{c}_{\mathfrak{g}}^!$$

to be the 1-affinization of the map χ of the lemma. And finally we arrived at the main result of the talk.

Theorem. Set $A = \mathbb{G}_a$. Assume that the derived group of G is simply-connected, or equivalently that $\check{G}$ has connected center.

Then there is a canonical mirror equivalence of 2-categories

$$2\mathrm{Shv}((BG)_{\mathrm{dR},A}^{ss}) \simeq 2\mathrm{Shv}(\mathfrak{c}_{\mathfrak{g}})^{\mathcal{L}}$$

relative to the enhanced Chevalley base $\mathfrak{c}_{\mathfrak{g}}^!$.

A sketch-proof of the theorem: One can identify $2\mathrm{Shv}((BG)_{\mathrm{dR},A}^{ss})$ as objects in $2\mathrm{Shv}(\mathfrak{c}_{\mathfrak{g}}^!)$ equivariant for the action of a relative Picard group $\mathcal{P}$, and then establish an isomorphism $\mathcal{P} \simeq \mathcal{L}$ compatible with their actions.

At the conclusion of the talk, in the ensuing discussion, we discussed the following question. Although $\mathbb{G}_a$ is not reductive, one can still apply the prior constructions to obtain $(B\mathbb{G}_a)_{\mathrm{dR},A}$. Here by construction, the Chevalley base is a point pt, and the map

$$\chi^{ss} : (B\mathbb{G}_a)_{\mathrm{dR},A}^{ss} \to pt$$

is in fact an equivalence. More generally, one can observe that the passage to the semisimplification $(BG)_{\mathrm{dR},A} \to (BG)_{\mathrm{dR},A}^{ss}$ collapses unipotent directions within G. This phenomenon motivates the following question.

Question. What is the relation between the semisimplification $(BG)^{ss}_{\mathrm{dR},A}$ and constructions of $\mathbb{A}^1$-homotopy theory (in the sense of [7])?

We also noted that for a torus T, the semisimplification $(BT)_{\mathrm{dR},A} \to (BT)^{ss}_{\mathrm{dR},A}$ is an equivalence, so is also suggestive of constructions of $\mathbb{A}^1$-homotopy theory.

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Toposes in Gestalten

Ko Aoki
(joint work with Robert Burklund)

This report summarizes my research talk on ongoing joint work with Robert Burklund. Our goal is to construct a functor from toposes to Scholze–Stefanich Gestalten [3] and to identify its left adjoint. It extends the sheaves–spectrum adjunction from my earlier work [1].

I began by recalling the 0-localic case. A *frame* is a poset F with finite meets and arbitrary joins such that finite meets distribute over arbitrary joins:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

Frames form the category Frm. The category Loc of locales is its opposite. If F is a frame, the category $\mathrm{Shv}(F)$ of sheaves on the associated locale is presentably symmetric monoidal under the cartesian product. The sheaves–spectrum adjunction [1] says that

$$\mathrm{Shv} \colon \mathsf{Frm} \longrightarrow \mathrm{CAlg}(\mathsf{Pr})$$

has a right adjoint, given by the frame of idempotent coalgebras, or equivalently smashing colocalizations. The stable case, where idempotent coalgebras are equivalent to idempotent algebras, is classical.

Note that Shv induces an equivalence between Loc and the full subcategory of Top spanned by 0-localic toposes. Hence, in geometric language, what we have seen is that the natural functor

$$\mathsf{Top}_{0\text{-loc}} \longrightarrow \mathsf{Gest}_{1\text{-aff}}$$

admits a left adjoint.

Our aim is to remove both truncations. Before proceeding, note the following size issue. Strictly speaking, the Scholze–Stefanich framework fixes the cardinal $\aleph_1$ on the Gestalten side. We therefore need to work with the non-full category of $\aleph_1$-compactly generated toposes on the left-hand side for the functor to be well defined. (Any uncountable regular cardinal would work as well.) This will be implicit in the following discussion.

Let $\mathcal{T}$ be a topos. For every $\aleph_1$-compact object $T \in \mathcal{T}$, consider the slice topos $\mathcal{T}_{/T}$. As T varies, the assignment

$$T \longmapsto \mathcal{T}_{/T}$$

forms a six-functor formalism in which every morphism is étale. Applying higher transmutation to this formalism produces a Gestalt $R(\mathcal{T})$ and hence a functor

$$R\colon \mathsf{Top} \longrightarrow \mathsf{Gest}.$$

At the 2-categorical level, $\mathcal{O}_2(R(\mathcal{T}))$ recovers the category of presentable categories internal to $\mathcal{T}$ studied by Martini–Wolf [2].

The main result presented in the talk was the following.

Theorem (A.-Burklund)**.** *The functor $R\colon \mathsf{Top} \to \mathsf{Gest}$ admits a left adjoint L. For a Gestalt X, the value of L is*

$$L(X) = \mathrm{Ind}_{\aleph_1}(\mathsf{Gest}_{/X}^{0\text{-ét}}),$$

which is a topos by the Scholze–Stefanich theorem.

The construction is also compatible with the localic and affine filtrations. An n-localic topos gives an $(n+1)$-affine Gestalt, leading to truncated functors

$$\mathsf{Top}_{n\text{-loc}} \longrightarrow \mathsf{Gest}_{(n+1)\text{-aff}}.$$

These finite stages connect the frame-theoretic description at height zero with internal higher category theory at the subsequent heights.

One motivation for the adjunction is the expected full faithfulness of R. Equivalently, one would like the counit $L(R(\mathcal{T})) \to \mathcal{T}$ to be an equivalence for every topos. Our calculations establish partial results in this direction, but a proof that R is fully faithful remains work in progress.

Remark. *After the talk, I learned that Marco Volpe and Sebastian Wolf were also studying the same construction. Their presentation of the right adjoint R describes its nth stage in terms of presentable n-categories internal to $\mathcal{T}$.*

References

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