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Calculus of Variations

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ABSTRACT. The Calculus of Variations is both a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and new applications. It is also a subject with a very wide scope, including geometric variational problems, optimal transportation, geometric inequalities and domain optimization, elliptic regularity, geometric measure theory, harmonic analysis, and free boundary problems. The workshop balanced the traditional interests of past conferences with emerging perspectives on weak convergence, concentration, rigidity and geometric regularity. The programme showed how ideas from one area can lead to progress in another, and how the study of physical models continues to inspire new mathematical questions.

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Introduction by the Organizers

The workshop *Calculus of Variations*, organized by Lia Bronsard (Hamilton), Filip Rindler (Coventry), Luca Spolaor (San Diego) and László Székelyhidi (Leipzig), took place from 30 August to 4 September 2026. The workshop was very well attended, with 48 participants, including 13 female researchers and 27 junior scientists. The programme included both theoretical and more applied problems. There were 31 talks in total, each lasting roughly 35 minutes, and most of the talks were given on the blackboard.

Multiple talks were focused on variational models arising in material science. In this context, such models describe for example phase transitions or defects in specific materials. Antonin Monteil's talk focused on Ginzburg–Landau energies with general compact vacuum manifolds. Luana Jost presented new energy growth bounds for entire minimizers of the Ginzburg–Landau energy. Dominik Stantejsky described an asymptotic expansion for harmonic-map energies with small inclusions. This expansion identifies Coulomb-like interactions between particles in nematic liquid crystals. Federico Luigi Dipasquale discussed two-dimensional ferromagnets and obtained a quantization result of the interface energy in the strong coupling regime.

Singular limits and geometric estimates were also used to study problems in elasticity and phase transitions. Cy Maor presented lower bounds for the bending energy of non-Euclidean plates which detect curvature concentration and account for curvature encoded by holes. Roberta Marziani presented a measure-valued Γ -limit for wrinkling in the Lamé problem, which describes the distribution of wrinkle frequencies beyond what is determined by energy scaling laws. Pascal Steinke discussed a second-order Γ -limit for the Cahn–Hilliard functional with Dirichlet data and subquadratic wells.

Another group of talks concerned shape optimization, including problems in which surface energies compete with additional variational terms. Matilde Gianocca presented a joint work with O. Chodosh which resolved Gamow's liquid-drop conjecture in dimension three. Balls are the unique minimizers up to a critical volume, while above this volume minimizers do not exist. Marc Pegon showed that screened interactions can produce nonball minimizers, including nonradial examples for suitable truncated Coulomb kernels. Ihsan Topaloglu studied stability and local minimality of balls for attractive–repulsive energies. Kenneth DeMason discussed partial regularity, nonexistence examples and open existence questions for weighted fractional clusters. David Ziener presented results on planar shape optimization with a specific focus on logarithmic capacity and torsional rigidity. He discussed existence and regularity of optimal convex domains, as well as a new shape-differentiability theorem for the logarithmic capacity.

The behaviour of nonlinear quantities under weak convergence is closely related to questions of convexity and rigidity. Gabriele Cassese presented counterexamples to the homogeneous Morrey problem, including a rank-one convex homogeneous quartic polynomial on 2×3 matrices which is nowhere quasiconvex. His construction makes use of a connection between rank-one convexity and martingales. Antonio De Rosa described constructions of currents with prescribed Gaussian images. These give a characterization of polyconvexity through quasiconvexity for multivalued maps of every multiplicity. Bernd Kirchheim discussed convexity and uniqueness in vectorial variational problems, including examples of nonuniqueness from nonlinear elasticity.

Differential inclusions and rigidity estimates were used to study several nonlinear PDEs and variational problems. Malte Borken characterized matrix submanifolds admitting Friesecke–James–Müller-type rigidity estimates. Cristiana De

Filippis used a duality with Born–Infeld-type energies to construct non-Lipschitz minimizers of autonomous strictly convex integrals. Riccardo Tione presented joint work with X. Lamy on regularity of Lipschitz solutions to planar degenerate elliptic equations, using in a clever way the theory of differential inclusions and hyperbolic methods in his proof. In fluid mechanics, Martina Zizza presented joint work with S. Schiffer on the Arnold variational formulation of incompressible flows. Their discrete pipe-based constructions yield a sharp linear form of Shnirelman’s inequality in dimensions at least three.

Other talks concerned the structure of measures and concentration under PDE constraints. Andrea Merlo presented a proof of the David–Semmes WALA conjecture. This characterizes uniform rectifiability of Ahlfors–David regular measures through an approximation of Lipschitz functions. Luca Gennaioli presented joint work with F. Rindler on concentration phenomena and the Vanishing Mass Conjecture. He described concentrations under first-order PDE constraints and some consequences for equiintegrability and lower semicontinuity. Paolo Bonicatto discussed joint work with G. Alberti and G. Del Nin on the strong locality of differential operators, relating this property to the algebraic structure of such operators. Xavier Lamy presented joint work with A. Guerra and K. Zemas on quantitative profile decompositions for almost-critical points, with sharp estimates for bubbling and energy quantization in critical Sobolev problems.

For geometric variational problems, a main question is how stationarity or minimality restricts the singular set. Anna Skorobogatova presented joint work with A. Bhattacharya and G. Orriols on Lipschitz Hamiltonian stationary Lagrangian graphs. The dimension of the singular set is at most $n - 5$, and new singular special Lagrangian graphs show that this bound is sharp. Jonas Hirsch presented joint work with L. Spolaor proving regularity almost everywhere for stationary integral cycles of codimension one with density below three. Aria Halavati discussed anisotropic minimization and min–max constructions, developed with A. De Rosa and L. Wang. New gluing and removable-singularity arguments give full regularity under some additional assumptions on the anisotropy. Michael Novack presented joint work with R. Resende proving the existence of flat blow-ups at smooth boundaries for anisotropic minimizing hypercurrents.

For free boundary problems, Guido De Philippis presented joint work with J. Hirsch and M. Nahon establishing optimal regularity of minimizers for vectorial and higher-order Bernoulli-type problems. Bozhidar Velichkov showed that branching points of planar two-phase Bernoulli minimizers are isolated. The proof uses a connection with thin two-membrane problems for the area functional. Salvatore Stuvard presented an ε -regularity theorem for Brakke flows near triple junctions in arbitrary dimension and codimension. Michael Struwe discussed the problem of prescribing Gauss curvature on the disc and geodesic curvature on its boundary. His approach uses a gradient flow and a concentration analysis to obtain existence results.

Workshop: Calculus of Variations

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Abstracts

On the WALA conjecture

ANDREA MERLO

Rademacher's theorem asserts that every Lipschitz function on Euclidean space is differentiable almost everywhere with respect to Lebesgue measure. Dorronsoro's theorem provides a quantitative counterpart, controlling affine approximation errors over locations and scales. The WALA conjecture of Guy David and Stephen Semmes asks whether a weak form of this approximation property characterizes uniform rectifiability among AD regular measures. In this talk, I present a proof of the conjecture and explain the geometric constructions underlying it.

Let μ be a k -AD regular Radon measure on $\mathbb{R}^n$. For a real-valued Lipschitz function f , consider the normalized affine approximation error

$$\Omega_\mu(f; x, r) = \inf_{A \text{ affine}} \frac{1}{r\mu(B(x, r))} \int_{B(x, r)} |f(y) - A(y)| d\mu(y).$$

The weak approximation of Lipschitz functions by affine functions condition, abbreviated WALA, requires that, for every $\varepsilon > 0$ and every 1-Lipschitz function f , the pairs (x, r) where $\Omega_\mu(f; x, r) > \varepsilon$ satisfy a Carleson packing estimate. More precisely,

$$\int_{B(x_0, R)} \int_0^R \mathbf{1}_{\{\Omega_\mu(f; x, r) > \varepsilon\}} \frac{dr}{r} d\mu(x) \leq C(\varepsilon) \mu(B(x_0, R)),$$

uniformly in f , x_0 and R in the admissible range. David and Semmes proved that uniformly rectifiable measures satisfy this condition and established the converse in dimension one. The result presented here proves the converse in every dimension and codimension. Thus weak affine approximation of all Lipschitz functions forces the support to contain, in every ball, a quantitatively large piece of a Lipschitz image of a subset of a ball.

The proof develops a quantitative counterpart of the decomposability bundle of measures introduced by Alberti and Marchese. The decomposability bundle consists of those directions generated by decompositions of a measure into rectifiable curves and controls the directional differentiability of Lipschitz functions. In a fixed ball $B(x, t)$, we instead detect invariant directions, i.e. those directions e for which for every point in $B(x, r)$ passes a curve with derivatives in a small cone around e which is extremely close to the support of μ inside the ball $B(x, r)$. Independent directions of this kind yield an approximate product structure. When their number is smaller than the dimension of the measure, and AD regularity ensures that in the orthogonal directions to the invariant ones, the set behaves like an AD-regular set in those directions.

The main difficulty is to turn this geometry into an obstruction to affine approximation on a family of balls with large total packing. Width functions, originating in the work of Alberti, Csörnyei and Preiss, provide the means to cancel transverse derivatives while controlling the Lipschitz constant. After this cancellation,

configurations of points transverse to invariant directions allow us to prescribe values that no affine function can approximate simultaneously. The approximate product structure allow us to force this impossibility of approximation of Lipschitz functions over sets carrying a definite amount of measure, giving a lower bound for the integral approximation error.

These modifications are iterated over separated scales. If the cubes with insufficiently many invariant directions do not satisfy a Carleson packing estimate, the construction would produce a single function, with uniformly bounded Lipschitz constant, whose approximation errors violate WALA. Consequently, all such families satisfy Carleson packing estimates. The remaining cubes carry enough independent directions to obtain the geometric approximation criterion for uniform rectifiability.

The argument also applies when affine functions are replaced by a fixed finite-dimensional space of locally integrable functions containing them.

I also briefly discuss applications to elliptic PDEs arising from joint work with Mihalis Mourgoglou. These concern the Regularity problem for uniformly elliptic divergence-form operators with bounded measurable coefficients, and show how estimates for gradients of solutions yield rectifiability and uniform rectifiability criteria for the boundary.

Sharp partial regularity of Hamiltonian stationary and special Lagrangian graphs

ANNA SKOROBOGATOVA

(joint work with Arunima Bhattacharya, Gerard Orriols)

Consider a Lagrangian submanifold L^n in a smooth almost Kähler manifold (M^{2n}, J, g, ω) , where $J^2 = -\text{Id}$, ω is the associated nondegenerate, closed symplectic 2-form, and $g = \omega(\cdot, J\cdot)$. Such a manifold comes with a canonical smooth volume $(n, 0)$ -form Ω . The Lagrangian condition means that L is a half-codimension submanifold for which ω satisfies $\omega|_{TL} \equiv 0$, namely, the J -rotation of any tangent plane to L lies entirely in the normal bundle of L . The model situation to bear in mind is the case $(\mathbb{C}^n, J_0, g_0, \omega_0)$, where J_0 is the canonical complex structure, ω_0 is the Kähler form, g_0 is the canonical compatible metric, and $\Omega_0 = dz_1 \wedge \cdots \wedge dz_n$.

Here, we focus on the graphical setting, in which case the Lagrangian condition amounts to requiring that $L = \text{graph}(Du)$ for a suitable scalar potential function u (defined locally through a Darboux chart in a general almost Kähler manifold). A natural quantity characterizing the geometric structure of a Lagrangian submanifold is the *Lagrangian angle* (or *phase*), defined on the normal bundle of L via $\Omega|_{T_p L} = e^{i\beta(p, T_p L)} \text{vol}_L$. When L has a gradient graph structure and the ambient space is $\mathbb{C}^n$, we obtain

$$(1) \quad \beta = \sum_{j=1}^n \arctan \lambda_j(D^2 u),$$

which yields the Lagrangian mean curvature equation for the potential when one prescribes some variable function $\beta(x)$ with values in $(-n\pi/2, n\pi/2)$ as the phase. In general, we merely get a more general fully nonlinear second order equation $\beta = \mathcal{F}(x, Du, D^2u)$.

Further constraints on β yield surfaces with certain geometric properties, that have been the subject of interest of many for decades. The most well-studied instance of this is the case when β is constant (the *special Lagrangian equation*), which corresponds to Lagrangian submanifold being calibrated and thus area-minimizing; such submanifolds are *special Lagrangians*. A natural fourth order variant of this are *Hamiltonian stationary submanifolds*, which are critical points of the area functional with respect to Hamiltonian variations through Lagrangian submanifolds. In the graphical setting, these are governed by the PDE $\Delta_g \beta = 0$ in ambient $\mathbb{C}^n$, where $g = \text{Id} + (D^2u)^t(D^2u)$ is the induced metric on the submanifold. In general, one instead obtains a more general divergence-form equation $\text{div}_g(\nabla_g \beta - \vec{b}) = 0$ in a weak sense. Together with (1), we thus obtain a fourth order equation that decouples into two second order ones, for the potential u associated to graphical Hamiltonian stationary submanifolds; this is the *Hamiltonian stationary equation*. Here, β is always assumed to be in $W_{\text{loc}}^{1,2}$ a priori, but elliptic regularity theory then allows us to deduce that it is in fact Hölder continuous, whenever the submanifold is Lipschitz.

We obtain a general sharp partial regularity result for such submanifolds:

Theorem 1. *Suppose that $\mathcal{U} \subset \mathbb{R}^n$ is a simply connected domain, and that $u \in C^{1,1}(\mathcal{U})$ is a solution to the Hamiltonian stationary equation. Then*

- (i) $u \in C^\infty(\mathcal{U} \setminus \Sigma_u)$ for a relatively closed set $\Sigma_u \subset \mathcal{U}$ with $\dim_{\mathcal{H}} \Sigma_u \leq n - 5$.
- (ii) This dimension bound is sharp: there exists a 2-homogeneous function $u \in C^{1,1} \setminus C^2$ that solves the special Lagrangian equation in $\mathbb{C}^n$ for every $n \geq 5$.

The conclusion (ii) can be equivalently formulated as the existence of a non-planar, Lipschitz graphical special Lagrangian cone in $\mathbb{C}^n$ for every $n \geq 5$. Thanks to the Hölder regularity of β that we are able to obtain in light of the Hessian bound on u , we can in turn establish an almost-monotonicity formula for area ratios for the submanifold in our setting. This in turn allows us to say that blow-ups are Lipschitz graphical *special Lagrangian cones*. When $n \leq 4$, it is known that any such cone is planar; this is the work of Yuan [11] when $n \leq 3$ and Nadirashvili-Vladut [8] when $n = 4$. On the other hand, our example provides the first instance of a singular Lipschitz graphical special Lagrangian cone, thus showing that they do indeed exist starting from dimension 5. Note that in the non-graphical setting, many singularity models have already been discovered, such as Schoen-Wolfson cones [10]. Moreover, in the non-graphical setting, there is in general no monotonicity formula for area ratios, even in the presence of a Legendrian lift, see [9].

On the other hand, ε -regularity around quadratic polynomials, using Fan's inhomogeneous version of Savin's "small perturbation" result [5], yields full regularity around all points with a quadratic polynomial blowup for the potential (i.e. a

planar blowup for its gradient graph). This allows for a dimension reduction approach à la Federer in our Lipschitz graphical class, to conclude part (i) of the theorem.

Remark 2.

- (1) When $n \leq 4$, full regularity (in ambient $\mathbb{C}^n$) was obtained by Bhattacharya-Warren [2] when $n = 2$ and Bhattacharya [1] when $n = 3, 4$. It is worth noting that at that point, neither the almost-monotonicity formula nor Fan's version of Savin's small perturbation result was available, so the blow-up procedure and ε -regularity required more work.
- (2) Partial regularity (with an $(n - 5)$ -dimensional singular set) for Lipschitz special Lagrangian graphs in $\mathbb{C}^n$ was established by Dimler [10], but no sharpness was known at that point.
- (3) One may compare Theorem 1 to the analogous sharp $(n - 4)$ -dimension bound on the singular set for Lipschitz minimal graphs. There, the Lawson-Osserman cone [6] is the relevant singular example yielding sharpness. In fact, the work of Dimler mentioned in (2) generalizes to Lipschitz minimal Lagrangian graphs, thus implying that the $(n - 4)$ -dimension bound improves to an $(n - 5)$ -dimension bound under such assumptions.
- (4) In general, the regularity theory depends on the phase; large enough phase yields suitable convexity properties. Indeed, the work of Chen-Warren [3] shows that $C^{1,1}$ regularity bootstraps to C^∞ in the supercritical (including critical) phase regime $\beta \geq (n - 2)\pi/2$. In particular, there is no supercritical phase variant of our singular conical example which is phase zero.

The basic idea behind finding the 5-dimensional phase-zero Lipschitz graphical special Lagrangian cone is as follows. Inspired by [7], we consider the ansatz $u = r^2 f$ that extends a real analytic scalar function f on $\mathbb{S}^4$, whose level sets agree with those of the link of the *Cartan cubic*

$$P(x) = x_1^3 + \frac{3x_1}{2}(x_3^2 + x_4^2 - 2x_5^2 - 2x_2^2) + \frac{3\sqrt{3}}{2}(x_2x_3^2 - x_2x_4^2 + 2x_3x_4x_5).$$

The latter is a degree 3 homogeneous polynomial whose zero level set is an isoparametric minimal cone, and whose level sets in $\mathbb{S}^4$ form a foliation by CMC surfaces. Note that the focal submanifolds $\{P = \pm 1\} \cap \mathbb{S}^4$ are Veronese surfaces.

Thanks to the $\mathrm{SO}(3)$ -invariance of this foliation, we can parameterize the level sets via the function $s(\theta) = \cos(3\theta)$, over the arc $\{r = 1, 0 \leq \theta \leq \frac{\pi}{3}\}$ in the $x_1 - x_2$ -plane which meets every $\mathrm{SO}(3)$ -orbit exactly once. We are thus reduced to seeking a function $f(s(\cdot))$ for $f \in C^\omega([-1, 1])$, such that the resulting 2-homogeneous potential $u = r^2 f$ solves the zero-phase special Lagrangian equation $\mathrm{Im} \det(\mathrm{Id} + iD^2u) = 0$. With this reduction, the equation reduces to an ODE

$$f''(s) = \frac{9sf'(s) - \frac{N(s, f, f')}{D(s, f, f')}}{9(1 - s^2)},$$

with fixed initial conditions $f(1) = A$, $f'(1) = B(A)$ to be determined, which is a priori singular. We begin by establishing local existence and uniqueness in the

analytic class near $s = 1$, for every initial data in a suitable interval. We then use computer-assisted ODE shooting, implemented in Python using the FLINT Arb library, to extend this analytically to a solution over $[0, 1]$, for initial data in a suitable, much smaller, subinterval. An odd extension then yields a solution over $[-1, 1]$, closing the argument. Particular care must be taken to ensure that the relevant uniform estimates on f and f' propagate iteratively in the ODE shooting mechanism, which is performed on short subintervals in the parameter s .

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Minimizing entire solutions of the Ginzburg–Landau equation on the plane and singular harmonic maps into compact manifolds

ANTONIN MONTEIL

(joint work with Rémy Rodiac, Jean Van Schaftingen, Kunlai Wu)

Let $\mathcal{N} \subset \mathbb{R}^n$ be a compact Riemannian manifold and $\Omega \subset \mathbb{R}^2$ a bounded simply connected domain. A basic variational problem is to minimize the Dirichlet energy

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2$$

among maps $u : \Omega \rightarrow \mathcal{N}$ with prescribed boundary data $u = \gamma_0$ on $\partial\Omega$. In dimension two, minimizing harmonic maps are smooth. Consequently, if the boundary

datum γ_0 is not null-homotopic in $\mathcal{N}$, there is no minimizer; in fact, there is no admissible map in $W^{1,2}(\Omega, \mathcal{N})$ attaining this boundary condition.

One way of relaxing this topological obstruction is to replace the constraint $u \in \mathcal{N}$ by a penalization. Let $F : \mathbb{R}^n \rightarrow [0, +\infty)$ vanish exactly on $\mathcal{N}$, and consider

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_\Omega F(u), \quad u = \gamma_0 \quad \text{on } \partial\Omega.$$

For $\varepsilon > 0$, minimizers exist without any topological restriction on the boundary datum γ_0 . As $\varepsilon \rightarrow 0$, minimizers converge, away from a finite set of points $a_1, \dots, a_k$, to a harmonic map $u_* : \Omega \setminus \{a_1, \dots, a_k\} \rightarrow \mathcal{N}$, as proved in [2]. The points a_i correspond to topological defects. This relaxation theory was initiated in the case $\mathcal{N} = \mathbb{S}^1$ and $F(u) = (1 - |u|^2)^2$ in [1], and was developed for general compact vacuum manifolds in [4, 5].

The local structure of u_* near a singularity a_i is described by a closed minimizing geodesic $\gamma_i : \mathbb{S}^1 \rightarrow \mathcal{N}$: its tangent maps at a_i are of the form

$$x \mapsto \gamma_i \left(\frac{x}{|x|} \right).$$

The corresponding defect carries a logarithmic energy cost $E(\gamma_i) \log(1/\varepsilon)$, where

$$E(\gamma) = \frac{1}{2} \int_{\mathbb{S}^1} |\gamma'|^2.$$

Consequently, the leading-order contribution to the energy is determined by the geodesics γ_i .

The relevant topological compatibility condition for the collection of geodesics $\gamma_1, \dots, \gamma_k$ is that there exists a continuous map on a punctured disk whose boundary values on the interior circles are given by $\gamma_1, \dots, \gamma_k$ and whose boundary value on the exterior circle is γ_0 . This naturally leads to the following variational problem:

$$E_{\text{sing}}(\gamma_0) = \min \left\{ \sum_{i=1}^k E(\gamma_i) : \gamma_1, \dots, \gamma_k \text{ are compatible with } \gamma_0 \right\}.$$

Since topological compatibility can be expressed as an algebraic relation in the fundamental group $\pi_1(\mathcal{N})$, which is finitely generated, this problem is essentially combinatorial.

The remaining $O(1)$ contribution to the energy consists of a renormalized energy, depending on the limiting singular harmonic map, and core energies associated with the defects and their profiles γ_i . A systematic construction of the renormalized energy and of the core energy for general compact target manifolds is given in [5], where several examples of vacuum manifolds $\mathcal{N}$ are also analyzed, including an exhaustive description of the possible singularity profiles.

We next turn to the microscopic scale of the Ginzburg–Landau problem. Blowing up a minimizer around a defect at the scale ε leads formally to an entire solution of

$$-\Delta u + \nabla F(u) = 0 \quad \text{in } \mathbb{R}^2.$$

The relevant solutions are minimizing entire solutions, meaning that

$$\int_B \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) \leq \int_B \left(\frac{1}{2} |\nabla v|^2 + F(v) \right)$$

for every ball $B \subset \mathbb{R}^2$ and every v such that $v - u \in H_0^1(B, \mathbb{R}^n)$.

In the classical Ginzburg–Landau case $\mathcal{N} = \mathbb{S}^1 \subset \mathbb{C}$ and $F(u) = \frac{1}{4}(1 - |u|^2)^2$, the situation is much more rigid. The only non trivial minimizing entire solutions have degree ± 1 at infinity, and each is unique up to translations and rotations. This classification is due to P. Mironescu [3] and E. Sandier [6].

This raises the question of what replaces the degree-one vortex for a general compact target manifold. The answer is related to the decomposition properties of closed geodesics in $\mathcal{N}$.

We say that a closed geodesic $\gamma : \mathbb{S}^1 \rightarrow \mathcal{N}$ is *strictly indecomposable* if, whenever it is topologically compatible with two closed curves γ_1, γ_2 , one has

$$E(\gamma) < E(\gamma_1) + E(\gamma_2).$$

The strict inequality means that splitting the defect represented by γ into two topologically non-trivial defects increases the total geodesic energy. This is reminiscent of the dichotomy phenomenon in concentration-compactness. More generally, the geometry and decomposition properties of closed geodesics in the target manifold determine which defects can remain as a single core and which ones split.

Theorem 1. *Let $\gamma : \mathbb{S}^1 \rightarrow \mathcal{N}$ be a strictly indecomposable minimizing closed geodesic. Then there exists a minimizing entire solution $u : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ of*

$$-\Delta u + \nabla F(u) = 0$$

such that $u(R \cdot) \rightarrow \gamma$ locally as $R \rightarrow +\infty$.

Conversely, any minimizing entire solution u converges to $\mathcal{N}$ at infinity and therefore determines a homotopy class at infinity, represented by a closed geodesic γ . This geodesic must be indecomposable, although not necessarily strictly. These properties are obtained using a Pohozaev identity together with Bochner-type arguments.

The intermediate case, where γ is indecomposable but not strictly indecomposable, is more subtle. Here the existence of a minimizing entire solution can depend on the potential F . Indeed, the coupling induced by F between the components of u may create an attractive interaction between the cores of the defects, leading to the existence of a minimizing entire solution, or a repulsive interaction, leading to its non-existence.

These results on minimizing entire solutions are joint work in progress with Kunlai Wu (Université Paris-Est Créteil) and Jean Van Schaftingen (UCLouvain).

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Polyconvexity, quasiconvexity and Gaussian images of multigraphs

ANTONIO DE ROSA

(joint work with Yucong Lei, Robert Young, and Daniele De Gennaro)

Let $\widetilde{\text{Gr}}(d, n)$ denote the oriented Grassmannian and, for $P \in \widetilde{\text{Gr}}(d, n)$, let $\omega_P \in \bigwedge^d \mathbb{R}^n$ be the associated unit simple d -vector. If T is a rectifiable d -current with underlying rectifiable set E , multiplicity θ , and oriented tangent plane map $\tau: E \rightarrow \widetilde{\text{Gr}}(d, n)$, its weighted Gaussian image is

$$\gamma_T := \tau_{\#}(\theta \mathcal{H}^d \llcorner E) \in \mathcal{M}^+(\widetilde{\text{Gr}}(d, n)).$$

Thus γ_T records the distribution of oriented tangent planes. The barycenter of a positive measure $\mu \in \mathcal{M}^+(\widetilde{\text{Gr}}(d, n))$ on the oriented Grassmannian is

$$b(\mu) := \int_{\widetilde{\text{Gr}}(d, n)} \omega_P d\mu(P).$$

We aim to approximate measures whose barycenter is a simple d -vector by Gaussian images of fillings of the boundary of a d -dimensional cube, and to use this approximation to obtain a variational characterization of polyconvexity.

This investigation was motivated by the atomic condition (AC), which we introduced in collaboration with De Philippis and Ghiraldin [3]. We proved that (AC) is the sharp condition ensuring that every varifold with positive lower density which is stationary for a C^1 anisotropic integrand is rectifiable [3]. This rectifiability theorem has found several applications, for instance in the solution of the anisotropic Plateau problem [7] and in anisotropic min-max constructions. The latter direction has proved particularly fruitful, leading to the resolution of Allard’s conjecture on the anisotropic min-max construction via the Almgren-Pitts scheme [4, 5], to the anisotropic min-max construction through phase transitions [12], and to an anisotropic Simon-Smith construction achieving genus control [9]. In codimension one, (AC) is equivalent to strict convexity, in spherical directions, of the one-homogeneous extension of the integrand [3]. In general codimension, (AC) is

much less understood. We proved that it implies Almgren's strict geometric ellipticity [10], while uniform forms of it yield interior and boundary $C^{1,\alpha}$ -regularity for anisotropic Lipschitz graphs [14, 13]. We refer to [8] for an overview on the topic.

Fillings with prescribed Gaussian image. In joint work with Lei and Young [11], we proved the following constructive approximation theorem. Fix $P_0 \in \widetilde{\text{Gr}}(d, n)$, let $D = [0, 1]^d \subset P_0$, and let $C = [0, 1]^n$. If $\mu \in \mathcal{M}^+(\widetilde{\text{Gr}}(d, n))$ satisfies $b(\mu) = 0$, there are polyhedral cycles T_j such that

$$\gamma_{T_j} \xrightarrow{*} \mu, \quad V_{T_j} \rightharpoonup (\mathcal{H}^n \llcorner C) \otimes p_{\#}\mu,$$

where $p : \widetilde{\text{Gr}}(d, n) \rightarrow \text{Gr}(d, n)$ is the projection on the non-oriented Grassmannian that forgets the orientation. If instead $b(\mu) = \omega_{P_0}$, there are polyhedral currents T_j with

$$\partial T_j = \partial[D], \quad \gamma_{T_j} \xrightarrow{*} \mu, \quad V_{T_j} \rightharpoonup (\mathcal{H}^d \llcorner D) \otimes p_{\#}\mu.$$

Moreover, if μ is supported on the planes positively oriented with respect to P_0 , the latter fillings can be constructed as Lipschitz multigraphs over P_0 . This strengthens the nonconstructive filling theorem of Burago and Ivanov [2] by providing an explicit construction and by achieving a multigraphical structure.

One consequence is that (AC) forces strict polyconvexity of the even extension $\tilde{F}$ of an anisotropic integrand F : for every $P_0 \in \widetilde{\text{Gr}}(d, n)$ and every $\mu \in \mathcal{M}^+(\widetilde{\text{Gr}}(d, n))$ satisfying $b(\mu) = \omega_{P_0}$,

$$\int \tilde{F}(P) d\mu(P) \geq \tilde{F}(P_0), \quad \text{with equality if and only if } \mu = \delta_{P_0}.$$

The converse fails in general codimension [15].

The graphical form of the filling theorem gives a characterization in the vectorial calculus of variations. Let $\psi : \mathbb{R}^{(n-d) \times d} \rightarrow [0, \infty)$ be continuous and associate to it, for each $Q \in \mathbb{N}$, the symmetric Q -integrand $\bar{\psi}_Q(X_1, \dots, X_Q) = \sum_{i=1}^Q \psi(X_i)$. Then ψ is polyconvex, i.e. convex as a function of all minors, if and only if $\bar{\psi}_Q$ is quasiconvex for Lipschitz Q -valued maps for every $Q \in \mathbb{N}$. Equivalently, ψ is polyconvex if and only if, for every $Q \in \mathbb{N}$, every $f \in \text{Lip}(D, \mathcal{A}_Q(\mathbb{R}^{n-d}))$ with $f|_{\partial D} = Q\delta_{a+Lx}$ satisfies

$$Q\psi(L) \leq \int_D \sum_{i=1}^Q \psi(Df_i(x)) dx.$$

For $Q = 1$ alone this is false, as shown by the quasiconvex, non-polyconvex examples of Šverák and Alibert-Dacorogna [16, 1].

Why every Q is necessary. In joint work with De Gennaro [6], we proved that the preceding approximation theorem is sharp. For every fixed $Q \geq 1$ there is $\mu_Q \in \mathcal{M}^+(\widetilde{\text{Gr}}(2, 4))$ whose barycenter is parallel to $e_1 \wedge e_2$, but which is not a weak-* limit of weighted Gaussian images of Lipschitz Q -valued maps

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathcal{A}_Q(\mathbb{R}^2), \quad f|_{\partial D} = Q\delta_0.$$

The measure μ_Q is constructed as the sum of three Dirac deltas: two are supported on almost horizontal planes P_1, P_2 (meaning that P_1, P_2 are almost parallel to $e_1 \wedge e_2$), and the third on an almost vertical plane P_3 (meaning that P_3 is almost parallel to, and positively oriented with respect to, $e_4 \wedge e_3$). In this way, the measure carries positive horizontal and vertical masses but no mixed mass. The construction extends to $\widetilde{\text{Gr}}(m, n)$ whenever $2 \leq m \leq n - 2$. Such an obstruction cannot occur in codimension one, where Minkowski's existence theorem gives density already for single-valued graphs.

The main geometric tool in the proof is a quantitative lower bound on the mass carried by genuinely mixed tangent planes in terms of the mass carried by almost vertical ones. After a suitable linear tilt, we write the current canonically associated to the Q -graph of f as $T_f = T_H + T_V + T_M$, according to whether the tangent plane is almost horizontal, almost vertical, or mixed, at a threshold scale ε . We then prove that there is a universal $c > 0$ such that, for $\varepsilon = \varepsilon(Q) > 0$ sufficiently small,

$$(1) \quad \mathbf{M}(T_M) \geq c \mathbf{M}(T_V).$$

The estimate follows from a density-growth argument combined with a Vitali covering argument. Since the tilted measure carries positive horizontal and vertical masses but no mixed mass, by (1) it cannot be approximated by Gaussian images of Q -graphs.

This obstruction also produces, for every $Q \geq 1$ and $p \geq 2$, a non-polyconvex Q -integrand $\mathcal{A}$ whose associated energy is weakly lower semicontinuous in $W^{1,p}$. Starting from a function ψ which vanishes exactly on the three slopes associated to P_1, P_2, P_3 , one sets

$$\mathcal{A}(a, X) := \inf_{\varphi \in \mathcal{T}(a, X; D)} \int_D \sum_{i=1}^Q \psi(D\varphi_i(x)) \, dx,$$

where the competitor class $\mathcal{T}(a, X; D)$ respects the maximal decomposition of the Q -point (a, X) . The integrand $\mathcal{A}$ is in general only lower semicontinuous: when distinct sheets collapse, the class of competitors enlarges. A blow-up argument together with a piecewise affine approximation lemma nevertheless yields weak lower semicontinuity in $W^{1,p}$ of the associated energy.

It remains open whether, for every fixed $Q \geq 1$, there exists a continuous non-polyconvex scalar integrand ψ , such that $\bar{\psi}_Q$ is quasiconvex.

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Min–Max Construction of Anisotropic Minimal Surfaces with Genus Bound

ARIA HALAVATI

(joint work with Antonio De Rosa and Ling Wang)

Let N be a closed Riemannian 3-manifold and let F be an even, uniformly elliptic parametric integrand. The anisotropic area of a smooth embedded surface $\Sigma \subset N$ is

$$(1) \quad \mathbf{F}(\Sigma) = \int_{\Sigma} F(x, T_x \Sigma) d\mathcal{H}^2 = \int_{\Sigma} G(x, \nu_{\Sigma}) d\mathcal{H}^2,$$

where G is the even, one-homogeneous extension of F and the second expression is understood locally when Σ is one-sided. We assume the standard uniform comparability, C^3 regularity, and convexity of G . Critical points of (1) will be called **F**-minimal surfaces. We are interested in the variational theory of **F**-minimal surfaces restricted to a fixed isotopy class. For the area functional this has been established by Meeks–Simon–Yau for minimizers and Simon–Smith for its minmax counterpart.

The goal of [4] is to adapt these theories to anisotropies the main obstruction is the absence of a monotonicity formula: tangent varifolds need not be cones and uniform density bounds are unavailable.

An anisotropic Meeks–Simon–Yau theorem. Let $\mathcal{G}(\Sigma_0)$ be the ambient isotopy class of a compact embedded surface Σ_0 [7], and suppose that

$$I = \inf\{\mathbf{F}(\Sigma) : \Sigma \in \mathcal{G}(\Sigma_0)\} > 0.$$

If $\Sigma_k \in \mathcal{G}(\Sigma_0)$ is any minimizing sequence, then, after passing to a subsequence,

$$(2) \quad \mathbf{v}(\Sigma_k) \longrightarrow V = \sum_{i=1}^R n_i \mathbf{v}(\Gamma^i) \quad \text{as varifolds,}$$

where the Γ^i are pairwise disjoint smooth embedded $\mathbf{F}$ -minimal surfaces and n_i are positive integers. Moreover, if $\mathcal{O}$ and $\mathcal{U}$ denote, respectively, the two-sided and one-sided components, then for all sufficiently large k ,

$$(3) \quad \sum_{i \in \mathcal{O}} n_i \text{genus}(\Gamma^i) + \frac{1}{2} \sum_{i \in \mathcal{U}} n_i (\text{genus}(\Gamma^i) - 1) \leq \text{genus}(\Sigma_k).$$

Every two-sided component in (2) is $\mathbf{F}$ -stable; if the approximating surfaces are two-sided, then all limiting components are stable. Boundary versions strengthen White’s existence theorem [9]: every minimizing sequence of disks with converging boundary has a smooth minimizing-disk limit. These results work for arbitrary regular $\mathbf{F}$. The proof completely avoids tangent-cone analysis. Finitely many neck-pinches reduce the local problem to stacked minimizing disks, for which we use Pitts’ [8] regularity scheme using replacements. We build these replacements using the existence result of White [9] and Hass–Scott [5].

The anisotropic Simon–Smith construction. Among generalized families $\{\Sigma_t\}_{t \in [0,1]}$ of sweepouts in a fixed isotopy class we define the width as

$$(4) \quad m_0(\Lambda) = \inf_{\{\Sigma_t\} \in \Lambda} \max_{t \in [0,1]} \mathbf{F}(\Sigma_t).$$

Using the usual Pull-tight and combinatorial selection we gain an almost minimizing varifold in the sense of Pitts’ for which the preceding isotopy theorem then provides smooth stable replacements. The central new point is to prove that two consecutive replacements glue smoothly across their common interface when the common sheet has arbitrary multiplicity.

The anisotropic constant-mean-curvature device of [2] does not apply here, since minimizers of that functional over isotopies can have only $C^{1,1}$ regularity. We instead induct on the interface multiplicity m . For $m \geq 2$, a third replacement is made between spherical caps meeting at a small angle, and a flux estimate bounds its total multiplicity by m . If splitting occurred, every branch would have smaller multiplicity and would glue by induction. The maximum principle and unique continuation then give a contradiction. Thus the min–max limit is smooth and embedded away from at most one point.

Two removable singularity theorems. To remove the remaining point, we establish two independent removable singularity theorems. They apply more generally to any properly embedded $\mathbf{F}$ -minimal surface in a punctured ball that is stable away

from the puncture. The first assumes the pointwise ellipticity bound

$$(5) \quad \kappa_F(x) := \frac{\max_{\nu} \lambda_{\max} \Psi_F(x, \nu)}{\min_{\nu} \lambda_{\min} \Psi_F(x, \nu)} < 8.$$

Stability initially gives only $|A|(x) \leq C d(x, p)^{-1}$. After blowing up around the puncture, we obtain a stable minimal lamination for the frozen integrand on $\mathbb{R}^3 \setminus \{0\}$. Under (5), a Bernstein theorem forces every leaf to be flat. This improves the curvature decay, controls the topology of small distance level sets, and gives a finite density bound. The puncture then has zero capacity, so stability and the equation extend across it.

The second criterion assumes, in normal coordinates at the puncture, that the extrinsic distance $r = |x|$ is a subsolution for the anisotropic Jacobi operator:

$$(6) \quad \mathcal{J}_F r \geq -C.$$

We provide a pointwise C^3 pinching condition that implies the bound above. If density were unbounded, small annuli would contain increasingly many graphical sheets. The separation $\mathbf{s}$ between adjacent sheets is an approximate positive Jacobi field. Hardy and Moser–De Giorgi estimates for $\mathbf{s}/r$ give double-exponential decay of its sublevel sets. This rules out excessive accumulation and shows that the density grows at most logarithmically. A log–log cutoff then proves zero capacity and removability.

Combining these ingredients gives the following conclusion. Assume either that (5) holds at every point of N , or that (6) is available locally at any possible puncture; Then there are a minimizing sequence $\{\Sigma_t^j\} \subset \Lambda$ and critical parameters t_j such that

$$(7) \quad \mathbf{v}(\Sigma_{t_j}^j) \longrightarrow V = \sum_{i=1}^L n_i \mathbf{v}(\Gamma^i), \quad \mathbf{F}(V) = m_0(\Lambda),$$

where the Γ^i are pairwise disjoint smooth embedded $\mathbf{F}$ -minimal surfaces. Writing $\mathcal{O}$ and $\mathcal{U}$ for the orientable and non-orientable components, respectively, one has the sharp genus estimate

$$(8) \quad \sum_{i \in \mathcal{O}} n_i \text{genus}(\Gamma^i) + \frac{1}{2} \sum_{i \in \mathcal{U}} n_i (\text{genus}(\Gamma^i) - 1) \leq \liminf_{j \rightarrow \infty} \liminf_{\tau \rightarrow t_j} \text{genus}(\Sigma_{\tau}^j).$$

The topological part is an anisotropic optimal-lifting argument followed by Ketover’s proof; (8) is optimal also in the presence of multiplicity and one-sided components. The same removable-singularity results apply to the finitely many possible singularities in multiparameter anisotropic Almgren–Pitts constructions, thus opening the way to the study of higher parameter sweepouts while removing the obstruction in [3] in ambient dimension 3 and towards a Yau conjecture regarding existence of infinitely many anisotropic minimal surfaces in a generic metric.

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Uniqueness in vectorial variational problems

BERND KIRCHHEIM

(joint work with Judith Campos Cordero, Jan Kristensen, Jan Kolář)

It is well known that, unlike in the scalar cases, convexity is not the notion characterising the weak lower semicontinuity nor the existence of minimisers in vectorial variational problems.

More precisely, if $E : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ is a (continuous) energy density of suitable growth and

$$\mathcal{E}u \in W_g^{1,p}(\Omega_n, \mathbb{R}^N) \mapsto \int_{\Omega} F(\nabla u(x)) \, dx$$

the existence of minimisers of $\mathcal{E}$ does not require that E or $\mathcal{E}$ is convex, but only that it satisfies the Jensen inequality with respect to probabilities on $\mathbb{R}^{N \times n}$ which is referred to as quasiconvexity in the sense of Morrey. A short formulation requires only that

$$\int_{(0,1)^n} E(X + \nabla \varphi(x)) \, dx \geq E(X)$$

holds for all $\varphi \in W_0^{1,\infty}([0,1]^n, \mathbb{R}^N)$. And $E(X) = \det(X)$ is the prototypical counterexample of non-convex but energy density.

However, more recently old characterisations of by Klee reemerged due to work of St. Raymond, see [1] for a convex optimization based approach and the references. It say, briefly, if not only $\mathcal{E}$ but also all perturbed functionals $\mathcal{E} + \varphi^*$, where φ^* runs through a weak*-dense and *convex* subset of the dual, have an *unique* minimizer then $\mathcal{E}$ must be convex on the Banach space considered, for us $W_0^{1,p}(\Omega, \mathbb{R}^n)(+g)$.

Therefore the question appears for which integrands E the corresponding functional is convex on all, or equivalently one, Dirchlet class. We call such E functionally convex, in general under quite mild assumption they have to be quasiconvex and also quasidecreasing in the sense of Scheffer [3]. We can show that under the particular $p \in [1, 2)$ growth condition such E has to be convex. For $p \geq 2$, and thus allowing determinants and other minors the situation is only very particularly understood and precise description can be given only under special structural assumptions. Our results show for instance that the usual strengthening of of quasiconvexity to strict polyconvexity is not giving functional convex integrands. We also observe that this class of integrand is not closed under taking pointwise maxima, which distinguish it from usual convexity notion in vectorial CV and explains why new ideas are needed for a characterization.

In any case we can deduce the question about functional convexity for the so called isochoric-volumetric split energies, see [2], in the negative. This leads to very interesting examples from mechanics where under suitable forcing energy minimizers do exist, but are not unique even with the class of orientation preserving homomorphisms.

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On the structure of the two-phase free boundaries in 2D

BOZHIDAR VELICHKOV

The two-phase Bernoulli problem – classical formulation.

Let D be an open subset of $\mathbb{R}^d$, $d \geq 2$. We say that a continuous function $u : D \rightarrow \mathbb{R}$ is a (classical) solution of the two-phase Bernoulli problem in D if:

- the boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are C^1 manifolds, where

$$\Omega_u^+ := D \cap \{u > 0\} \quad \text{and} \quad \Omega_u^- := D \cap \{u < 0\};$$

- the following functions are differentiable at each point of the sets $D \cap \overline{\Omega_u^\pm}$:

$$u_+ = \max\{u, 0\} : D \cap \overline{\Omega_u^+} \rightarrow \mathbb{R} \quad \text{and} \quad u_- = \min\{u, 0\} : D \cap \overline{\Omega_u^-} \rightarrow \mathbb{R};$$

- u solves the following system for some real constants $\lambda_{\pm} > 0$:

$$\begin{aligned} \Delta u &= 0 & \text{in } \Omega_u^+ \cup \Omega_u^-, \\ |\nabla u_+| &= \lambda_+ & \text{on } \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap D, \\ |\nabla u_-| &= \lambda_- & \text{on } \partial\Omega_u^- \setminus \partial\Omega_u^+ \cap D, \\ |\nabla u_+|^2 - |\nabla u_-|^2 &= \lambda_+^2 - \lambda_-^2 & \text{and } |\nabla u_{\pm}| \geq \lambda_{\pm} \text{ on } \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D. \end{aligned}$$

Variational formulation. Consider the functional

$$J(u, D) := \int_D \left(|\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u<0\}} \right) dx,$$

where for a set A , χ_A denotes characteristic function of A . In [1] Alt, Caffarelli and Friedman showed that if $u \in H^1(D)$ is a minimizer of J in D among all functions with prescribed boundary datum $u = g$ on D , then u is Lipschitz continuous in D . Furthermore, they noticed that if the boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are $C^{1,\alpha}$ manifolds, then u is a classical solution of the two-phase Bernoulli system.

Regularity of the free boundary. In [8], we used an epiperimetric inequality to prove that if u is a minimizer of J in an open set $D \subset \mathbb{R}^2$, then the boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are $C^{1,\alpha}$ regular, for some $\alpha > 0$. Later, we extended this result to any dimension $d \geq 2$. Precisely, in [3] we proved that if u is a minimizer of J in $D \subset \mathbb{R}^d$, then the boundaries $\partial\Omega_u^{\pm}$ are $C^{1,\alpha}$ manifolds, for some $\alpha > 0$, in a neighborhood of any two-phase point $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D$. As a consequence of this result and the classical one-phase regularity theory (see [10]), we obtained that up to dimension $d = 4$ the free boundaries $\partial\Omega_u^{\pm} \cap D$ are $C^{1,\alpha}$ manifolds.

Examples of branching in dimension $d = 2$. The regularity of the free boundaries $\partial\Omega_u^{\pm}$, by itself, does not imply that the whole set $\Omega_u^+ \cup \Omega_u^-$ is regular; for instance, the interior of the zero set $\{u = 0\}$ might have cusps. Examples of such branching-type singularities were constructed in [6] and [2].

Theorem 1 (Theorem 1.8 of [6]). *Let $\lambda_+ = \lambda_- = 1$. For any $m \in \mathbb{N}$, $m \geq 1$, there is a minimizer u of J in the open set $D := (-1, 1)^2$ such that the free boundaries $\partial\Omega_u^{\pm} \cap D$ are graphs of $C^{1,\alpha}$ functions $\eta_{\pm} : (-1, 1) \rightarrow \mathbb{R}$ satisfying*

$$\begin{aligned} \eta_+(x) &= \eta_-(x) & \text{for all } -1 < x \leq 0, \\ cx^{2m-\frac{1}{2}} &\leq \eta_+(x) - \eta_-(x) \leq Cx^{2m-\frac{1}{2}} & \text{for all } 0 \leq x < 1, \end{aligned}$$

for some constants $C \geq c > 0$.

Solutions with a single contact points are much easier to construct, indeed, we have the following

Theorem 2. *For any $\lambda_{\pm} > 0$ and any $m \in \mathbb{N}$, $m \geq 1$, there is a minimizers u of J in $D := (-1, 1)^2$ for which the free boundaries $\partial\Omega_u^{\pm} \cap D$ are the graphs of analytic functions $\eta_{\pm} : (-1, 1) \rightarrow \mathbb{R}$ such that:*

$$cx^{2m} \leq \eta_+(x) - \eta_-(x) \leq Cx^{2m} \quad \text{for all } x \in (-1, 1),$$

for some constants $C \geq c > 0$.

Proof. Take two analytic functions η_+ and η_- satisfying the above inequalities. Use the theorem from the appendix of [4] to construct two one-phase solutions u_1 and u_2 with free boundaries given by η_+ and η_- . Thanks to the appendix of [6], the function $u := u_1 - u_2$ is a minimizer of the two-phase functional J . $\square$

Definition 3. Given a minimizer u of the two-phase functional in a set $D \subset \mathbb{R}^d$, we will say that a two-phase point $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D$ is a point of branching if

$$|B_r(x_0) \cap \{u = 0\}| \neq 0 \quad \text{for all } r > 0.$$

Question A. *Is it true that, in $\mathbb{R}^2$, the free boundaries around branching points can be described by the model examples from Theorems 1 and 2?*

In [6] we gave a positive answer to Question A for isolated branching points.

Theorem 4 (Corollary 1.7 from [6]). *Let $\lambda_{\pm} = 1$ and let u be a minimizer of J in an open set $D \subset \mathbb{R}^2$. If $0 \in D$ is an isolated branching point for u , then up to a rotation and for $\delta > 0$ small enough, the free boundaries $\partial\Omega_u^{\pm}$ in $(-\delta, \delta)^2$ are the graphs of $C^{1,\alpha}$ functions $\eta_{\pm} : (-\delta, \delta) \rightarrow \mathbb{R}$ and one of the following holds:*

- *there are $m \in \mathbb{N}$ and constants $C \geq c > 0$ such that:*

$$\begin{aligned} \eta_+(x) &= \eta_-(x) \quad \text{for all } -\delta < x \leq 0, \\ cx^{2m-\frac{1}{2}} &\leq \eta_+(x) - \eta_-(x) \leq Cx^{2m-\frac{1}{2}} \quad \text{for all } 0 \leq x < \delta; \end{aligned}$$

- *there are $m \in \mathbb{N}$ and constants $C \geq c > 0$ such that:*

$$cx^{2m} \leq \eta_+(x) - \eta_-(x) \leq Cx^{2m} \quad \text{for all } x \in (-\delta, \delta).$$

The main remaining question at this point is the following:

Question B. *Is it true that all the branching points of all the two-phase minimizers in $\mathbb{R}^2$ are isolated?*

The next theorem answers this question.

Theorem 5 ([5]). *Let $\lambda_{\pm} > 0$. Then, for any minimizer u of the two-phase functional J in an open set $D \subset \mathbb{R}^2$, the branching set consist of isolated points.*

The new idea, that led to the proof of the above theorem, is the use of the Weierstrass-Enneper transform, which allows to transform the two-phase Bernoulli problem into a two-membrane problem for the area functional. This connection has already been noticed and exploited by Hauswirth, Hélein and Pacard [7], and by Traizet [9] for the construction and the classification of the global one-phase solutions in 2D. In order to illustrate the key idea behind the proof of Theorem 5, we focus on the case $\lambda_+ = \lambda_- = 1$. Thanks to the regularity theorems [8, 3], by restricting D to a sufficiently small neighborhood of 0, we can suppose that Ω_u^+ and Ω_u^- are smooth simply connected sets. Consider the differential forms

$$\alpha_1 = \frac{1}{2} \left(1 - (\partial_x u)^2 + (\partial_y u)^2 \right) dx - \partial_x u \partial_y u dy,$$

$$\alpha_2 = -\partial_x u \partial_y u dx + \frac{1}{2} \left(1 + (\partial_x u)^2 - (\partial_y u)^2 \right) dy.$$

A direct computation gives that α_1 and α_2 are closed on each of the sets $\Omega_u^\pm$. Since $\Omega_u^\pm$ are simply connected, the forms $\alpha_{1,2}$ admit potentials

$$w_1^\pm : \overline{\Omega_u^\pm} \cap D \rightarrow \mathbb{R} \quad \text{and} \quad w_2^\pm : \overline{\Omega_u^\pm} \cap D \rightarrow \mathbb{R}.$$

The value of $w_{1,2}^\pm$ at a free boundary point $z \in \partial\Omega_u^\pm \cap D$ can be computed by integrating $\alpha_{1,2}$ along a curve joining 0 and z and lying in $\partial\Omega_u^\pm \cap D$. Since u is a classical solution, the free boundary conditions imply that $w_1^+(z) = w_1^-(z)$ and $w_2^+(z) = w_2^-(z)$ for all $z \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D$. Thus, the parametric surfaces

$$\Phi_\pm : \overline{\Omega_u^\pm} \cap D \rightarrow \mathbb{R}^3, \quad \Phi_\pm(z) := (w_1^\pm(z), w_2^\pm(z), u(z)).$$

solve a thin two-membrane problem in $\mathbb{R}^3$, precisely:

- $\mathcal{S}_\pm := \Phi_\pm(\overline{\Omega_u^\pm} \cap D)$ is a $C^{1,\alpha}$ surface with boundary $\Gamma_\pm := \Phi_\pm(\partial\Omega_u^\pm \cap D)$;
- $\mathcal{S}_\pm \setminus \Gamma_\pm = \Phi_\pm(\Omega_u^\pm \cap D)$ is a minimal surface contained in $\{\pm x_3 > 0\}$;
- the two surfaces intersect (only) in the hyperplane $\{x_3 = 0\}$ and we have

$$\Gamma_+ \cap \Gamma_- = \Phi_+(\partial\Omega_u^+ \cap \partial\Omega_u^- \cap D) = \Phi_-(\partial\Omega_u^+ \cap \partial\Omega_u^- \cap D);$$

- $\Gamma_+ \setminus \Gamma_- = \Phi_+(\partial\Omega_u^+ \setminus \partial\Omega_u^- \cap D)$ and at any $z \in \Gamma_+ \setminus \Gamma_-$ the surface $\mathcal{S}_+$ is orthogonal to e_3 ; analogously, at any $z \in \Gamma_- \setminus \Gamma_+$, $\mathcal{S}_-$ is orthogonal to e_3 .

Now, by the argument from [4] (see [5, Section 4]), the contact set $\Gamma_+ \cap \Gamma_-$ is locally composed of finitely many arcs. Since $\Gamma_+ \cap \Gamma_-$ is the image of the two-phase part of the free boundary $\partial\Omega_u^+ \cap \partial\Omega_u^-$, this shows that $\partial\Omega_u^+ \cap \partial\Omega_u^-$ is composed of finitely many arcs. Thus, the branching set consists of isolated points.

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On the Born-Infeld model (transversally)

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In their influential 1970 paper [16], Ladyzhenskaya & Ural'tseva conjectured that no universal structural criterion for pointwise interior gradient estimates in nonuniformly elliptic PDEs can be formulated. We settle this issue, cf. [7]. Our (positive) answer is sharply encoded in the family of nonparametric area-type functionals¹

$$(1) \quad w \mapsto \int_{\Omega} (1 + |Dw|^p)^{\frac{1}{p}} dx, \quad 1 < p < \infty,$$

whose Euler-Lagrange equations are

$$(2) \quad -\operatorname{div} \left(\frac{|Du|^{p-2} Du}{(1 + |Du|^p)^{1-1/p}} \right) = 0 \quad \text{in } \Omega.$$

Specifically for any $2 < p < \infty$, the integrand in (1) satisfies all structural hypotheses of [16, Theorem 1] except the crucial ellipticity/nonuniformity compensation condition [16, (1.12)], and the conclusion fails. In fact, our main result is the following theorem.

Theorem 1. *Let $n \geq 2$ and $2 < p < \infty$. There exists a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, and a function $v_0 \in C^\infty(\bar{\Omega})$ such that the unique solution $v \in v_0 + W_0^{1,1}(\Omega)$ to Dirichlet problem*

$$(3) \quad v_0 + W_0^{1,1}(\Omega) \ni v \mapsto \min_{w \in v_0 + W_0^{1,1}(\Omega)} \int_{\Omega} (1 + |Dw|^p)^{\frac{1}{p}} dx,$$

and unique generalized minimizer of integral (1) in class $v_0 + W_0^{1,1}(\Omega)$, satisfies $v \in C^{0,1-\frac{2}{p}}(\bar{\Omega}) \cap W^{1,\frac{p}{2},\infty}(\Omega) \setminus W_{\text{loc}}^{1,\infty}(\Omega)$. In particular, the set of non-Lebesgue points of Dv has positive $\mathcal{H}^{n-1}$ -measure.

The proof of Theorem 1 rests on a duality - up to a sign, and on suitable function classes - between minima of (1) and maxima of the singular/degenerate Born-Infeld type energy [2],

$$(4) \quad w \mapsto \int_{\Omega} (1 - |Dw|^q)^{\frac{1}{q}} dx,$$

where $q := p/(p-1)$, so that $p > 2$ corresponds to $1 < q < 2$. Although (4) is autonomous, its integrand displays the singular set of maximizers explicitly: the set of light rays, that is, the family of maximally extended segments along which maxima are affine with unit slope. Crucially, light rays are not singularities of the maximum, which admissibility forces to be 1-Lipschitz, but lines along

¹Here, $\Omega \subset \mathbb{R}^n$ will always be an open, bounded domain with Lipschitz boundary.

which the associated stress field blows up - and for $q < 2$ that blow-up may still be integrable, which is exactly what convex duality lets us exploit. In dimension $n = 2$, with variables (x, y) , we take as maximizer an affine map perturbed quadratically in the transversal y -direction, and produce the perturbation via the Cauchy-Kovalevskaya theorem. The resulting maximizer has a unique, maximally extended light ray contained in $\{y = 0\}$, with stress field singular of order $|y|^{-2/p}$, integrable precisely because $p > 2$. Manipulating the stress field then yields a globally Hölder continuous, non-Lipschitz minimum of (1) whose gradient blows up in the interior exactly on the light ray. As a direct consequence of Theorem 1, the celebrated regularity theory of Finn [9] and Bombieri & De Giorgi & Miranda [6] for nonparametric minimal surfaces - $p = 2$ in (1)-(2) - does not extend beyond Bernstein genre 2. Since [16], the necessity of Ladyzhenskaya & Ural'tseva's sufficient conditions for pointwise interior gradient estimates in nonuniformly elliptic problems has been raised repeatedly in the literature, see Giaquinta & Modica & Souček [13, p. 157], Ivanov [15, p. 11-14], and, more recently, Bildhauer [5, p. 119-122], Mingione [20, Section 6.2], Beck & Schmidt [3, p. 121] and [4, p. 87], and Schmidt [21, p. 215]. Prior to Theorem 1, irregular behavior of minima of autonomous, strictly convex functionals was known to occur essentially in two ways:

- Finn's nonattainment of boundary data for solutions of the minimal surfaces equation on nonconvex domains [10];
- Giaquinta [12], Marcellini [19], and Hong's [14] unbounded minimizers of the anisotropic energies like

$$W^{1,2}(\Omega) \ni w \mapsto \int_{\Omega} |Dw|^2 + |\partial_n w|^4 \, dx, \quad n \geq 6.$$

Interior singularities of scalar minima of autonomous, strictly convex integrals are hard to produce. On a smooth domain with regular datum the singularity must form genuinely inside; autonomy offers no coefficients for a priori prescribing its location; minimality carries full energy and comparison information, and hence none of the flexibility of general weak solutions; and partial regularity already confines the singular set to a small one. In contrast, ever since the classical examples of Lavrentiev [17] and Manià [18], coefficients have been known to dramatically affect both relaxation and regularity. A very simple example of Giaquinta, Modica & Souček [13, Example 3.2] shows how the lack of compensation between ellipticity and nonuniformity impacts the regularity of minima. Let $p > 2$, $m > 0$ satisfy

$$\int_{-1}^1 \left((1+x^2)^{\frac{p}{p-1}} - 1 \right)^{-\frac{1}{p}} \, dx < m,$$

which is possible because $p > 2$ makes finite the integral on the left-hand side above. Then, problem

$$(5) \quad BV(-1, 1) \ni u \mapsto \begin{cases} \inf_{w \in W^{1,1}(-1,1)} \int_{-1}^1 (1 + (1+x^2)|w'|^p)^{1/p} \, dx \\ w(-1) = -m, \quad w(1) = m, \end{cases}$$

admits a generalized solution $u \in BV(-1, 1)$ that is smooth in $(-1, 1) \setminus \{0\}$, and attains its boundary data, but jumps at the origin; in particular, $u \in BV(-1, 1) \setminus W_{\text{loc}}^{1,1}(-1, 1)$. Although the density in (5) looks like an elementary nonautonomous counterpart of the one in (1), the apparently harmless coefficient $(1+x^2)$ is exactly what generates and localizes the interior singularity, see [13, Section 3]: since $x \mapsto 1+x^2$ has a strict minimum at $x=0$ jumping at the origin is cheaper than jumping at the endpoints, so that $x=0$ acts as an “artificial boundary” and Finn’s loss of boundary trace [10] is transferred to the interior. The effect of coefficients can be devastating already in classical uniformly elliptic models. Zhikov [22] introduced a series of basic functionals such as

$$(6) \quad w \mapsto \int_{\Omega} |Dw|^p + a(x)|Dw|^q dx$$

$$1 < p < q < \infty, \quad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1],$$

and showed that the Lavrentiev phenomenon occurs as soon as the gap $q-p$ is sufficiently large compared to α . Building on Zhikov’s pioneering work, Esposito & Leonetti & Mingione [8], Fonseca & Malý & Mingione [11], and Balci & Diening & Surnachev [1] exhibited coefficients a and boundary data $u_0 \in W^{1,\infty}(\Omega)$ for which the solution $u \in u_0 + W_0^{1,p}(\Omega)$ to Dirichlet problem

$$u_0 + W_0^{1,p}(\Omega) \ni u \mapsto \min_{w \in u_0 + W_0^{1,p}(\Omega)} \left(\int_{\Omega} |Dw|^p + a(x)|Dw|^q dx \right),$$

does not belong to $W_{\text{loc}}^{1,q}(\Omega)$. The geometry of a is what drives these constructions: in [11, 1] the set of essential discontinuity points of u is a Cantor-type fractal of (almost) maximal Hausdorff dimension, so that $u \notin W_{\text{loc}}^{1,t}(\Omega)$ for all $t > p$. Although built on different models, the examples in [13] and [22, 8, 11, 1] share one mechanism: through its set of minimum points, the coefficient singles out interior regions where singular variation or large gradients are energetically cheap.

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Curvature lower bounds in non-Euclidean elasticity

CY MAOR

(joint work with Raz Kupferman, David Padilla-Garza)

Non-Euclidean elasticity is a model for prestrained elastic bodies (e.g., elastic bodies that undergo inhomogeneous growth/shrinkage) [1]. This model has been very successful in describing various phenomena, from the shape of leaves [7] to shape transitions in molecular assemblies [8, 6].

In this model, the elastic body is a domain $\Omega \subset \mathbb{R}^3$, endowed with a smooth Riemannian metric G . A prototypical elastic energy associated with a configuration $F : \Omega \rightarrow \mathbb{R}^3$, satisfying $\det DF > 0$, is

$$E_{(\Omega, G)}^{3D}(F) = \int_{\Omega} |DF^T DF - G|^2 dV_G.$$

That is, the energy measures stretching: the deviation of the metric induced by F from the reference metric G . The curvature of G is an obstruction to relaxing this energy: If the Riemannian curvature of G does not vanish, then $\inf E^{3D} > 0$ [2, 5]. A main open question is obtaining effective lower bounds for E^{3D} in terms of the curvature [1, 3].

When we consider *thin, non-Euclidean elastic plates*, we assume that the body is thin $\Omega = \Omega_t = \omega \times (-\frac{t}{2}, \frac{t}{2})$, for some $\omega \subset \mathbb{R}^2$, and

$$G(x', x_3) = g(x') + dx_3 \otimes dx_3,$$

where g is a Riemannian metric on ω . In this case it can be shown that, generically, $\inf E_{(\Omega_t, G)}^{3D} \approx t^3$, and

$$\frac{1}{t^3} E_{(\Omega_t, G)}^{3D} \xrightarrow{\Gamma} E_{(\omega, g)}^{\text{Bend}},$$

where

$$E_{(\omega, g)}^{\text{Bend}}(f) = \int_{\omega} |\Pi_f|^2 dV_g, \quad f \in W^{2,2}(\omega, \mathbb{R}^3), \quad Df^T Df = g \text{ a.e.}$$

is the *bending energy* of $W^{2,2}$ isometric immersions of the metric g ; here Π_f is the second fundamental form of the immersion f . This energy is equivalent to the Willmore functional. A natural question is thus to establish effective lower bounds of $E_{(\omega, g)}^{\text{Bend}}$ in terms of the Gauss curvature K of g . An immediate bound, using Gauss' Theorema Egregium, is

$$\inf E_{(\omega, g)}^{\text{Bend}} \geq 2\|K\|_{L^1}.$$

This lower bound, however, is far from being tight, and fails in the following cases:

- (1) **Curvature concentration:** A hemisphere has much smaller energy than a (regularized) cone with the same total curvature and volume. In fact, the bending energy is expected to blow up when the curvature becomes more concentrated.
- (2) **Curvature “in holes”:** When a neighborhood of the tip of a cone is removed, the bending energy still blows up, yet $\|K\|_{L^1} = 0$.
- (3) **Super-extensibility:** The energy of (ω, g) is, in general, much larger than the sum of the energies of a partition of ω , yet $\|K\|_{L^1}$ is additive.

We present a new lower bound for $E_{(\omega, g)}^{\text{Bend}}$ that detects both curvature concentration and curvature in the holes [4]:

$$(1) \quad \inf E_{(\omega, g)}^{\text{Bend}} \geq \frac{4\pi - \|K\|_{L^1}}{\|K\|_{L^1}} \|K\|_{H^{-1}}^2.$$

In the case of multiply-connected ω , both $\|K\|_{L^1}$ and $\|K\|_{H^{-1}}$ include the curvature “in the holes” via the Gauss–Bonnet formula. In the non-simply connected case, the infimum above applies to *extendable* isometric immersions, namely isometric immersions that can be extended to an immersion of a simply-connected domain containing ω ; for non-extendable isometric immersions, as slightly different lower bound holds.

The proof of (1) is based on two main ingredients: (i) interpreting the Gaussian curvature as an electric charge, foliating ω by level sets of an appropriate electric potential, and (ii) applying a new isoperimetric inequality of framed-loops to the Gauss map of the immersion f along these level sets.

Beyond the geometric motivation, inequality (1), as well as a similar lower bound based on circulation (Burgers' vector) presented in [4], is useful for obtaining optimal energy scaling laws for thin plates with defects — disclinations and dislocations — in the pure-bending regime.

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Logarithmic capacity and torsion in planar shape optimization

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We study a family of scale-invariant functionals involving torsion, logarithmic capacity, area and perimeter. For a bounded open set $\Omega \subset \mathbb{R}^2$, let $u_\Omega \in H_0^1(\Omega)$ be the solution to $-\Delta u_\Omega = 1$ and define its torsional rigidity by $T(\Omega) = \int_\Omega u_\Omega$. For a compact set $K \subset \mathbb{R}^2$, define the logarithmic potential energy and logarithmic capacity as

$$I(K) = \inf_{\mu \in \mathcal{P}(K)} \iint_{K \times K} \log \frac{1}{|x - y|} d\mu(x) d\mu(y), \quad \text{Cap}(K) = e^{-I(K)}.$$

It is easy to see that the functional

$$H_\alpha(\Omega) := \frac{T(\Omega)^{1/2} \text{Cap}(\Omega)}{|\Omega|^\alpha P(\Omega)^{3-2\alpha}}, \quad 0 \leq \alpha \leq \frac{3}{2},$$

is unchanged by dilations. Following the variational problem introduced by van den Berg and Malchiodi [1], we maximize H_α over nonempty bounded open convex subsets of $\mathbb{R}^2$. We are interested in understanding whether maximizers exist, for which parameters the disc is (uniquely) maximizing, and how regular the boundary of a maximizer must be.

Existence at the critical exponent. For $0 \leq \alpha < 3/2$, existence follows from the compactness results in [1]. At $\alpha = 3/2$ there is no perimeter term, so controlling the diameter of a maximizing sequence requires an additional argument.

We give a short sketch for the existence proof here.

Makai's inequality for convex sets, together with $P(\Omega) \geq 2 \operatorname{diam}(\Omega)$, gives

$$T(\Omega) \leq \frac{2|\Omega|^3}{3P(\Omega)^2} \leq \frac{|\Omega|^3}{6 \operatorname{diam}(\Omega)^2}.$$

Normalize $|\Omega| = \pi$. If a sequence of convex sets has diameter tending to infinity, rescaling each set to diameter one produces, after passing to a subsequence, a segment of length one. Continuity of logarithmic capacity on connected compact sets [2] and the fact that the capacity of this segment equals $1/4$, imply

$$\frac{\operatorname{Cap}(\Omega_n)}{\operatorname{diam}(\Omega_n)} \longrightarrow \frac{1}{4}, \quad \limsup_{n \rightarrow \infty} H_{3/2}(\Omega_n) \leq \frac{1}{4\sqrt{6}} < \frac{1}{2\sqrt{2}\pi} = H_{3/2}(D),$$

where D is the unit disc. Hence, such a sequence cannot be maximizing. The remaining sequences have bounded diameter, and the direct method yields a maximizer. Consequently, a maximizer exists for every $\alpha \in [0, 3/2]$.

For which α is the disc optimal? For $0 \leq \alpha \leq 1$ the disc is the unique maximizer even among Jordan domains, up to translations and dilations. Indeed, after normalizing $|\Omega| = \pi$, Saint-Venant's inequality bounds the torsion by that of D , while the classical Pólya bound $\operatorname{Cap}(\Omega) \leq P(\Omega)/(2\pi)$ and the isoperimetric inequality give $H_\alpha(\Omega) \leq H_\alpha(D)$.

For $\alpha > 1$, it is not difficult to see that the supremum over Jordan domains is infinite. Convexity allows us to extend the disc maximality range beyond 1.

More precisely, the disc is the unique maximizer among bounded open convex sets for $0 \leq \alpha \leq 9/8$. To see this, one can combine a sharpened capacity-perimeter inequality [3] and the Saint-Venant and Makai inequalities. Whether the disc is optimal all the way to $\alpha = 3/2$ is open.

Regularity. Every convex maximizer of H_α has a $C^{1,1}$ boundary if $0 \leq \alpha < 3/2$. The main ingredient for this is a differentiability statement of the logarithmic capacity under Lipschitz perturbations. If E is a nonpolar compact set, $\Phi_h = \operatorname{Id} + h$ and $\operatorname{Lip}(h) < 1$, then $\mathcal{I}(h) := I(\Phi_h(E))$ is C^1 as a function of $h \in W^{1,\infty}(E; \mathbb{R}^2)$. Writing μ_h for the equilibrium measure pulled back to E , its derivative is

$$D\mathcal{I}(h)[q] = - \iint_{E \times E} \frac{(\Phi_h(x) - \Phi_h(y)) \cdot (q(x) - q(y))}{|\Phi_h(x) - \Phi_h(y)|^2} d\mu_h(x) d\mu_h(y).$$

For a convex body E , the equilibrium measure has a boundary density $\mu_E = f_E \mathcal{H}^1|_{\partial E}$ with $f_E \in L^p(\partial E)$ for some $p > 2$, and the formula becomes

$$D\mathcal{I}(0)[q] = -2\pi \int_{\partial E} f_E^2(q \cdot \nu) d\mathcal{H}^1.$$

Representing convex sets by a gauge function $u \in W^{1,\infty}(\mathbb{T})$ turns convexity into $u'' + u \geq 0$. The derivative above supplies an $L^{p/2}$ density for the capacity term in the Euler-Lagrange equation. For $\alpha < 3/2$, the perimeter term has strictly positive coefficient $3 - 2\alpha$ and is strictly convex in u' . The regularity criterion of Lamboley, Novruzli and Pierre [4] first yields $u \in W^{2,p/2}$ and a simple bootstrapping gives $u \in W^{2,\infty}$. This proves the asserted $C^{1,1}$ regularity. This argument of course does not

extend to $\alpha = 3/2$, where the perimeter coefficient vanishes. The differentiability statement for the logarithmic capacity might be of independent interest. It can, for example, be used to give an alternative proof of the $C^{1,1}$ regularity of volume-constrained convex minimizers of $P(E) + Q^2 I(E)$ established by Goldman, Novaga and Ruffini [5].

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Asymptotic expansion of harmonic maps with multiple inclusions

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(joint work with Lia Bronsard, Xavier Lamy and Raghav Venkatraman)

This work investigates the asymptotics of the minimal Dirichlet energy for $\mathbb{S}^2$ -valued maps n outside a finite number of particles of size ρ in $\mathbb{R}^3$ under general anchoring conditions at the particle boundaries as $\rho \rightarrow 0$. To be precise, we consider the exterior domain

$$\Omega_\rho = \mathbb{R}^3 \setminus \bigcup_{j=1}^N \omega_{j,\rho}, \quad \omega_{j,\rho} = x_j + \rho \hat{\omega}_j,$$

where $x_1, \dots, x_N \in \mathbb{R}^3$ are fixed particle centers and $\hat{\omega}_j \subset \mathbb{R}^3$ represent the particles after zooming in at scale ρ . We also impose far-field alignment along a fixed orientation $n_\infty \in \mathbb{S}^2$ via

$$(1) \quad \int_{\Omega_\rho} \frac{|n(x) - n_\infty|^2}{1 + |x|^2} dx < \infty.$$

We then minimize the energy

$$(2) \quad E_\rho(n) := \frac{1}{\rho} \int_{\Omega_\rho} |\nabla n|^2 dx + \sum_{j=1}^N F_{j,\rho}(n|_{\partial\omega_{j,\rho}}),$$

where $F_{j,\rho}$ represents anchoring energies on the particle surfaces. They include strong anchoring Dirichlet conditions and weak anchoring (enforced by a surface energy).

In the case of a single particle ($N = 1$), let us define μ_j to be the minimum of the energy $\hat{E}_j(n) := \int_{\mathbb{R}^3 \setminus \hat{\omega}_j} |\nabla n|^2 dx + \hat{F}_j(n|_{\partial\hat{\omega}_j})$, where $\hat{F}_j(n|_{\partial\hat{\omega}_j}) = F_{j,\rho}(n(x_j +$

$\rho \cdot)_{|\partial\omega_{j,\rho})}$ subject to $\int_{\mathbb{R}^3 \setminus \hat{\omega}_j} \frac{|n - n_\infty|^2}{1 + |x|^2} dx < \infty$. It is shown in [1] that any minimizer $\hat{m}_j$ of $\hat{E}_j$ has a far-field expansion

$$(3) \quad \hat{m}_j(x) = n_\infty + \frac{v_j}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \rightarrow +\infty,$$

for some $v_j \in \mathbb{R}^3$ orthogonal to n_∞ . The vector v_j can be interpreted as a torque applied by the particle $\hat{\omega}_j$ on the nematic background [2], see also [1, Theorem 2].

For $N > 1$, we our main theorem can be stated as follows:

Theorem 1 ([3]). *There exist minimizers $\hat{m}_j$ of the single-particle energy $\hat{E}_j$ such that the minimum of E_ρ over maps $n: \Omega_\rho \rightarrow \mathbb{S}^2$ with far-field alignment (1) satisfies*

$$(4) \quad \min_n E_\rho(n) = \sum_{j=1}^N \mu_j - 4\pi\rho \sum_{i \neq j} \frac{\langle v_i, v_j \rangle}{|x_i - x_j|} + o(\rho) \quad \text{as } \rho \rightarrow 0,$$

where $\mu_j = \hat{E}_j(\hat{m}_j)$ is the minimal single-particle energy, and $v_j \in n_\infty^\perp$ is defined by the asymptotic expansion (3) of $\hat{m}_j$.

In other words, the leading order term of the expansion consists of the minimal energy after zooming in at scale ρ around each particle, the next order is a Coulomb-like interaction that agrees with the electrostatic analogy depending on the far-field behavior of the corresponding single-particle minimizer.

The idea of this linearization is as follows: the nonlinear behaviour of the original problem is concentrated in a vicinity of the particles and encoded through the presence of μ_j and v_j , determined by the single-particle problem. Away from the particles, the harmonic map equation $-\Delta n_\rho = |\nabla n_\rho|^2 n_\rho$ is linearized around the uniform state n_∞ , which corresponds to writing $n_\rho \approx n_\infty + u_\rho$ and $-\Delta u_\rho \approx 0$. The effect of the particle $\omega_{j,\rho}$ is replaced by a singular source term at x_j , and that source term is chosen to match the far-field expansion (3) generated by the single particle. Thus, u solves the following Poisson equation

$$\Delta u_\rho = 4\pi\rho \sum_{j=1}^N v_j \delta_{x_j} \quad \text{in } \mathbb{R}^3, \quad \text{that is, } u_\rho(x) = \rho \sum_{j=1}^N \frac{v_j}{|x - x_j|}.$$

The infinite energy of u_ρ can be renormalized to give

$$\lim_{\sigma \rightarrow 0} \left(\frac{1}{\rho} \int_{\mathbb{R}^3 \setminus \bigcup B_\sigma(x_j)} |\nabla u_\rho|^2 dx - \frac{4\pi}{\sigma} \rho \sum_{j=1}^N |v_j|^2 \right) = -4\pi\rho \sum_{i \neq j} \frac{\langle v_i, v_j \rangle}{|x_i - x_j|}.$$

The approximation (4) is commonly used in the physics literature for colloid interactions in nematic liquid crystals [2, 4]. For the first time a precise estimate of the energy error introduced by that linearization is derived, by developing new tools that address the lack of convergence rate when zooming in at scale ρ .

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Strong coupling in two-dimensional ferronematics

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(joint work with Yoshihiro Tonegawa)

Ferronematics are composite materials formed by dispersing magnetic nanoparticles into a nematic liquid crystals host [5]. For thin films of ferronematics, a simplified phenomenological model recently proposed in the physical literature [4] is given by the functional

$$(1) \quad \mathcal{F}_\varepsilon(\mathbf{Q}, \mathbf{M}) := \int_\Omega \left(\frac{1}{2} |\nabla \mathbf{Q}|^2 + \frac{\varepsilon}{2} |\nabla \mathbf{M}|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \right) dx.$$

In (1), the energy functional is presented already in a non-dimensional form, Ω is a bounded, simply connected, open set in $\mathbb{R}^2$ with smooth boundary, ε is a (small) positive parameter, the map $\mathbf{Q} : \Omega \rightarrow \mathcal{S}_0$, where $\mathcal{S}_0$ denotes the space of 2×2 , symmetric, traceless matrices with real entries, describes the nematic part of the system, and the vector field $\mathbf{M} : \Omega \rightarrow \mathbb{R}^2$ accounts for the average polarisation vector generated by the included particles. The potential $f_\varepsilon(\mathbf{Q}, \mathbf{M})$ is given by

$$f_\varepsilon(\mathbf{Q}, \mathbf{M}) := \frac{1}{4} (|\mathbf{Q}|^2 - 1)^2 + \frac{\varepsilon}{4} (|\mathbf{M}|^2 - 1)^2 - \varepsilon \beta \mathbf{Q} \mathbf{M} \cdot \mathbf{M} + \chi_\varepsilon,$$

where χ_ε depends only on ε and ensures that $f_\varepsilon \geq 0$ pointwise, while $\beta > 0$ is a material-dependent parameter measuring the strength of the coupling between $\mathbf{Q}$ and $\mathbf{M}$. As it is immediately checked, for $\beta > 0$ the coupling term promotes alignment between $\mathbf{M}$ and the eigendirection of $\mathbf{Q}$ relative to its positive eigenvalue.

Physically, the scaling in (1) is reminiscent of a suitable ‘superdilute regime’ for the magnetic particles; after non-dimensionalisation, the limit $\varepsilon \rightarrow 0$ corresponds to a large-domain limit. In this regime, the magnetic energy and the interaction energy are expected to be lower order perturbations to the nematic contribution.

The Euler-Lagrange equations of the functional (1) are given by the following two systems:

$$(2) \quad -\Delta \mathbf{Q}_\varepsilon + \frac{1}{\varepsilon^2} (|\mathbf{Q}_\varepsilon|^2 - 1) \mathbf{Q}_\varepsilon - \frac{\beta}{\varepsilon} \left(\mathbf{M}_\varepsilon \otimes \mathbf{M}_\varepsilon - \frac{|\mathbf{M}_\varepsilon|^2}{2} \mathbf{I} \right) = 0,$$

$$(3) \quad -\Delta \mathbf{M}_\varepsilon + \frac{1}{\varepsilon^2} (|\mathbf{M}_\varepsilon|^2 - 1) \mathbf{M}_\varepsilon - \frac{2\beta}{\varepsilon^2} \mathbf{Q}_\varepsilon \mathbf{M}_\varepsilon = 0.$$

The aim of [6, 7, 8] is to study the asymptotic behaviour of critical pairs $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ subject to appropriate ε -independent boundary conditions inducing a non-trivial topological behaviour of critical points, similar to those considered in the physical literature [9, 4]. Such a non-trivial behaviour is basically embodied in the following energy bounds, which are consequences of the Euler-Lagrange equations and the boundary conditions and that confirm the aforementioned expectations: there exist a constant $C > 0$, depending on the data but independent of ε , such that the inequalities

$$(4) \quad \int_{\Omega} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \leq C, \quad \varepsilon \int_{\Omega} |\nabla \mathbf{M}_\varepsilon|^2 \leq C$$

hold for every $\varepsilon > 0$ and, moreover,

$$(5) \quad \int_{\Omega} |\nabla \mathbf{Q}_\varepsilon|^2 \sim |\log \varepsilon|,$$

so that, overall, $\mathcal{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \sim |\log \varepsilon|$.

The competition between (4) and (5) gives rise to a peculiar energy concentration phenomenon, studied in [6, 7] through a detailed analysis of the asymptotic behaviour of the energy densities

$$\begin{aligned} \mu_\varepsilon &:= \frac{1}{|\log \varepsilon|} \left(\frac{1}{2} |\nabla \mathbf{Q}_\varepsilon|^2 + \frac{\varepsilon}{2} |\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \right), \\ \nu_\varepsilon &:= \frac{\varepsilon}{2} |\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon). \end{aligned}$$

It has been shown in [6, 7] that, as $\varepsilon \rightarrow 0$, the energy of critical pairs $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ concentrates on *distinct singular sets*. More precisely, as $\varepsilon \rightarrow 0$, there holds

$$\mu_\varepsilon \rightharpoonup^* \mu_\star, \quad \nu_\varepsilon \rightharpoonup^* \nu_\star,$$

where $\mu_\star, \nu_\star$ are Radon measures in Ω , $\text{spt } \mu_\star$ is a finite set of points and $\text{spt } \nu_\star$ is a union of segments with locally constant density. These results were obtained by separate (although tightly intertwined) analysis of the ‘ $\mathbf{Q}$ -component’ (i.e., of the system (2)) and of the ‘ $\mathbf{M}$ -component’ (i.e., of (3)). Concerning (2), it has been shown in [6] that a careful treatment of the coupling term allows one to pursue the classical strategy traced, for the pure Ginzburg-Landau theory, in the seminal monograph [3] by F. Bethuel, H. Brezis and F. Hélein. This comes with suitable convergence results for $\mathbf{Q}_\varepsilon$ towards a limiting map $\mathbf{Q}_\star$, a clearing-out result as typical of the standard Ginzburg-Landau theory, and a priori estimates yielding, in particular, the structure properties of $\text{spt } \mu_\star$ above.

The situation is far more involved for the system (3). Although at first sight it looks like a perturbed Ginzburg-Landau system, it has instead the structure of a *perturbed Allen-Cahn system*. This is revealed by introducing the auxiliary energy functional

$$E_\varepsilon(\mathbf{M}_\varepsilon) := \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon} V(\mathbf{M}_\varepsilon) \right) dx,$$

where

$$V(\mathbf{M}_\varepsilon) := \ell(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) - \inf_{y \in \mathbb{R}^2} \ell(\mathbf{Q}_\varepsilon, y),$$

$$\ell(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) := \frac{1}{4} (|\mathbf{M}_\varepsilon|^2 - 1)^2 - \beta \mathbf{Q}_\varepsilon \mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon + \frac{1}{2} (\beta^2 + \sqrt{2}\beta),$$

and noticing that the Euler-Lagrange equation of E_ε , i.e.,

$$-\varepsilon \Delta \mathbf{M}_\varepsilon + \frac{1}{\varepsilon} \nabla_{\mathbf{M}} V(\mathbf{M}_\varepsilon) = 0,$$

coincides *exactly* with (3). In addition, the energy bounds (4), (5) yield a uniform bound on $E_\varepsilon(\mathbf{M}_\varepsilon)$. The claimed perturbation to a pure Allen-Cahn system comes from the interaction between $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$, which makes the wells of the potential dependent on $\mathbf{Q}_\varepsilon$. Indeed, at fixed ε and x and for $\mathbf{Q} = \mathbf{Q}_\varepsilon(x) \neq 0$, it is easily shown that the zeroes of the function $\mathbf{M} \mapsto V(\mathbf{M})$ are given by $\mathbf{M}_\pm := \pm (1 + \sqrt{2}\beta|\mathbf{Q}|)^{1/2} \mathbf{n}$, where $\mathbf{n}$ denotes a unit eigenvector relative to the positive eigenvalue of $\mathbf{Q}$. While the moving wells prevent us from applying directly the asymptotic analysis for critical points of the Allen-Cahn system in 2d satisfying a uniform energy bound contained in the recent breakthrough paper [2], in [7] we adapted the same strategy, combining it with the results in [6] to control the movement of the wells accurately.

Following [2], it has been shown in [7] that, as $\varepsilon \rightarrow 0$, the potential energy densities $\zeta_\varepsilon := \frac{1}{\varepsilon} V(\mathbf{M}_\varepsilon)$ converge to a limiting Radon measure $\zeta_\star$ and that $\zeta_\star$ is $\mathcal{H}^1$ -rectifiable. More precisely, calling $\mathfrak{S}_\star := \text{spt } \nu_\star \setminus \text{spt } \mu_\star$, it turns out (cf. [7, Theorem B]) that:

- (i) The support of $\zeta_\star$ coincides with $\mathfrak{S}_\star$;
- (ii) $\zeta_\star = \Theta_\star \mathcal{H}^1 \llcorner \mathfrak{S}_\star$, where the density $\Theta_\star$ is defined at every point $x_0 \in \Omega$ and satisfies $0 < c_K \leq \Theta_\star(x) \leq C_K$ at any point $x \in \mathfrak{S}_\star \cap K$, where K is any compact set in $\Omega \setminus \text{spt } \mu_\star$;
- (iii) $\zeta_\star$ is the weight measure of the $\mathcal{H}^1$ -rectifiable varifold $\mathbf{V}_\star := \mathbf{v}(\mathfrak{S}_\star, \Theta_\star)$;
- (iv) the varifold $\mathbf{V}_\star$ is stationary in $\Omega \setminus \text{spt } \mu_\star$.

Combined with [1, 2], the above results imply in particular that $\text{spt } \nu_\star$ is a union of segments with locally constant density, as claimed before.

The purpose of the talk is to illustrate the following theorem, obtained in [8].

Theorem 1. *Let $\sigma_\beta := \frac{\sqrt{2}}{3} (1 + \sqrt{2}\beta)^{3/2}$. If $\sqrt{2}\beta > 1$, then the varifold*

$$\sigma_\beta^{-1} \mathbf{V}_\star := \mathbf{v}(\mathfrak{S}_\star, \sigma_\beta^{-1} \Theta_\star)$$

is integral. In other words, the function $\sigma_\beta^{-1} \Theta_\star$ takes its values into $\mathbb{N}$.

The idea of the proof is as follows. By projecting (3) onto an eigenframe $(\mathbf{n}_\varepsilon, \mathbf{m}_\varepsilon)$ of $\mathbf{Q}_\varepsilon$ (which is well-defined in any simply connected subdomain G of $\Omega \setminus \text{spt } \mu_\star$ by the clearing-out result for $\mathbf{Q}_\varepsilon$), one sees that, if $\sqrt{2}\beta > 1$, the equation governing $\mathbf{M}_\varepsilon \cdot \mathbf{m}_\varepsilon$, the component of $\mathbf{M}_\varepsilon$ orthogonal to $\mathbf{Q}_\varepsilon$, guarantees an

extremely rapid decay of such a component as $\varepsilon \rightarrow 0$, of the type

$$(6) \quad \int_G \left(\varepsilon |\nabla(\mathbf{M}_\varepsilon \cdot \mathbf{m}_\varepsilon)|^2 + \frac{1}{\varepsilon} (\mathbf{M}_\varepsilon \cdot \mathbf{m}_\varepsilon)^2 \right) dx \leq C_\alpha \varepsilon^\alpha, \quad \forall \alpha \in (0, 2).$$

At the same time, the parallel component $\mathbf{M}_\varepsilon \cdot \mathbf{n}_\varepsilon$ satisfies a perturbed *scalar* Allen-Cahn equation with a perturbation that, thanks to (6), satisfies the assumptions of the integrality results in [10] (see also [11]). The conclusion then follows by showing, again relying on (6), that the difference between ζ_ε and the scalar Allen-Cahn potential $\frac{1}{4\varepsilon} ((\mathbf{M}_\varepsilon \cdot \mathbf{n}_\varepsilon)^2 - 1 - \sqrt{2}\beta)^2$ tends to zero in $L^1_{\text{loc}}(\Omega \setminus \text{spt } \mu_\star)$ as $\varepsilon \rightarrow 0$ and recalling that $\Theta_\star(a) = 0$ for any $a \in \text{spt } \mu_\star$.

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The homogeneous Morrey problem

GABRIELE CASSESE

Morrey’s problem is one of the central problems in the calculus of variations. It asks the following question:

Is a rank-one convex integrand $f : \mathbb{R}^{d \times m} \rightarrow \mathbb{R}$ quasiconvex?

It has been known since the ’90s that this implication is false in dimensions 3×2 and higher, but the 2-component case remained open. We focus on a more specific, but nonetheless far-reaching, version of Morrey’s problem:

Given $p \in [1, \infty)$, is a rank-one convex p -homogeneous integrand $f : \mathbb{R}^{d \times m} \rightarrow \mathbb{R}$ quasiconvex at 0?

When $p = 1$, the implication holds: this is the main result of [4].

Note that the previously known counterexamples to Morrey's problem ([5] and [3]) do not address this version of the problem. The former is not homogeneous; the latter is homogeneous but nonnegative, and hence quasiconvex at 0.

The main tool we introduce is the following connection with martingales: for real-linear maps $P, Q : \mathbb{R}^{d \times m} \rightarrow X$, where X is a finite-dimensional normed space, and $C \geq 0$, set $g_C(A) = C^p \|P(A)\|_X^p - \|Q(A)\|_X^p$. Its rank-one convex envelope satisfies $g_C^{\text{rc}}(0) = 0$ if and only if

$$\|Q(M_N)\|_{L^p(X)} \leq C \|P(M_N)\|_{L^p(X)}$$

for every finite martingale starting at 0 with increments $M_k - M_{k-1} = \varepsilon_k A_k$, where the ε_k are independent symmetric signs and $A_k = A_k(\varepsilon_1, \dots, \varepsilon_{k-1})$ has rank at most one. In contrast, quasiconvexity of g_C at 0 is equivalent to

$$\|Q(\nabla u)\|_{L^p(X)} \leq C \|P(\nabla u)\|_{L^p(X)} \quad (u \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^d)).$$

When the first estimate holds but the second fails, g_C^{rc} is a real-valued, p -homogeneous counterexample [2].

Thus, it suffices to construct integrands where the martingale-transforms¹ are well behaved while the differential transform is not. This is very easy to do in 2 components, provided the domain-dimension is large enough:

Theorem. *For every $p \in (1, \infty)$ there exists $m_0(p)$ such that, for all $m \geq m_0(p)$, there is a p -homogeneous rank-one convex integrand $f : \mathbb{R}^{2 \times m} \rightarrow \mathbb{R}$ which is nowhere quasiconvex.*

For the construction, take $m = 2n$, $q = p + 3$ and $X = \ell_q^n(\mathbb{C})$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and $\mathbb{R}^2$ with $\mathbb{C}$, write

$$Az = \sum_{j=1}^n (\alpha_j z_j + \beta_j \bar{z}_j), \quad P(A) = (\alpha_j)_{j=1}^n, \quad Q(A) = (\beta_j)_{j=1}^n.$$

For rank-one R , we have $Q(R) = \lambda(R) \overline{P(R)}$ with $|\lambda(R)| = 1$; in particular, $\|P(R)\|_X = \|Q(R)\|_X$. The UMD property of $\ell_q(\mathbb{C})$ [6] therefore gives the martingale estimate with a constant independent of n . On the other hand, taking

$$u_n(z_1, \dots, z_n) = \prod_{j=1}^n f(z_j), \quad f(z) = \bar{z} e^{-|z|^2/2},$$

an elementary law of large numbers argument gives

$$\frac{\|Q(\nabla u_n)\|_{L^p(X)}}{\|P(\nabla u_n)\|_{L^p(X)}} \longrightarrow \infty.$$

Cutting off at infinity gives compactly supported witnesses, proving the theorem [2].

¹for which there is an extremely rich and well-developed theory started by Burkholder [1]

To get low-dimensional examples, the trick is to chain martingale transforms. Indeed, this allows us to construct a 2×3 polynomial counterexample to the homogeneous Morrey problem: take $n = 2$ and write $X = \overline{P}$, $Y = Q$. Restricting to the subspace $Y^2 = X^1$ produces the chain

$$Z^0 = X^2 \mapsto Z^1 = Y^2 = X^1 \mapsto Z^2 = Y^1.$$

Along rank-one martingales in this subspace, the increments satisfy $dZ = \varepsilon a(1, \lambda, \lambda^2)$, with $|\lambda| = 1$. Consequently, the quadratic observables

$$D(Z) = Z^0 Z^2 - (Z^1)^2, \quad H(Z) = |Z^0|^2 + |Z^2|^2 - 2|Z^1|^2$$

are themselves martingales, with $|dH| \leq 2|dD|$. Thus $4|D|^2 - H^2$ is rank-one convex on this subspace.

Setting the fourth column to zero gives

$$L = \left\{ \begin{pmatrix} a & c & a+d \\ b & d & c-b \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

on which the quartic becomes $h(A) = -\det(AJA^T)$, where $J = \text{diag}(1, 1, -1)$. The gradient constraint is $u_t = 2\overline{\partial}_w u$, where $w = x + iy$. For a witness, let $\zeta(w, t)$ be the root in the unit disc of

$$w + 2(t + i)\zeta + \bar{w}\zeta^2 = 0, \quad \text{and set} \quad u = \bar{\zeta} + \frac{\zeta^2}{2}.$$

Then u is smooth and bounded, $\nabla u \in L$, and $h(\nabla u) = -|\partial_w \zeta|^4(1 - |\zeta|^2)^6 < 0$ is integrable. An extension argument, combined with a suitable cutoff, gives a rank-one convex homogeneous quartic polynomial on $\mathbb{R}^{2 \times 3}$ which is nowhere quasiconvex [2].

Note that the transpose of this integrand also provides a counterexample (though this is of course not true in general [7]) in dimension 3×2 , thus answering the homogeneous Morrey problem there too (and providing a new proof of Šverák's result).

We expect that these methods, if pushed in a more refined and quantitative way, will also provide counterexamples in low dimension for any $p \in (1, \infty)$. Whether the homogeneous (or not) Morrey problem has a positive answer in 2×2 is anybody's guess.

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Optimal regularity for higher order free boundary problems

GUIDO DE PHILIPPIS

(joint work with Mickael Nahon, Jonas Hirsch)

The goal of the talk is to present a series of new optimal regularity results for *higher order or vectorial* free boundary problems of Bernoulli type. More precisely, we consider the following variational problems:

$$(HOB) \quad \min \int Q(\nabla^m u) + |\{\nabla^\ell u \neq 0\}|.$$

where $u \in H_{loc}^m(\mathbb{R}^d; \mathbb{R}^p)$ and $Q \in \mathbb{R}^p \otimes (\mathbb{R}^d)^{\otimes m}$ is a quadratic elliptic form:

$$Q(\eta \otimes (\xi)^{\otimes m}) \gtrsim |\eta| |\xi|^m.$$

The problem arises in the study of various shape optimization problems and recently has attracted the attention of several researchers, [6, 7, 10]. The first order isotropic case:

$$(B) \quad \min \int |\nabla u| + |\{u \neq 0\}|.$$

has been widely studied since the seminal work of Alt-Caffarelli [1] (where they study the *one-phase case* $u > 0$) and Alt-Caffarelli-Friedman (for the two phase case)[2]. More recently, vectorial problems, $u \in H^1(\mathbb{R}^d; \mathbb{R}^p)$, have been studied in conjunction with optimization of Laplacian eigenvalues [9, 10, 6, 7, 3]. In order to study local behavior of minimizers of (HOB) next to free boundary points, a first key step is to establish *optimal regularity* of u . A simple dimensional analysis argument shows that the expected regularity should be $u \in W^{m,\infty}$. Regularity of this type was established for minimizers of (B) in [1], for the two phase case in [2] and for the vectorial case in [10]. All these proofs strongly rely either on the use of a Harnack type inequality for positive solutions of an elliptic PDE or on the *Alt-Caffarelli-Friedman* monotonicity formula, [2]. In particular, it has been unknown how to extend these results to minimizers of (HOB) and to date the best known result is the *almost optimal regularity* of [5]. We introduce a new variational technique to establish optimal regularity that relies neither on the maximum principle nor on the monotonicity formula. This allows us to prove optimal regularity for local minimizers of (HOB) and also for problems involving side differential constraints, as those appearing in the optimal drag problem. These ideas can be coupled with harmonic analysis techniques to also show Lipschitz regularity of *two dimensional stationary* points of (B), recently studied in [8].

In a nutshell, the main idea behind the proof is to locally compare minimizers of (HOB) with the Q -harmonic replacement $v_{x_0,R}$ in a ball $B_R(x_0)$. This comparison is used to deduce that

$$\int_{B_R(x_0)} |\nabla^m u - \nabla^m v_{x_0,R}|^2 \lesssim |\{u = 0\}|.$$

By integrating the left hand side only on the set where $u = 0$ and exploiting elliptic estimates for v , one deduces a dichotomy of the form:

- (a) either $\int \nabla^m u = O(1)$
- (b) or $|\{u = 0\} \cap B_{\frac{R}{2}}| \ll |\{u = 0\} \cap B_R|$.

Since the latter alternative can not hold for infinitely many scales, we deduce the desired regularity, see [4] for the details.

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Stability and minimality of the ball for weakly repulsive nonlocal energies

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(joint work with Garrett Moore)

Since their introduction in [1], geometric minimization problems that involve non-local interaction energies of the form

$$(1) \quad \mathcal{E}(E) = \int_E \int_E K(x - y) \, dx \, dy$$

defined over *measurable sets* $E \subset \mathbb{R}^d$ of finite measure $|E| = m$ have attracted significant attention (see [3, 7, 8, 9, 10]). These works consider interaction kernels K in the power-law form

$$K(x) = \frac{|x|^q}{q} - \frac{|x|^p}{p}$$

with various choices of powers $-d < p < q$. The advantage of these power-law kernels is that they not only capture different type of singularity behavior at the origin (strong versus weak repulsion) but also different tail behavior at infinity (strong versus weak attraction) (see Figure 1). Another advantage of working with power-law kernels is that the attractive and repulsive effects are separated as the term with power q corresponds to attractive forces and the term with power p to repulsive forces.

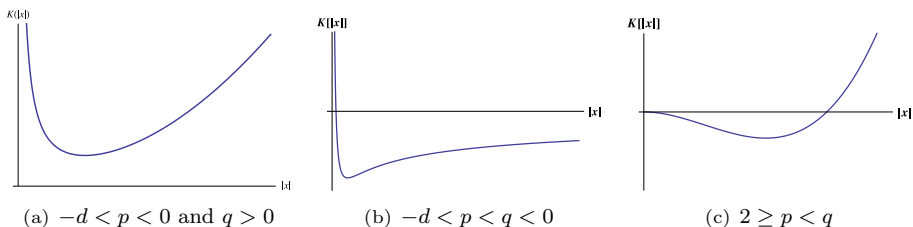


FIGURE 1. Generic examples of K for various values of q and p .

In the strong repulsion and strong attraction regime ($-d < p < 0 < q$), Frank and Lieb [8] prove that for all $q > 0$ and $1 - d < p < 0$ there is a threshold $m_{\text{ball}} \in (0, \infty)$ such that the ball with volume m is the only (up to translation) minimizer for $m > m_{\text{ball}}$. It is also observed in [8], for $-d < p < 1 - d$ balls are never minimizers. Concerning the small volume regime, in [1] it is shown that for $q = 2$ and $-d < p \leq 2 - d$ the energy $\mathcal{E}$ does not admit a minimizer for m sufficiently small. In [7] the same is proved for $d = 3$, $p = -1$, and any $q > 0$. In a joint work with Cristoferi and Bonacini [2], we investigate the *stability and local optimality of the ball*, and characterize the exact range of volumes for which the ball is a volume-constrained stable set for $\mathcal{E}$ (in the sense of nonnegativity of the second variation of the energy with respect to smooth perturbations of the boundary of the ball). Moreover, we prove that the strict stability of the ball yields its (quantitative) local minimality among sets with the same volume and whose boundary is contained in a small uniform neighborhood of the boundary of the ball.

In the weak repulsion and strong attraction regime ($2 \leq p < q$) the picture is more interesting. In a joint work with Carazzato and Pratelli [4], we prove that these energies admit minimizers when m is sufficiently small (or even for every m , in particular cases). For some choices of the parameters p and q we are able to give a certain characterization of minimizing sets. Our results are closely related to the seminal work by Davies, Kim, and McCann [5, 6] where they determine conditions

on p, q when minimizers of interaction energies over probability measures are purely atomic or concentrate on spheres of $(d - 1)$ -dimensions. Specifically, we prove the following.

Theorem 1 (Theorems 3.6 and 3.8 in [4]).

- (1) *Let $p = 2$ and $2 < q < 4$. For every $m > 0$, the unique (up to translations) minimizer of $\mathcal{E}$ among sets of volume m is either an annulus or a ball.*
- (2) *Let $q > p \geq 2$, $q \geq 2$, and $(q, p) \neq (4, 2)$. For m sufficiently small, the unique (up to translations) minimizer of $\mathcal{E}$ among sets of volume m consists of $d + 1$ convex components contained in a small neighborhood of a vertex of the d -simplex.*

On the other hand, also in this regime it is possible to apply the arguments of Frank and Lieb to obtain that when m is sufficiently large the unique minimizer of $\mathcal{E}$ (up to translations) is given by $B[m]$, the ball of volume m . This leaves the questions of whether minimizers exist, and if so what they look like, for intermediate values of m . As a first step in this direction, in a joint work with Garrett Moore, we investigate the stability and local minimality of $B[m]$ in the regime $2 \leq p < q$. For this we rescale the energy, and consider

$$\mathcal{E}_\gamma(E) = \int_E \int_E \gamma |x - y|^q - |x - y|^p \, dx \, dy$$

over sets $|E| = |B_1|$, where B_1 denotes the unit ball in $\mathbb{R}^d$.

Calculating the second variation around B_1 via a volume-preserving flow, and expanding in spherical harmonics, one obtains that B_1 is a volume-constrained stable critical set for $\mathcal{E}_\gamma$ if and only if

$$\gamma \geq \gamma_* = \sup_{k \geq 2} \frac{\mu_1(p) - \mu_k(p)}{\mu_1(q) - \mu_k(q)},$$

where μ_k are the eigenvalues corresponding to the attractive and repulsive part of the energy. Up to a constant that depends on d, p , and q , the stability threshold γ_* is determined by the supremum of some explicitly given sequence X_k . Using the properties of this sequence we show that in the regime where the minimizers are either annuli or balls for all m , this supremum is one, and it is not achieved by a finite mode, indicating a fine mixing of oscillations, whereas in the regime where the minimizers consist of disconnected components, the supremum of this sequence is achieved by the third mode, indicating the direction at which B_1 loses its stability. More specifically, we prove the following.

Theorem 2.

- (1) *Let $p = 2$ and $2 < q < 4$. Then $X_k < 1$ for all $k \geq 2$, $(X_k)_{k \geq 2}$ is monotone increasing, and $\sup_{k \geq 2} X_k = 1$.*
- (2) *Let $q > p \geq 2$, $q \geq 2$, and $(q, p) \neq (4, 2)$. Then $\sup_{k \geq 2} X_k = X_3$.*

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Measure 0 of the singular set for 2-valued stationary hypercurrents

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(joint work with Jonas Hirsch, Luca Spolaor)

1. INTRODUCTION

The main focus of our research is to investigate the optimal regularity of integral cycles whose associated integral varifold is stationary – a measure-theoretic generalization of oriented minimal hypersurfaces – without any further variational assumption, such as minimality or stability.

In his groundbreaking work [1], Allard proved that the singular set of stationary integral varifolds is closed and nowhere dense. Since then little to no progress has been made on the question of the optimal dimension of the singular set for stationary integral varifolds (see [2, 3]). In this note we address this question under two assumptions: a density bound, $\Theta < 3$, and the fact that the varifold is induced by an integral cycle with co-dimension one. We do this by applying Almgren’s strategy in the stationary setting, that is without any minimizing (nor stability) assumption or any other additional topological assumption.

Recall that $p \in \text{spt}(T) \cap U$ is a *regular* point if in a neighbourhood of p the current T coincides with an integer multiple of the current of integration over a smooth embedded minimal hypersurface, and we set $\text{reg}_U(T)$ for the set of such points and $\text{sing}_U(T) = (\text{spt}(T) \cap U) \setminus \text{reg}_U(T)$, which is relatively closed in U .

Our main result is the following:

Theorem 1. *Let $U \subset \mathbb{R}^{m+1}$ be open and let $T \in \mathbf{I}_m(U)$ satisfy*

$$(\partial T) \llcorner U = 0 \quad \text{and} \quad \delta V(T) = 0 \quad \text{in } U.$$

Suppose moreover that

$$\Theta_{V(T)}(p) < 3 \quad \forall p \in \text{spt}(T) \cap U,$$

where $\Theta_{V(T)}(p)$ denotes the density of the associated varifold. Then

(i)

$$\mathcal{H}^m(\text{sing}_U(T)) = 0.$$

in particular, $\text{reg}_U(T)$ is open and dense in $\text{spt}(T) \cap U$, and the local multiplicity there is either 1 or 2.

(ii) *More precisely, there is an open set $\mathcal{O} \subset U$ such that $\|T\|(U \setminus \mathcal{O}) = 0$ and*

$$\dim_{\mathcal{H}}(\text{sing}_U(T) \cap \mathcal{O}) \leq m - 1.$$

This open set $\mathcal{O}$

The set $\mathcal{O}$ in (ii) is the (open) set of points admitting a neighborhood in which the hypotheses of Theorem 7 hold, i.e. in which the current T is close to a multiplicity two plane coherently oriented; its complement is $\|T\|$ -negligible but is exactly where a catenoid-type degeneration as in Remark 3 is not excluded.

Several remarks are in order.

Remark 2 (Ambient manifolds). *We expect the conclusion to extend, with the corresponding density and stationarity hypotheses, to currents in a $C^{2,\alpha}$ Riemannian ambient manifold. The present paper proves only the Euclidean statement.*

Remark 3 (General stationary varifolds). *Using the results of this paper with a simple modification of [2] in the construction of the center manifolds, would establish the following fact: if V is a stationary integral varifold in general dimension m and codimension n with $\Theta_V < 3$ satisfying a reverse-Hölder-type estimate,*

$$(1) \quad \mathbf{e}_2(B_x) \leq C \left(\frac{\mathbf{e}_2(2B_x)}{|2B_x|} \right)^{1/2} \mathbf{e}_1(2B_x),$$

for each ball $B_x \subset U$ satisfying additional assumptions on the flatness, where $\mathbf{e}_1, \mathbf{e}_2$ are the following un-oriented excess measures

$$\begin{aligned} \mathbf{e}_2(A) &:= \int_{A \times \mathbb{R}} \frac{1}{2} |\mathbf{p}_{T_z V(T)} - \mathbf{p}_0|^2 d\|T\|(z) \\ \mathbf{e}_1(A) &:= \int_{A \times \mathbb{R}} \left(\frac{1}{2} |\mathbf{p}_{T_z V(T)} - \mathbf{p}_0|^2 \right)^{1/2} d\|T\|(z). \end{aligned}$$

Then the dimension of its singular set is at most $m - 1$. However such assumption is not valid along the conjectured example of a sequence of unstable catenoids blowing down to a double plane (see for instance [5]), so we decided not to derive our main theorem as a corollary of this more general, but conditional, result.

In our setting we use the current structure to show that in a neighborhood of almost every point of multiplicity 2 the topology of the multiplicity 1 part is trivial, from which (1) follows. The observation that topological control leads to “regularity” has already been used in [2] and [3]. In the former, this follows naturally since Lipschitz graphs are investigated; in the latter, it is imposed.

Remark 4 (Brakke’s example). *Brakke [6] constructed a two-dimensional integral varifold in $\mathbb{R}^3$ with bounded mean curvature and density at most 2 whose singular set has positive $\mathcal{H}^2$ -measure. Thus it is essential in Theorem 1 that the mean curvature is vanishing.*

Prior to our result here, we are aware of only two results on the dimension of the singular set of general stationary varifolds; both make some additional regularity or topological assumptions.

- In [2] we prove that the singular set of a 2-valued Lipschitz graph of dimension m that is stationary for the area is of dimension at most $m - 1$. The strategy there is similar to the one employed in this paper, but our assumptions here are more general, in particular we do not exclude the presence of infinite topology neither on the multiplicity 1 nor on the multiplicity 2 parts of the current.
- In [3] the authors prove an Allard type regularity result for stationary varifolds close to multiplicity 2 planes, under a suitable topological condition, which essentially rules out complicated topology in the multiplicity 1 parts close to points of multiplicity 2 of the varifold. In some sense their result should be compared to Remark 3: [3] proves an Allard’s regularity result plus optimal dimensional bound of the singular set in the multiplicity two case, while we could prove only the optimal dimensional bound of the singular set, but with no multiplicity assumptions. However, as mentioned above, both of these results are conditional on a topological/analytical assumption that is known to fail, in general. There are, however, situations where they can be verified, and we refer to [3] for the interested reader.

Remark 5. *Combining the present work, and the results in [3] and [4] it is possible to show that the dimension of the branching set in $\mathcal{O}$ is at most $(m-2)$ dimensional. However, we are not able to conclude the same dimensional bound for the whole branching set, which a priori is only of measure zero.*

2. SKETCH OF THE PROOF

The proof has four steps. (a) by the classical Almgren-Federer stratification up to the set of dimension $m - 1$ $V(T)$ has a flat tangent cone, i.e. we find a cylinder $\mathbf{C} = \mathbf{C}_r(x)$ in which the un-oriented excess is arbitrary small. Since by assumption $\Theta_{V(T)}(x) < 3$ we have either $\|T\|(\mathbf{C}) \leq (1 + \epsilon)\omega_m r^m$, in which case Allard’s regularity theorem applies, or $|\frac{\|T\|(\mathbf{C})}{\omega_m r^m} - 2| \leq \epsilon$. In the latter case, we distinguish the only two possibilities: $(\mathbf{p}_0)_\# T = 2 \llbracket B_r(x) \rrbracket$ and $(\mathbf{p}_0)_\# T = 0$. This follows from classical slicing theory of currents. The latter set has $\|T\|$ -measure 0 by Lebesgue

differentiation, since $\frac{dT}{d\|T\|}$ exists $\|T\|$ -a.e. . The union of such cylinders where one of the first two possibilities occurs is the open set $\mathcal{O}$ of Theorem 1.

(b) It remains to analyse the middle situation i.e. a cylinder $\mathbf{C}$ in which the non-oriented excess is small and $\|T\|(\mathbf{C}_r(x)) \leq (2 + \epsilon)\omega_m r^m$, both together give small oriented excess. In such a cylinder we verify (1) for every ball $B_x \subset \mathbf{C}$ satisfying the flatness assumptions by distinguishing two cases: if B_x contains a point of density $\Theta_{V(T)} \geq 2$, the estimate follows from a Hardt–Simon type estimate, [7, Theorem 2.2] or [5]; if not, then $\Theta_{V(T)} < 2$ on B_x and we can exploit homological structure of the current to deduce that T separates into two disjoint smooth sheets (Lemma 6), for which (1) is a consequence of the standard excess estimate. (c) Gehring’s lemma upgrades (1) to a higher integrability of the excess density. (d) With this gain the center-manifold and frequency-function scheme of [2] applies with minor modifications and gives the optimal dimensional bound for $\text{sing}(T) \cap \mathbf{C}$, see Theorem 7.

These four steps establish indeed the theorem since: $\text{sing}(T)$ is covered by the stratification set (dimension $\leq m - 1$), the projection-0 set ($\mathcal{H}^m$ -0) and countably many “good” cylinders (dimension $\leq m - 1$).

As mentioned above the key insight is the following “a priori” topological control:

Lemma 6 (Separation lemma). *There exists $\epsilon_{or}(m) > 0$ such that if T is a stationary current with*

- (1) $(\partial T) \llcorner \mathbf{C}_8 = 0$;
- (2) $(\mathbf{p}_0)_\# T = 2 \llbracket B_8 \rrbracket$ as currents in B_8 ;
- (3) $\mathbf{E}_{or}(T, \mathbf{C}_8, \pi_0) \leq \epsilon_{or}$;
- (4) $\Theta_{V(T)}(x) < 2$ for all $x \in \text{spt}(T) \cap \mathbf{C}_2$;

then there exist disjoint smooth embedded m -dimensional manifolds M_1, M_2 such that $T \llcorner \mathbf{C}_1 = \llbracket M_1 \rrbracket + \llbracket M_2 \rrbracket$.

As outlined above it can be combined with a Hardt–Simon estimate and a Gehring type lemma to obtain a higher integrability of the excess density. And then following with small modifications the strategy developed in [2], to obtain the following optimal dimension bound for the small oriented excess regime:

Theorem 7 (Small oriented excess regime). *There exists $\epsilon_{or} = \epsilon_{or}(m) > 0$ with the following property. Let $T \in \mathbf{I}_m(\mathbf{C}_{R_*})$ be an integral cycle with $\|T\|(\mathbf{C}_{R_*}) < \infty$ and*

$$\text{spt}(T) \cap \mathbf{C}_{R_*} \subset B_{R_*} \times [-L, L] \quad \text{for some } L < \infty.$$

Suppose that $V(T)$ is stationary in $\mathbf{C}_{R_*}$. Assume that, as currents in B_{R_*} ,

$$(2) \quad (\mathbf{p}_0)_\# T = 2 \llbracket B_{R_*} \rrbracket,$$

that

$$(3) \quad \mathbf{E}_{or}(T, \mathbf{C}_{R_*}, \pi_0) < \epsilon_{or},$$

and that

$$(4) \quad \Theta_{V(T)}(p) < 3 \quad \text{for every } p \in \text{spt}(T) \cap \mathbf{C}_{R_*}.$$

Then

$$\dim_{\mathcal{H}}(\text{sing}_{\mathbf{C}_{R_*}}(T) \cap \mathbf{C}_1) \leq m - 1.$$

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Existence, Non-existence, and Regularity of Minimizers in Weighted Fractional Cluster Problems

KENNETH DEMASON

(joint work with Lia Bronsard, Vincent Millot)

In recent years there has been a resurgence of attention concerning minimizing clusters in $\mathbb{R}^n$. We call an N -cluster in $\Omega \subset \mathbb{R}^n$ (either an open, bounded, smooth set or $\mathbb{R}^n$ itself), denoted by $\mathcal{E}$, a collection of N -many sets of finite perimeter $\{E_i\}_{i=1}^N$, called the chambers of $\mathcal{E}$, and denote by $E_0 = \Omega \setminus \bigcup_{i=1}^N E_i$ the exterior chamber. The weighted perimeter of a cluster, relative to Ω , is defined as

$$P^\sigma(\mathcal{E}) := \frac{1}{2} \sum_{i \neq j} \sigma_{ij} \mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega)$$

where the $\sigma_{ij} = \sigma_{ji} > 0$ for $i \neq j$, $\sigma_{ii} = 0$. The volume of a cluster $|\mathcal{E}|$ is the vector of volumes of its chambers $(|E_1|, \dots, |E_N|)$. Sullivan conjectured in [11] that with $\sigma_{ij} = 1$ and $N \leq n + 1$, volume-constrained minimizers of P^σ with $\Omega = \mathbb{R}^n$ are exactly so-called standard bubbles; this was confirmed for $N = 2$ by [6] in dimension $n = 3$ and [10] in dimension $n \geq 4$, and for $N = 3, 4, 5$ by [8, 9].

One natural way to interpret P^σ is via a vectorial Allen-Cahn problem. Specifically, for $u : \Omega \rightarrow \mathbb{R}^d$ and $W : \mathbb{R}^d \rightarrow [0, \infty)$, with $\{W = 0\} = \{a_0, \dots, a_N\}$, the Allen-Cahn energy is

$$J_\epsilon(u) = \int_{\Omega} \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(u) dx \quad \text{with } \epsilon \in (0, 1).$$

whose critical points satisfy a vectorial Allen-Cahn equation

$$-\Delta u_\epsilon + \frac{1}{\epsilon^2} \nabla W(u_\epsilon) = 0 \quad \text{in } \Omega.$$

The functional P^σ is obtained in the following way: we have $J_\epsilon \xrightarrow{\Gamma} P^\sigma$ with

$$\sigma_{ij} = \inf \left\{ \int_0^1 2\sqrt{W}(\gamma(t)) |\gamma'(t)| dt \mid \gamma \in C^1([0, 1]; \mathbb{R}^d), \gamma(0) = a_i, \gamma(1) = a_j \right\}$$

and roughly speaking $E_i = \{u = a_i\}$.

In [4], Gabard and Millot consider a fractional version of the Allen-Cahn problem, where critical points of an appropriate non-local Allen-Cahn energy $J_{2s, \epsilon}$ satisfy instead

$$(-\Delta)^s u_\epsilon + \frac{1}{\epsilon^2} \nabla W(u_\epsilon) = 0 \quad \text{in } \Omega.$$

Here, we have $J_{2s, \epsilon} \xrightarrow{\Gamma} P_{2s}^\sigma$ where, for an N -cluster $\mathcal{E}$ in Ω ,

$$P_{2s}^\sigma(\mathcal{E}) = \frac{1}{2} \sum_{i \neq j} \sigma_{ij} I_{2s}(E_i, E_j), \quad I_{2s}(A, B) := \int_A \int_B \frac{dx dy}{|x - y|^{n+2s}},$$

and in contrast to the classical case, $\sigma_{ij} = |a_i - a_j|^2$.

We highlight two features of the fractional vectorial Allen-Cahn energy: first, there is a useful monotonicity formula, unlike in the classical setting, which allows to deduce the standard regularity theory [4]; second, as $s \rightarrow 1/2^-$, after multiplying by $(1 - 2s)$, the energy Γ -converges to a “relaxed” weighted classical perimeter P^σ , as was shown in [5]. It is therefore interesting to study volume-constrained minimizers of P_{2s}^σ as a way of studying classical isoperimetric clusters. This is made accessible by the regularity theory of the fractional vectorial Allen-Cahn problem.

In [1], Almgren proved the existence of volume-constrained minimizers in $\Omega = \mathbb{R}^n$ of P^σ whenever the σ_{ij} satisfy a triangle inequality: $\sigma_{ij} \leq \sigma_{ik} + \sigma_{jk}$. It is natural to assume such a condition in light of the connection to the vectorial Allen-Cahn problem, where the weights are obtained by a suitable geodesic distance. The partial regularity of minimizers was obtained in [7] when $\sigma_{ij} < \sigma_{ik} + \sigma_{jk}$ by using an infiltration lemma. In the fractional setting, we mention the works [2] and [3], who prove for specific choices of σ_{ij} the existence (in $\mathbb{R}^n$) and regularity of volume-constrained minimizers of P_{2s}^σ .

By viewing P_{2s}^σ as the Γ -limit of $J_{2s, \epsilon}$, we are able to appeal to the regularity theory developed by [4] for the fractional vectorial Allen-Cahn problem and use this to deduce the expected partial regularity of stationary clusters for P_{2s}^σ . That is, we show that $\partial \mathcal{E} := \bigcup_{i=1}^N \partial^* E_i$ is locally smooth away from a set of Hausdorff dimension at most $(n - 2)$. Again in contrast to the classical setting, we are able to do this even when a strict triangle inequality for the weights fails; in this case we show that $\partial^* E_i \cap \partial^* E_j$ is part of the singular set. As for the existence of volume-constrained minimizing fractional clusters, in $\mathbb{R}^n$ this presents a real challenge since the cluster could separate; the non-local interaction then forces these pieces to diverge, and hence there is no compactness. When $\sigma_{ij} < \sigma_{i0} + \sigma_{j0}$ we believe this is not possible, and thus we would have existence. For $N = 2$, when $\sigma_{12} > \sigma_{10} + \sigma_{20}$, the cluster separates into two balls whose centers diverge, and hence there is no existence.

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An a priori estimate for the energy growth of minimizing entire solutions of the Ginzburg-Landau equation in $\mathbb{R}^n$, $n \geq 2$.

LUANA JOST

(joint work with Étienne Sandier)

In this talk, we study entire solutions $u : \mathbb{R}^n \rightarrow \mathbb{R}^2$, $n \geq 2$, of the Ginzburg-Landau equation

$$(1) \quad -\Delta u = u(1 - |u|^2) \quad \text{in } \mathbb{R}^n$$

that are local minimizers of the associated Ginzburg-Landau energy

$$E(u, \Omega) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2 dx,$$

defined for every open and bounded set $\Omega \subset \mathbb{R}^n$.

One motivation for studying these solutions is their classification. In two dimensions, it follows from results of Mironescu [2] and Sandier [3] that, up to symmetry, every solution is either constant or equal to the so-called vortex solution $u_0(r, \theta) = f(r)e^{\pm i\theta}$, where f is the unique solution to the ODE corresponding to (1).

In higher dimensions much less is known. In the presence of a magnetic field, De Philippis, Halavati and Pigati showed in [1] that if a local minimizer (u, ∇) of the self-dual $U(1)$ -Yang-Mills-Higgs model satisfies an upper bound of the form

$$E_{\text{Mag}}(u, \nabla, B_R^n) \leq (2\pi + \delta)\omega_{n-2}R^{n-2},$$

then u and ∇ depend on two variables only. Even though the corresponding result in the absence of a magnetic field is not yet known, it is natural to establish an upper energy bound as a first step towards some classification result.

Following the strategy in [3], we prove the following theorem.

Theorem 1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^2$, $n \geq 2$, be a local minimizer of the Ginzburg-Landau energy. Then, for any $\delta > 0$,*

$$\lim_{R \rightarrow \infty} \frac{E(u, B_R^n)}{R^{n-\frac{4}{3}+\delta}} = 0.$$

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Concentration phenomena and the Vanishing Mass Conjecture

LUCA GENNAIOLI

(joint work with Filip Rindler)

The study of the structure of measures satisfying PDEs has a long history. One of the first influential results dates back to 1993 (see [1]), in the form of the following.

Theorem 1 (Rank-One Theorem). *Let $u \in \text{BV}(\mathbb{R}^d; \mathbb{R}^n)$, then*

$$\text{rank} \left(\frac{dDu^s}{d|Du|^s}(x) \right) \leq 1, \quad \text{for } |Du|^s - \text{almost every } x \in \mathbb{R}^n.$$

After roughly 23 years, a more general result has been proven by De Philippis and Rindler (cf. [2, Theorem 1.1]).

Theorem 2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^m)$ be a finite Radon measure such that*

$$\mathcal{A}\mu = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^m),$$

where $\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$ is a k -homogeneous, linear and constant coefficients differential operator. Then

$$\frac{d\mu^s}{d|\mu|^s}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu|^s - \text{almost every } x,$$

where $\Lambda_{\mathcal{A}}$ is the wave cone associated to $\mathcal{A}$, defined as

$$\Lambda_{\mathcal{A}} = \bigcup_{|\xi|=1} \text{Ker} \mathbb{A}(\xi), \quad \text{with } \mathbb{A}(\xi) = \sum_{|\alpha|=k} A_\alpha \xi_\alpha.$$

One should observe that the choice $\mathcal{A} = \text{curl}$ in the above Theorem recovers the Rank-One Theorem for BV functions.

The philosophy behind this result is that the singularities of measures satisfying PDEs are constrained to lie where the PDE “stops from being elliptic” (namely in the wave cone $\Lambda_{\mathcal{A}}$).

In our recent preprint, together with Filip Rindler, we study this question for sequences of functions which behave in a singular way and satisfy some PDE. More precisely, we consider a sequence of functions $(u_j)_{j \in \mathbb{N}} \subset C^\infty(Q; V)$ (where Q is the unit cube of $\mathbb{R}^d$ and V is a Euclidean vector space) such that $\|u_j\|_{L^1} = 1$ for all $j \in \mathbb{N}$, $|u_j| \rightarrow 0$ in measure, and $\mathcal{A}u_j = 0$ for all $j \in \mathbb{N}$ for a linear, constant-coefficients, differential operator $\mathcal{A}$ of *first order*. Then we have the following (cf. [3, Theorem 1.5]).

Theorem 3. *Let $(u_j)_{j \in \mathbb{N}}$ be a sequence as above and consider any measure $\mu \in \mathcal{P}(\mathbb{S}_V)$ arising as (possibly up to a subsequence)*

$$\mu = w^* - \lim_{j \rightarrow \infty} \left(\frac{u_j}{|u_j|} \right) (|u_j| dx_Q).$$

Then there exists $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{S}_V))$ such that

$$\mu = \int_{\mathbb{S}_V} \nu d\pi(\nu),$$

where

$$[\nu] = \int_{\mathbb{S}_V} \text{Ad}\nu(A) \in \Lambda_{\mathcal{A}} \quad \text{for } \pi\text{-almost every } \nu.$$

Theorem 3, despite being for first order operators only, is more general than Theorem 2, since it allows to describe behaviors which are called *diffuse concentrations*, i.e. sequences u_j as above whose weak*-limit is absolutely continuous with respect to the Lebesgue measure (while Theorem 2 only addresses singular measures). What Theorem 3 is ultimately saying is that concentrations can be described in elementary terms as superposition of certain predefined and simpler structures. To highlight the importance of Theorem 3 we write two of its consequences. The first one is the following (cf. [3, Theorem 1.8])

Theorem 4. *Let $\mu_j \in \mathcal{M}(Q; V)$ be a sequence of measures such that $\mathcal{A}\mu_j = 0$. Let $K \subset V$ be a closed convex cone such that $K \cap (-K) = \{0\}$ and such that $K \cap \Lambda_{\mathcal{A}} = \{0\}$. Assume that*

$$\lim_{j \rightarrow \infty} \int_{\Omega} \text{dist} \left(\frac{d\mu_j}{d|\mu_j|}(x), K \right) d|\mu_j|(x) = 0.$$

Then, for every $\Omega' \subseteq \Omega$, writing $\mu_j|_{\Omega'} = g_j dx|_{\Omega'} + \mu_j^s$ with $g_j \in L^1(\Omega')$, it holds that the family $(g_j)_j$ is equi-integrable in $L^1(\Omega'; V)$ and $|\mu_j^s|(\Omega') \rightarrow 0$ as $j \rightarrow \infty$. Moreover, if even

$$\frac{d\mu_j}{d|\mu_j|}(x) \in K \quad \text{for } |\mu_j|\text{-a.e. } x \text{ and for all } j \in \mathbb{N},$$

then $\mu_j^s = 0$ and there is $p = p(K) \in (1, \infty)$ (implicitly also depending on $\mathcal{A}$) such that for all $\Omega' \subseteq \Omega$,

$$\sup_{j \in \mathbb{N}} \|\mu_j\|_{L^p(\Omega')} < \infty.$$

The second one instead concerns the validity of Morrey's conjecture for linear growth $\Lambda_{\mathcal{A}}$ -convex functions when tested against concentrations (cf. [3, Corollary 1.10])

Theorem 5. *Let $(u_j)_j$ and $\mathcal{A}$ be as above, with the additional assumptions that $\mathcal{A}$ is of constant rank and $\Lambda_{\mathcal{A}}$ is spanning. Then, for every $f : V \rightarrow \mathbb{R}$ that is $\Lambda_{\mathcal{A}}$ -convex, non-negative on $\Lambda_{\mathcal{A}}$, and for which f^∞ exists, it holds that*

$$\liminf_{j \rightarrow \infty} \int_Q f(A + u_j) dx \geq f(A) \quad \text{for all } A \in V.$$

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A characterization of submanifolds of $\mathbb{R}^{m \times n}$ satisfying Frieesecke–James–Müller-type estimates

MALTE BORKEN

In 2002, Frieesecke, James and Müller [1] proved that $\text{SO}(n)$ satisfies the following optimal rigidity estimate: If $u \in H^1(\Omega; \mathbb{R}^n)$ is a vector field defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, then it satisfies

$$\|Du - (Du)_\Omega\|_{L^2(\Omega)} \leq C(\Omega) \|\text{dist}_{\text{SO}(n)}(Du)\|_{L^2(\Omega)},$$

where $(Du)_\Omega := \int_\Omega Du$ and $\text{dist}_{\text{SO}(n)}(Du(x)) := \text{dist}(Du(x), \text{SO}(n))$. We are interested in the following question: Exactly which structural properties of the set $\text{SO}(n)$ enable it to satisfy the aforementioned estimate? To make this precise, we consider the following notions of rigidity:

Definition 1. *Let $H \subset \mathbb{R}^{m \times n}$ be closed and $p \in (1, \infty)$. Denote $Q := (0, 1)^n$. We say that H satisfies*

- (i) *exact rigidity if every map $u \in W^{1,\infty}(Q; \mathbb{R}^m)$ satisfying the differential inclusion $Du \in H$ is affine.*
- (ii) *sequential rigidity if for every weakly- $*$ convergent sequence $(u_j)_{j \in \mathbb{N}} \subset W^{1,\infty}(Q; \mathbb{R}^m)$ satisfying*

$$\text{dist}_H(Du_j) \rightarrow 0 \quad \text{in measure,}$$

there exists $A \in H$ such that

$$Du_j \rightarrow A \quad \text{in measure.}$$

(iii) L^p quantitative rigidity if there exists $C > 0$ such that

$$\|Du - (Du)_Q\|_{L^p} \leq C \|\text{dist}_H(Du)\|_{L^p} \quad \forall u \in W^{1,p}(Q; \mathbb{R}^m).$$

The more precise question is then: *What are necessary and sufficient conditions for a compact submanifold $K \subset \mathbb{R}^{m \times n}$ to satisfy L^p quantitative rigidity?*

The case of smooth, connected 1-manifolds of $\mathbb{R}^{2 \times 2}$ has recently been settled by Lamy, Lorent and Peng [3], however, their proof relies heavily on the 2-dimensional setting. We will discuss a new approach to the problem, which allows one to prove necessary and sufficient conditions for a general compact C^1 -submanifold of $\mathbb{R}^{m \times n}$ to satisfy such a rigidity estimate.

Theorem 2. *Let $K \subset \mathbb{R}^{m \times n}$ be a compact C^1 -submanifold with boundary and $p \in (1, \infty)$. Then K satisfies L^p quantitative rigidity if and only if the following hold.*

- (a) *K satisfies sequential rigidity.*
- (b) *For every $A \in K$, the tangent space to K at A satisfies exact rigidity.*

Furthermore, this approach allows one to show that L^p quantitative rigidity is stable under small graphical perturbations of the submanifold.

Theorem 3. *Let $K \subset \mathbb{R}^{m \times n}$ be a compact C^1 -submanifold with boundary satisfying L^p quantitative rigidity, where $p \in (1, \infty)$. Then there exists a constant $\rho(K, p) > 0$ such that the following holds. If $\Phi : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ is bi-Lipschitz with*

$$\text{Lip}(\Phi - \text{id}) \leq \rho,$$

then $\Phi(K)$ satisfies L^p quantitative rigidity as well. Moreover, the constant in the rigidity estimate can be controlled uniformly.

The idea of the proof of the “if” statement in Theorem 2 is as follows. We linearize dist_K around some suitably chosen point $A \in K$, with the goal of eventually reducing the rigidity estimate for K to a rigidity estimate for $T_A K$. The latter follows from condition (b) by singular integral estimates. We begin by splitting Q into a superlevel set

$$S := \{x \in Q : |Du(x) - (Du)_Q| > \delta\},$$

and the corresponding sublevel set $Q \setminus S$. The constant δ is chosen sufficiently small so that on $Q \setminus S$, the remainder term arising from the linearization can be controlled. The main difficulty lies in showing that $|Du - (Du)_Q|^p$ cannot concentrate too much on the superlevel set S . To this end, consider

$$\zeta(s) := \sup_{\|\text{dist}_K(Du)\|_{L^p} \leq s} \|Du - (Du)_Q\|_{L^p},$$

and note that (a) is equivalent to

$$\zeta(s) \rightarrow 0 \quad \text{as } s \rightarrow 0.$$

Moreover, by scaling invariance we have

$$(1) \quad \left(\int_{\tilde{Q}} |Du - (Du)_{\tilde{Q}}|^p dx \right)^{\frac{1}{p}} \leq \zeta \left(\left[\int_{\tilde{Q}} \text{dist}_K^p(Du) dx \right]^{\frac{1}{p}} \right)$$

for all cubes $\tilde{Q} = rQ + y \subset Q$ and $u \in W^{1,p}(Q; \mathbb{R}^m)$. Letting

$$E := \{x \in Q : M[\text{dist}_K^p(Du)](x) > s_0^p\}$$

be a small bad set where the maximal function of $\text{dist}_K^p(Du)$ is (relatively) large, we may formally interpret (1) as

$$\|Du\|_{\text{BMO}(Q \setminus E)} \leq \zeta(s_0).$$

This can be made rigorous by proving that Du satisfies an adapted version of the John-Nirenberg inequality [2], which in its original form states that

$$|\{x \in Q : |f(x) - (f)_Q| > t\}| \leq C \exp \left(-\frac{t}{C \|f\|_{\text{BMO}(Q)}} \right) \quad \forall f \in \text{BMO}(Q).$$

This turns out to be just enough to control the concentration of $|Du - (Du)_Q|^p$ on S . The talk is based on the author's master's thesis, which he wrote at the University of Bonn under the supervision of Konstantinos Zemas and Sergio Conti.

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Symmetry breaking in the liquid drop model with screened interactions

MARC PEGON

(joint work with Lia Bronsard, Michael Goldman, Benoît Merlet)

Gamow introduced in 1928 his *liquid drop model* for the atomic nucleus, in which one minimizes, among sets $E \subset \mathbb{R}^3$ of prescribed volume (or mass) m , the energy

$$\mathcal{E}(E) = P(E) + \int_{E \times E} \frac{dx dy}{|x - y|},$$

where P denotes the perimeter. The two terms are in competition: the perimeter favours aggregation into a single round drop, while the Coulomb repulsion favours spreading out and splitting. More generally, for $n \geq 2$ and a nonnegative, radial, locally integrable kernel G on $\mathbb{R}^n$, one considers

$$(1) \quad e_G(m) = \inf \left\{ \mathcal{E}_G(E) := P(E) + \int_{E \times E} G(x-y) dx dy : E \subset \mathbb{R}^n, |E| = m \right\}.$$

The *generalized* liquid drop model usually refers to the case of Riesz kernels $R_\alpha(x) = |x|^{-\alpha}$, $0 < \alpha < n$.

For Riesz kernels, the expected behaviour is governed by a single threshold. Let m_{ball} be the largest mass below which balls minimize (1), let m_{ex} be the threshold above which no minimizer exists, and let m_{crit} be the mass at which one ball of mass m has the same energy as two balls of mass $m/2$. It is conjectured that $m_{\text{ball}} = m_{\text{ex}} = m_{\text{crit}}$, that is, balls are the unique minimizers up to the critical mass, and above it minimizing sequences split into pieces which escape to infinity, so that the problem admits no minimizer. In the physically relevant case $n = 3$, $\alpha = 1$, this conjecture has just been settled by Chodosh and Gianocca [CG26], who show that balls uniquely minimize the energy for $m \leq m_{\text{crit}}$ and that no minimizer exists for $V > m_{\text{crit}}$.

The question addressed in this talk is whether this picture is specific to Riesz interactions, or whether it persists for general repulsive radial kernels. It turns out that it already fails when G decays faster than Riesz kernels at infinity, the relevant quantity being the first moment $\int_{\mathbb{R}^n} |x| G(x) dx$.

If G is integrable with first moment *below* an explicit dimensional constant, then (1) admits minimizers of arbitrarily large mass [Peg21], in sharp contrast with the Riesz case. In addition, after renormalization, minimizers converge to the ball as $m \rightarrow \infty$. In fact, a uniform regularity theory for the associated almost-minimizers shows that, in any dimension $n \geq 2$, they are exactly balls above some critical mass [GMP24]. Fast decay therefore restores existence, but does not seem to produce symmetry-breaking, at least for large masses.

Above the same explicit threshold on the first moment, large balls become unstable. For compactly supported kernels, for which (1) always admits minimizers, this yields large-mass minimizers which are not balls [Peg21]. This is not fully satisfactory either: nothing prevents such minimizers from being disjoint unions of balls, in which case the picture is still essentially Gamow's, the only difference being that the pieces need not be sent to infinity, but merely far apart.

This leaves the real question: for kernels that are radial and radially nonincreasing (so that no anisotropy is built into the energy), can one produce *connected* minimizers which are not balls, and even non-radial ones? Both answers are positive in dimension 3 for screened Coulomb interactions [BMP25]. Consider the truncated Coulomb potentials $R_{1,\kappa}(x) = \mathbf{1}_{|x| < \kappa} |x|^{-1}$ and the Yukawa potentials $Y_{1,\kappa}(x) = e^{-|x|/\kappa} |x|^{-1}$. For both, and for suitable values of the screening length κ , there are masses m for which (1) admits connected minimizers, none of which is a ball and at such masses, minimizers are neither balls nor disjoint countable unions of balls. Connected minimizers which are not balls had in fact already been reached in dimension 2, for Yukawa potentials, by Muratov, Novaga and Simon [MNS25]: in a large-mass, near-critical screening regime, the rescaled energy Γ -converges to a combination of the perimeter and Euler's elastica functional, so that minimizers are no longer disks for suitable parameters. Their analysis suggests, however, that these minimizers are rings, hence *radial*. By contrast, we go further and show that for truncated Coulomb potentials, for some good choice of

the screening parameter and some mass, the problem admits non-radial connected minimizers, and thus obtain genuine symmetry breaking. To our knowledge, these are the first examples of non-radial minimizers in dimension three, and the first non-radial ones in any dimension, within the class of repulsive radial kernels.

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Variational formulation of Euler Equations: new perspectives and open questions

MARTINA ZIZZA

(joint work with Stefan Schiffer)

When describing the flow of an incompressible, inviscid fluid in a vessel $M \subset \mathbb{R}^\nu$, $\nu \geq 2$, which is a compact, simply connected domain with Lipschitz boundary, there are two classical viewpoints: one way is to describe it via the **Eulerian formulation**, i.e. looking at the fluid's velocity given at a fixed point $x \in M$ at times $t > 0$. The Eulerian formulation for example gives rise to the incompressible Euler equations, in the Cauchy Problem formulation:

$$(1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0, \\ u_{t=0} = u_0, \\ u \cdot \hat{n} = 0 \text{ a.e. on } \partial M. \end{cases}$$

Here, $u : [0, T] \times M \rightarrow \mathbb{R}^\nu$ is the fluid velocity, $u_0 : M \rightarrow \mathbb{R}^\nu$ is the initial fluid velocity and $\hat{n}$ is the exterior unit normal. The quantity $p : \mathbb{R} \times M \rightarrow \mathbb{R}$ is the hydrodynamic pressure and its role in the equations is to enforce the incompressibility condition ($\operatorname{div} u = 0$).

Conversely, the **Lagrangian viewpoint**—pioneered by V. Arnold in the 1960s—shifts the focus to the individual trajectories of fluid particles. In this framework, fluid motion is interpreted as the geodesic flow on $\mathcal{D}(M)$, the infinite-dimensional group of volume-preserving diffeomorphisms. More precisely, for each particle $x \in M$, $\phi_t(x) \in M$ represents the position reached by the particle $x \in M$ at

time t , while $t \mapsto \phi_t(x)$ is its *trajectory*. The fluid configuration at time t is a volume-preserving diffeomorphism $\phi_t \in \mathcal{D}(M)$.

The group $\mathcal{D}(M)$ is a metric space, with metric given by

$$(2) \quad \text{dist}_{\mathcal{D}(M)}(f, g) = \inf_{\substack{\{\phi_t\} \subset \mathcal{D}(M), \\ \phi_0=f, \phi_1=g}} \ell\{\phi_t\}_0^1,$$

where ℓ is the *length functional*

$$(3) \quad \ell\{\phi_t\}_0^1 = \begin{cases} \int_0^1 \|\dot{\phi}_t\|_{L^2(M)} dt & \text{if } \phi \in W^{1,1}((0, 1); L^2(M)), \\ \infty & \text{otherwise.} \end{cases}$$

Arnold's profound insight [1] was demonstrating that the critical points of the length functional parametrized with constant speed (the geodesics in $\mathcal{D}(M)$) correspond exactly to smooth solutions of the Euler equations (1).

Despite the elegance of Arnold's perspective, the transition from finite to infinite dimensions introduces incredible challenges, since we do not have a good picture of the geometry of $\mathcal{D}(M)$. Moreover, it is still unclear if we can deduce properties of the two-point boundary value formulation of the Euler Equations (i.e. determine whether there exist geodesics connecting fluid configurations) from the PDE approach. For example, the global well-posedness of Euler Equations has been a classical result for a long time in dimension $\nu = 2$ (see for example [6]), while the existence of geodesics in $\mathcal{D}(M)$ is a wide open problem.

Therefore, determining the geometry of $\mathcal{D}(M)$ can shed some light on the behavior of fluid motion.

1. Bridging Discrete Mathematics and Fluid Dynamics. To address these challenges, we turn to a powerful yet underexplored perspective: the interpretation of fluid configurations as permutations of particles, pioneered by Shnirelman [5] and Brenier [2]. This approach, significantly extended and developed in [7, 3], constructs a conceptual bridge between the continuous realm of Fluid Dynamics and the combinatorial world of Discrete Mathematics.

In this framework, we consider the embedding of $\mathcal{D}(M)$ into the space $L^2(M; \mathbb{R}^\nu)$ of vector functions on $M = [0, 1]^\nu$. The core of the difficulty lies in the regularity of this immersion. We fix $\nu \geq 3$, then in [4] Shnirelman proved that there exists $C > 0$ such that, for every $f, g \in \mathcal{D}(M)$ it holds

$$\|f - g\|_{L^2(M)} \leq \text{dist}_{\mathcal{D}(M)}(f, g) \leq C \|f - g\|_{L^2(M)}^{\frac{2}{4+\nu}}$$

and conjectured this exponent to be sharp. In [3], we proved the following

Theorem 1 (Sharp Shnirelman's inequality). *There exists $C > 0$ such that, for every $f, g \in \mathcal{D}(M)$, it holds*

$$\text{dist}_{\mathcal{D}(M)}(f, g) \leq C \|f - g\|_{L^2(M)}.$$

While this bound is trivial for smooth, finite-dimensional compact manifolds, its validity in the infinite-dimensional setting was long believed to be false. This result reveals that $\mathcal{D}(M)$ behaves as a metrically regular object, contradicting the

previous geometric intuition. The implications of this regularity are profound. If $\mathcal{D}(M)$ is metrically *well-behaved*, what can we infer on its infinitesimal structure?
Is it possible to give a metric description of $\mathcal{D}(M)$?

2. Pipe-flow on discrete networks. In order to solve Theorem 1, in [3] we re-interpreted Shnirelman's inequality as a problem on a discrete network (a graph), translating a purely infinite-dimensional problem into a finite-dimensional one. The discretization of the elements of $\mathcal{D}(M)$ is given by the permutations of subcubes of M (Lax Theorem).

We then transfer the problem on a lattice graph $G = (V, E)$, where each vertex is a subcube of the previous tiling, and two vertices are connected if they are two adjacent cubes. Every permutation is therefore a bijection $\sigma : V \rightarrow V$ onto the set of vertices V . A *flow* in this graph is then represented by a path (or multiple paths) going from a vertex v to $\sigma(v)$.

One has to picture a path in the network as a **pipe**: the portion of fluid contained in a cube travels in a thick pipe (in an incompressible way) in order to reach its final destination. The advantage of the pipe-flow is that the motion of the fluid inside the pipe does not affect the motion of the other particles in the fluid.

Because of the incompressibility condition of the velocity field, and because of the bounded motion (everything should occur inside M), we ask for the pipes to have non-constant thickness. We indicate as $\rho(e, \gamma)$ the thickness of the pipe γ along the edge $e \in E$. Then, we express the incompressibility of the fluid velocity as: $\rho(e, \gamma)u(e, \gamma) = \text{const}$, where $u(e, \gamma)$ represents the velocity of the fluid in the pipe γ along the edge $e \in E$.

The discrete length functional becomes

$$(4) \quad \left(\sum_{\gamma \in \Gamma} \sum_{e \in E(\gamma)} \rho(e, \gamma) |u(e, \gamma)|^2 \right)^{1/2} = \left(\sum_{\gamma \in \Gamma} \sum_{e \in E(\gamma)} \rho(e, \gamma)^{-1} \right)^{1/2},$$

which corresponds to the quantity $\int_0^1 \|\dot{\phi}_t\|_{L_x^2} dt$.

We solve the discrete problem by finding a minimizer (up to a constant) for this functional. Its solution leads to the formulation of the sharp Shnirelman's inequality as soon as we can guarantee:

- **Incompressible motion** in every pipe;
- **Capacity constraint**: The pipes do not intersect and the sum of their thicknesses is small enough to be confined in M ;
- **Time constraint**: Fluid particles travel in time $t = 1$ to their final destination.

Once the discrete flow problem on the graph is solved, we can effectively reconstruct the continuous space-time fluid flow. This approach tackles fluid dynamics by translating continuous infinite-dimensional problems into discrete flow problems on graphs. By visualizing the paths on our graph as *pipes*, we can rewrite

the length functional in terms of their thicknesses. Once the discrete flow problem on the graph is solved, it is possible to effectively reconstruct the continuous space-time fluid flow from this discrete solution.

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No Compromise in the Liquid Drop Model

MATILDE GIANOCCA

(joint work with Otis Chodosh)

The liquid drop model was introduced by Gamow [15] to describe atomic nuclei. For a measurable set $\Omega \subset \mathbb{R}^3$ of finite perimeter and volume $|\Omega| = V > 0$, the energy is

$$E(\Omega) = P(\Omega) + D(\Omega), \quad D(\Omega) = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{dx dy}{|x - y|}.$$

The first term is the perimeter, which is uniquely minimized at fixed volume by balls, up to translations and null sets, by the isoperimetric inequality [17]. The second term describes a nonlocal repulsive interaction. Its infimum at fixed positive volume is zero, approached by dispersing the mass, and is not attained. In fact, balls maximize the Coulomb energy at fixed volume. Thus the two terms favor opposite geometries; see [9] for background.

Let B_V be a ball of volume V . At the critical volume

$$V_* = 5 \frac{2 - 2^{2/3}}{2^{2/3} - 1} \approx 3.51, \quad E(B_{V_*}) = 2E(B_{V_*/2}),$$

a single ball has the same energy as the limiting configuration of two equal balls whose separation tends to infinity. For $V > V_*$, this two-ball configuration has lower limiting energy. The following “no compromise” conjecture, discussed for instance by Frank and Lieb [13] (see also [10, 14]), reflects the physical expectation.

Conjecture 1 (No compromise). *For $0 < V \leq V_*$, balls of volume V uniquely minimize E among sets of volume V . For $V > V_*$, no minimizer exists.*

In joint work with Otis Chodosh [7], we prove this conjecture. The proof was obtained by a publicly available large language model over a series of conversations. We have checked and reworked the argument, whose fundamental strategy remains close to the model's original output.

The rough strategy is as follows. A minimizer Ω has a bounded representative with C^3 boundary [6] and is connected. As in Chodosh–Ruohonieminen [8], we start from its Euler–Lagrange equation and the scaling identity:

$$(1) \quad \begin{aligned} H + v_\Omega &= \lambda \quad \text{on } \partial\Omega, & v_\Omega(x) &= \int_\Omega \frac{dy}{|x-y|}, \\ 3V\lambda &= 2P(\Omega) + 5D(\Omega) = 5E(\Omega) - 3P(\Omega). \end{aligned}$$

Here H is the sum of the principal curvatures with respect to the outward normal. Let U_∞ be the unbounded component of $\mathbb{R}^3 \setminus \Omega$ and set $K = \mathbb{R}^3 \setminus U_\infty$, thereby filling any bounded holes. Then $\partial K \subset \partial\Omega$ and $P(K) \leq P(\Omega)$. Consider the capacitary potential u of K , normalized by

$$\Delta u = 0 \text{ in } \mathbb{R}^3 \setminus K, \quad u = 1 \text{ on } \partial K, \quad u(x) \longrightarrow 0 \text{ as } |x| \longrightarrow \infty,$$

$$\text{Cap}(K) = \frac{1}{4\pi} \int_{\partial K} |\nabla u| \, dS.$$

The key idea is to integrate the Euler–Lagrange equation over ∂K with weight $|\nabla u|$. Since $\Delta v_\Omega = -4\pi \mathbf{1}_\Omega$, two integrations by parts give

$$\int_{\partial K} v_\Omega |\nabla u| \, dS = 4\pi V,$$

and hence

$$(2) \quad \int_{\partial K} H |\nabla u| \, dS = 4\pi (\lambda \text{Cap}(K) - V).$$

Using the potential-theoretic techniques of Agostiniani and Mazzieri [3], strengthened by applying Gauss–Bonnet to the connected regular level sets of u , we obtain

$$\int_{\partial K} H |\nabla u| \, dS \geq 4 \int_{\partial K} |\nabla u|^2 \, dS - 8\pi \geq 8\pi.$$

The underlying monotone quantity also appears in Israel's static uniqueness argument [16]; see also Robinson [18]. Related capacitary approaches to geometric inequalities appear in [1, 2, 4, 5, 11]. Combining these estimates with Cauchy–Schwarz yields the following key proposition; see [7, Proposition 2.4].

Proposition 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded connected C^3 stationary domain of volume V , and let K be its filled hull. Then*

$$\lambda \text{Cap}(K) - V \geq \max \left\{ 2, \frac{16\pi \text{Cap}(K)^2}{P(\Omega)} - 2 \right\}.$$

Consequently,

$$(3) \quad \lambda \sqrt{\frac{P(\Omega)}{4\pi}} \geq \begin{cases} V + 2, & 0 < V \leq 6, \\ 4\sqrt{V - 2}, & V \geq 6. \end{cases}$$

The proof of the conjecture then follows by combining (1) and (3) with comparison against the one- and two-ball configurations. For $V \leq V_*$, existence is known by Frank–Nam [14], and the one-ball comparison forces equality in the isoperimetric inequality, hence Ω is a ball. For $V_* < V \leq 8$, the two-ball comparison contradicts (3); for $V > 8$, nonexistence follows from Frank–Killip–Nam [12].

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Existence of flat blowups at the boundary for anisotropic minimizing hypercurrents

MICHAEL NOVACK

(joint work with Reinaldo Resende)

We examine the boundary behavior of codimension-one currents that minimize anisotropic energies. More precisely, fix $m \geq 2$ and $F : \mathbb{R}^{m+1} \rightarrow [0, \infty)$ which is $C^{2,\text{Dini}}$ away from 0, positively one-homogeneous, and uniformly convex in directions tangent to the sphere, so that there are $0 < a < A$ such that

$$a|\tau|^2 \leq D^2 F(\nu)[\tau, \tau] \leq A|\tau|^2, \quad \forall \nu \in \mathbb{S}^m, \quad \tau \in \nu^\perp.$$

The anisotropic energy of an m -current T on $B_1(0)$ is

$$\mathbb{F}(T) := \int_{B_1(0)} F(\star \vec{T}) d\|T\|.$$

For a fixed $C^{1,\alpha}$ $(m-1)$ -dimensional submanifold $\Gamma \subset B_1(0)$, we consider an integer rectifiable m -current T with $\partial T \llcorner B_1(0) = \llbracket \Gamma \rrbracket$ that minimizes $\mathbb{F}$ on $B_1(0)$, that is

$$\mathbb{F}(T) \leq \mathbb{F}(T + X)$$

whenever X is an integer rectifiable m -current such that $\text{spt } X \subset\subset B_1(0)$ and $\partial X = 0$. We are interested in the following question:

Is T regular near the boundary Γ ?

In the isotropic case $F(\nu) = |\nu|$, this question was answered affirmatively by Hardt and Simon in the seminal work [3]. They proved that every $x \in \Gamma$ is either a *one-sided* or *two-sided* regular point. Briefly, in a neighborhood of a one-sided point, $\text{spt } T$ is a minimal surface with boundary, whereas in a neighborhood of a two-sided point, $\text{spt } T$ is a minimal surface without boundary, and the multiplicity of T jumps across Γ . This distinction can be understood via the following example in $\mathbb{R}^3$; see Figure 1. Let c_1 and c_2 be the 1-currents, both oriented counterclockwise, induced by concentric circles of radii 1 and 2 in the xy -plane, respectively. Then the current T given by the sum of the currents induced by the corresponding disks of radii 1 and 2 (each counted once with the standard orientation) is area minimizing with boundary $c_1 + c_2$. Every point on the circle of radius 1 is a two-sided point for T .

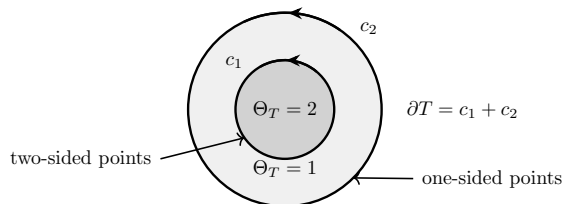


FIGURE 1. A minimizer with two-sided points and one-sided points.

The argument of Hardt and Simon has two main components: proving every boundary point has a one-sided or two-sided flat blowup and proving an ε -regularity theorem for T near a flat blowup. In forthcoming joint work with Reinaldo Resende, we extend the existence of flat blowups to the anisotropic setting in the following theorem [4].

Theorem 1 (Existence of flat blowups). *Let F , T , and Γ be as above, and let $x \in \Gamma$. Then there exist $r_j \rightarrow 0$, an integer $Q \geq 1$, and an oriented hyperplane $\Pi \supset T_x \Gamma$ such that, if Π^+ and Π^- denote the two half-hyperplanes into which $T_x \Gamma$ divides Π , each with the orientation induced by Π and labeled so that $\partial[\Pi^+] = \llbracket T_x \Gamma \rrbracket$, then*

$$T_{x,r_j} \rightarrow Q[\Pi^+] + (Q-1)[\Pi^-] = (Q-1)[\Pi] + [\Pi^+]$$

as currents, where $T_{x,r} := (\eta_{x,r})_{\#} T$ for $\eta_{x,r}(y) := (y-x)/r$.

Previously, by work of Hardt [2], the existence of flat blowups in the anisotropic codimension-one case was only known at points $x \in \Gamma$ at which $\text{spt } T$ admits a supporting hyperplane. A one-sided ε -regularity theorem for minimizers was also proved by Hardt in [2], and extended to almost-minimizers under weaker assumptions on the integrand by Duzaar and Steffen in [1]. Establishing an anisotropic ε -regularity theorem near two-sided points remains an open problem.

In [3], the authors first prove the ε -regularity theorem, and then use it to classify an arbitrary blowup T_0 as flat. The second crucial ingredient of their proof is the fact that in the isotropic case, blowups are cones, which is a consequence of the monotonicity formula. These two tools allow Hardt and Simon to carefully analyze the “polar angle” function $\theta : \text{Reg } T_0 \rightarrow \mathbb{R}$, which is defined as follows, in coordinates where $\partial T_0 = \llbracket \mathbb{R}^{m-1} \times \{0\}^2 \rrbracket$. Given connected components $\{C_i\}_i$ of $\text{Reg } T_0$ and “base points” $y^i = (y_1^i, \dots, y_{m-1}^i, r_i \cos \theta_i, r_i \sin \theta_i) \in C_i$, we let

$$\theta(x) = \theta_i + \text{Im} \left(\int_{\gamma_m + i\gamma_{m+1}} \frac{dz}{z} \right)$$

if $x \in C_i$ and $\gamma : [0, 1] \rightarrow C_i$ is any C^1 curve with $\gamma(0) = y^i$ and $\gamma(1) = x$. (One can show that $\theta(x)$ is independent of the choice of γ .) Hardt and Simon show that θ is continuous on the link $\mathbb{S}^m \cap \text{Reg } T_0$, and conclude flatness using the maximum principle at a point of maximum θ -value and an induction on the dimension.

Since the two-sided ε -regularity theorem is not known in the anisotropic case and the monotonicity formula only holds when $F(\nu) = |\Lambda \nu|$ for symmetric positive definite A , the proof of Theorem 1 is inspired by [3] but must proceed differently. The first step is to show that after taking one blowup and obtaining a minimizer T_0 on $\mathbb{R}^{m+1}$ with flat boundary, $\theta \in L^\infty(B_{1/2}(0) \cap \text{Reg } T_0)$. This is achieved by the construction of a novel barrier set. The second step combines a parametric analogue of the classical Hopf lemma with an iterated blowup procedure to conclude the existence of a flat blowup.

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Prescribed curvature on the disc

MICHAEL STRUWE

Prescribed curvature. Beginning with the work of Berger [1] and Kazdan-Warner [8] the problem of finding conformal metrics g on a surface M having prescribed Gauss curvature $K_g = f$ for a given function f has attracted geometric analysts. In particular, Nirenberg’s problem, that is, the study of the case when $M = S^2$, has attracted geometric analysts because of the complications due to the action of the (non-compact) conformal group of S^2 on the problem. This difficulty also arises in the case when M is the unit disc $B = B_1(0) \subset \mathbb{R}^2$, where in addition to the Gauss curvature K_g of the metric we also would like to prescribe the geodesic curvature k_g of the boundary. Here, we report on our work [12] on this problem.

Writing $g = e^{2u}g_0$ for the conformal metric, where g_0 is the Euclidean background metric, in order to achieve $K_g = f$ and $k_g = j$ for given functions f and j , we need to solve the nonlinear equation

$$(1) \quad -\Delta u = f e^{2u} \text{ in } B$$

with the nonlinear Neumann boundary condition

$$(2) \quad \frac{\partial u}{\partial \nu_0} + 1 = j e^u \text{ on } \partial B = S^1,$$

where ν_0 is the outward unit normal on ∂B .

Variational characterization. Thanks to the work of Cruz-Blazquez and Ruiz [5], with the help of an auxiliary variable $0 < \rho < \pi$ solutions to (1), (2) may be characterized as critical points of the functional

$$(3) \quad \begin{aligned} E(u, \rho) = E_{f,j}(u, \rho) = & \frac{1}{2} \int_B |\nabla u|^2 dz + \int_{\partial B} u ds_0 \\ & - \rho \log \left(\int_B f e^{2u} dz \right) - 2(\pi - \rho) \log \left(\int_{\partial B} j e^u ds_0 \right) \\ & + 2(\pi - \rho) \log (2(\pi - \rho)) + \rho + \rho \log(2\rho). \end{aligned}$$

Flow approach. As in our study of Nirenberg's problem in [11] we attempt to solve problem (1), (2) by following the negative gradient flow of E . That is, we seek to solve the equations

$$(4) \quad \frac{du}{dt} = \alpha f - K = \alpha f + e^{-2u} \Delta u \text{ in } B \times [0, \infty[$$

and

$$(5) \quad \frac{du}{dt} = \beta j - k = \beta j - e^{-u} \left(\frac{\partial u}{\partial \nu_0} + 1 \right) \text{ on } \partial B \times [0, \infty[$$

as well as

$$(6) \quad \frac{d\rho}{dt} = \log(\beta^2/\alpha) = -\partial_\rho E(u, \rho) \text{ on } [0, \infty[$$

for given initial condition

$$(7) \quad (u, \rho)|_{t=0} = (u_0, \rho_0),$$

where we let

$$(8) \quad \alpha = \alpha(t) = \frac{2\rho}{\int_B f e^{2u} dz}, \quad \beta = \beta(t) = \frac{2(\pi - \rho)}{\int_{\partial B} j e^u ds_0},$$

for all $t > 0$. For brevity, in the following we let $u_t = \frac{du}{dt}$ and so on.

Note that the coupling of equation (4) with the boundary condition (5) involves a Neumann type boundary condition of second order, and its treatment requires special care. For constant f and j , however, equations similar to (4) and (5) were already proposed by Brendle [2], who proved global existence and exponential convergence of the flow towards a conformal metric having both constant Gauss curvature and constant geodesic boundary curvature. For non-constant f and j , we may use Brendle's analysis to obtain local existence; however, as we shall see below, the behavior of the flow (4) - (6) for large time may be quite subtle, and also equation (6) plays an important role.

Energy identity. Multiplying both (4) and (5) with u_t and integrating with respect to g , and also multiplying (6) with ρ_t , we find the identity

$$(9) \quad \int_B e^{2u} u_t^2 dz + \int_{\partial B} e^u u_t^2 ds_0 + \rho_t^2 + \frac{d}{dt} E(u) = 0.$$

In particular, the energy E is non-increasing in time. Integrating also in time, and using that E is bounded from below, for a global smooth solution (u, ρ) of (4) - (6) we thus infer

$$(10) \quad \begin{aligned} \int_0^\infty \int_B |\alpha f - K|^2 d\mu_g dt + \int_0^\infty \int_{\partial B} |\beta j - k|^2 ds_g dt + \int_0^\infty \rho_t^2 dt \\ = E(u_0, \rho_0) - \lim_{T \rightarrow \infty} E(u(T), \rho(T)) < \infty, \end{aligned}$$

and there exists a sequence $t_l \rightarrow \infty$ such that

$$(11) \quad \int_{B \times \{t_l\}} |\alpha f - K|^2 d\mu_g + \int_{\partial B \times \{t_l\}} |\beta j - k|^2 ds_g + \rho_t^2(t_l) \rightarrow 0 \quad (l \rightarrow \infty).$$

In fact, we can show that (11) holds unconditionally as $t \rightarrow \infty$.

Concentration-convergence. Similar to a result of Jevnikar et al. in [7] we then obtain that as $t_l \rightarrow \infty$ either a subsequence of the conformal metrics $g_l = e^{2u(t_l)}g_0$ converges in $H^{3/2}(B) \cap H^1(\partial B)$ and uniformly to a metric $g_\infty = e^{2u_\infty}g_0$ inducing a solution of (1), (2), or the metrics $g(t_l)$ subsequentially concentrate at a point $z_0 \in \partial B$ in the sense of measures, exhibiting blow-up in a spherical cap whose size is determined by the value $J(z_0)$, where

$$J = j + \sqrt{j^2 + f}$$

with j extended as a harmonic function to B . In the latter case, moreover, similar to our analysis of the prescribed curvature flow on S^2 in [11], we can show that the flow (4) - (6) is essentially governed by a 2-dimensional system of ordinary differential equations describing the motion of the center of mass of the evolving metrics. Arguing indirectly, in the following we suppose that no solution to problem (1), (2) exists and seek to derive a contradiction from the study of the motion of the center of mass.

Center of mass, normalized flow. For any $0 < R < 1$, we let

$$\psi_R(z) := \frac{2Rz}{1 + |Rz|^2} = \pi_{\mathbb{R}^2} \Psi_R(z),$$

where $\Psi_R(z) = \Psi(Rz)$ is the scaled inverse $\Psi: \mathbb{R}^2 \rightarrow S^2$ of stereographic projection, and where $\pi_{\mathbb{R}^2}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the orthogonal projection. Then given any $u \in H^1(B)$ inducing a metric $g = e^{2u}g_0$, following the proof of Chang-Liu [3], Theorem 3.1, Onofri [9], p.324, or Chang-Yang [4], Appendix, for any $0 < R < 1$ we can find a Möbius map

$$\Phi_a(z) = \frac{z + a}{1 + \bar{a}z}$$

of the disc B such that (with $\Phi = \Phi_a$) for

$$(12) \quad v = u \circ \Phi + \frac{1}{2} \log(\det d\Phi) \text{ in } B, \quad h = \Phi^*g = e^{2v}g_0.$$

there holds

$$(13) \quad \frac{1}{2} \int_B \psi_R(z) e^{2v} dz + \int_{\partial B} \psi_R(z) e^v ds_0 = 0.$$

We interpret this condition as fixing the center of mass of the normalized metric $h = e^{2v}g_0$ lifted to the sphere S^2 by means of Ψ_R , and we call $p := \Phi_a(0) = a$ the center of mass of the metric g .

Note that Φ_a is unique up to rotations and, given a time $t_0 \gg 1$, we can rotate our coordinates to achieve $a > 0$ at time t_0 . Writing $p(t) = a(t)e^{i\phi(t)}$ with functions a and ϕ such that $a(t) > 0$ and $\phi(t_0) = 0$, for t close to t_0 with the normalized distance $\varepsilon = \varepsilon(t) = \frac{1-a}{1+a} < 1$ to the boundary of the disc and with constants $C_i = C_i(z_0) > 0$, $1 \leq i \leq 2$, we then have the equation

$$(14) \quad \left(\frac{da}{dt}, \frac{d\phi}{dt} \right) \Big|_{t=t_0} + \varepsilon^2 \left(C_1 \frac{\partial J(z_0)}{\partial x}, C_2 \frac{\partial J(z_0)}{\partial y} \right) = o(\varepsilon^2).$$

Main result. Our main result again shows the dominant role of the function J for problem (1), (2),

Theorem 1. *i) Suppose that there holds $\partial J/\partial\nu_0 > 0$ on ∂B . Then there exists a solution to problem (1), (2).*

ii) Suppose that there holds $\partial J/\partial\nu_0 < 0$ on ∂B . Then there exists a solution to problem (1), (2).

iii) Suppose that there exist points $z_i^\pm = e^{i\phi_i^\pm} \in \partial B$, $i = 1, 2$, such that

$$0 \leq \phi_1^+ < \phi_1^- < \phi_2^+ < \phi_2^- < 2\pi$$

and with

$$\frac{\partial J}{\partial\nu_0}(z_i^-) < 0 < \frac{\partial J}{\partial\nu_0}(z_i^+), \quad i = 1, 2.$$

Then there exists a solution to problem (1), (2).

The result improves a result by Ruiz [10] obtained by degree theory and may be best possible.

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On the strong locality of differential operators

PAOLO BONICATTO

(joint work with Giovanni Alberti, Giacomo Del Nin)

If $u \in C^1(\mathbb{R}^d; \mathbb{R})$ vanishes identically on an open set $U \subset \mathbb{R}^d$, then

$$Du = 0 \quad \text{on } U.$$

This expresses a *locality* property of the gradient: the value of Du at a point x only depends on the values of u in an arbitrarily small neighbourhood of x . One can say even more: If $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and $u = 0$ almost everywhere on an arbitrary measurable set E , then

$$Du = 0 \quad \text{a.e. on } E.$$

We refer to this property, valid on arbitrary measurable sets, as *strong locality* of the gradient operator. A natural question is whether an analogous statement holds for other differential operators.

A first indication that the answer can be quite different is provided by Alberti's Lusin theorem for gradients [1]. We recall a convenient version of it.

Theorem 1 (Alberti). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $f \in L^1(\Omega; \mathbb{R}^d)$. There exist a Borel set $B \subset \Omega$ and a function $g \in C_0^1(\Omega)$ such that*

$$|B| \leq \frac{1}{2}|\Omega|, \quad Dg = f \quad \text{on } \Omega \setminus B, \quad \|Dg\|_{L^1(\Omega)} \leq C(d)\|f\|_{L^1(\Omega)}.$$

As an immediate and somewhat surprising corollary, we obtain that the curl operator is not strongly local. Indeed, take a smooth vector field f on the unit ball $\Omega \subset \mathbb{R}^3$ with $\text{curl } f \neq 0$ everywhere, and let $g \in C_0^1(\Omega)$ be the function given by Alberti's theorem. The (continuous) vector field

$$v := f - Dg$$

vanishes on the set $\Omega \setminus B$, which has positive measure, whilst it can be easily checked that, in the sense of distributions, it holds

$$\text{curl } v = \text{curl } f \neq 0.$$

Thus $\text{curl } v$ does not vanish on the zero set of v .

In dimension two, divergence and curl are closely related, so the divergence operator is non-local as well. In this regard, we point out a remarkable connection with the linear transport equation, brought to light by Alberti, Bianchini and Crippa. For a suitable class of planar vector fields, the locality property of the divergence can be characterized in terms of a *weak Sard property* of an associated Hamiltonian potential. It turns out that the very same property governs uniqueness questions for the two-dimensional continuity equation, see [2].

We can now formulate the general question. Let

$$\mathcal{A} : C^\infty(\mathbb{R}^d; \mathbb{R}^m) \longrightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^n)$$

be a first-order homogeneous linear differential operator with constant coefficients. We write

$$(\mathcal{A}u)^\ell = \langle A^\ell, Du \rangle, \quad \ell = 1, \dots, n,$$

where $A^\ell \in \mathbb{R}^{m \times d}$ and $\langle \cdot, \cdot \rangle$ denotes the Frobenius inner product on $\mathbb{R}^{m \times d}$. By interpreting derivatives distributionally, one can let $\mathcal{A}$ act naturally on L^1_{loc} functions.

We say that $\mathcal{A}$ is *strongly local* if, whenever $u \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ and $\mathcal{A}u \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ the condition $u = 0$ a.e. on a measurable set E implies $\mathcal{A}u = 0$ a.e. on E .

The problem is to understand which algebraic or analytical features of the operator $\mathcal{A}$ determine this locality property.

This matrix representation introduces some natural degrees of freedom which should be factored out, just as different choices of coordinates lead to different matrix representations of the same linear map. We thus regard two operators as equivalent if one can be obtained from the other by invertible linear changes of basis in the domain and target spaces. Strong locality is, of course, invariant under this equivalence relation.

In the scalar case ($n = 1$), the following characterization holds true.

Theorem 2. *Assume $n = 1$. Then $\mathcal{A}$ is represented by a single matrix $A \in \mathbb{R}^{m \times d}$. If $r = \text{rank } A$, then $\mathcal{A}$ is equivalent to*

$$v \mapsto \partial_1 v^1 + \dots + \partial_r v^r.$$

In particular,

$$\mathcal{A} \text{ is strongly local} \iff \text{rank } A \leq 1.$$

Thus, in this case, a purely algebraic quantity (the rank of the coefficient matrix) completely determines locality.

The vector-valued case is more subtle, since no analogous canonical form for the coefficient tensor is available in general. The first non-trivial situation in which a complete reduction can nevertheless be carried out is $d = m = n = 2$.

Theorem 3. *Let $d = m = n = 2$. Up to the equivalence relation above, every first-order homogeneous operator is represented by one of the following types:*

$$(\partial_1 u^1, \partial_i u^j), \quad (\text{div } u, \text{curl } u), \quad (\text{div } u, 0), \quad (\text{div } u, \partial_2 u^1),$$

with $i, j \in \{1, 2\}$. The first two families are strongly local, while the remaining two types are not.

The div-curl example is particularly instructive: although divergence and curl separately fail to be strongly local, putting the two together restores locality. This illustrates that strong locality is governed by a delicate interplay between linear algebra, cancellations and fine properties of functions on measurable sets. More general characterizations, as well as quantitative versions of locality, remain natural directions for further investigation.

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Second-Order Γ -limit for the Cahn–Hilliard Functional with Dirichlet Boundary Conditions

PASCAL STEINKE

(joint work with Francesco Colasanto)

The Cahn–Hilliard functional introduced in [5]

$$(1) \quad E_\varepsilon^{(0)}(u) := \int_{\Omega} W(u) + \varepsilon^2 |\nabla u|^2 \, dx$$

for a double-well potential W is used as a phase-field approximation for sharp interface problems, like the mixtures of two immiscible fluids. It is closely related to the Ginzburg–Landau functional ([4]), the Ambrosio–Tortorelli approximation ([1]) and is the energy underlying the L^2 -gradient flow structure of the Allen–Cahn equation, which yields an approximation to mean curvature flow (e.g. [11]). The Γ -convergence of its first order rescaling $E_\varepsilon^{(1)} := E_\varepsilon^{(0)}/\varepsilon$ has first been considered in [15] and has been extensively studied, both under mass constraints [14, 18, 16, 10], the multiphase case [3] and Dirichlet boundary conditions [17].

Higher-order Γ -expansions have been introduced in [2] as a means to more closely understand the energetic behaviour of minimizers of functionals like $E_\varepsilon^{(0)}$. The first second-order expansion for $E_\varepsilon^{(0)}$ has been done in [7] and has been followed up by resolving the mass-constrained case in [12, 13]. The analysis of the Dirichlet case has been initiated in [8, 9].

In this talk, we discussed the results of [6], which contributes to the understanding of the second-order functional under Dirichlet constraints by considering the case of a subquadratic potential W in (1). The boundary data is incorporated as usual by setting $E_\varepsilon^{(0)}(u) = \infty$ if we do not have $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$. As a starting point, [17] showed that if $g_\varepsilon \rightarrow g$, then the first-order Γ -limit of the first-order energy $E_\varepsilon^{(1)} = E_\varepsilon^{(0)}/\varepsilon$ under the constraint $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$ is

$$(2) \quad \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} E_\varepsilon^{(1)} = E^{(1)}$$

with

$$E^{(1)}(u) := \begin{cases} C_W \operatorname{Per}(\{u = a\}) + \int_{\partial\Omega} d_W(g, u) \, d\mathcal{H}^{n-1}, & \text{if } u \in \operatorname{BV}(\Omega; \{a, b\}), \\ \infty, & \text{else.} \end{cases}$$

The topology is given by strong convergence on $L^1(\Omega)$, and d_W is the geodesic distance induced by W . The main result we discussed is the following limit for the second-order functional.

Theorem 1. Assume that $W: \mathbb{R} \rightarrow [0, \infty)$ with $\{W = 0\} = \{a, b\}$ has subquadratic growth at the wells. Let $g: \partial\Omega \rightarrow \mathbb{R}$ be sufficiently smooth and be chosen such that the constant function b is the unique minimizer of the first-order Γ -limit (2). Then the second-order Γ -limit of the Cahn–Hilliard energy (1) with Dirichlet data g_ε is given by

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \frac{E_\varepsilon^{(1)} - \min E^{(1)}}{\varepsilon}(b) = \int_{\partial\Omega} -\kappa(y) \int_0^\infty d_W(z_{g(y)}(t), b) dt d\mathcal{H}^{n-1},$$

where d_W is the geodesic distance induced by W , κ is the mean curvature of $\partial\Omega$ and z_α is the optimal transition profile defined in (3).

The behaviour at the wells of W is crucial for the second-order limit since it determines the properties of the one-dimensional optimal profile function

$$(3) \quad z'_\alpha(t) = \sqrt{W(z_\alpha(t))} \quad \text{and} \quad z_\alpha(0) = \alpha.$$

In the case of a subquadratic potential, it reaches the wells in finite time and does not need additional energy to move out of the wells, while in the quadratic case, it approaches both wells with exponential speed, but never reaches them. We discussed how this means that a different scaling for the second-order Γ -limit occurs in the subquadratic case compared to the quadratic case.

To arrive at Theorem 1, we utilized slicing techniques as in [8] and one-dimensional comparison principles.

To conclude the talk, we discussed limitations and future prospects. Namely, we pointed out how the slicing technique is not appropriate to handle more irregular boundary data, since energy might sit in the tangential directions of the boundary. Moreover we related this problem to the task of finding the second-order Γ -limit in the case when $u = b$ might not be the unique minimizer.

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Hyperbolic regularization effects for degenerate elliptic equations

RICCARDO TIONE

(joint work with X. Lamy)

We consider the issue of regularity for two-dimensional Lipschitz solutions to the elliptic equation

$$(1) \quad \operatorname{div}(G(Du)) = 0 \quad \text{in } B_1 \subset \mathbb{R}^2,$$

in the sense of distributions, for a general continuous field $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is strictly monotone:

$$(2) \quad (G(b) - G(a), b - a) > 0 \quad \forall a \neq b \in \mathbb{R}^2.$$

Specifically, we focus on the fundamental question:

$$(Q) \quad \text{are Lipschitz solutions of (1) } C^1, \text{ or at least partially } C^1?$$

Here, partially C^1 means C^1 in the complement of a closed Lebesgue-negligible set. For a continuous field G , (2) is, up to a change of sign, the weakest assumption under which Question (Q) is reasonable. Indeed, if $G(b) - G(a) \perp b - a$ for some $b \neq a$, then there exist Lipschitz solutions u of (1) with wildly discontinuous gradient, obtained via the so-called *simple laminate* construction.

Of particular interest are gradient fields $G = Df$ with f strictly convex. In that case, (1) is the Euler–Lagrange equation satisfied by minima of the energy:

$$\int_{B_1} f(Du) \, dx.$$

It is well-known that solutions are smooth provided f is smooth and uniformly convex, that is,

$$(3) \quad \Lambda^{-1} \text{id} \leq D^2 f(\xi) \leq \Lambda \text{id}, \text{ in the sense of quadratic forms.}$$

In higher dimensions, this is due to the celebrated theorems of De Giorgi–Nash–Moser, while in two dimensions it has been known since the work of Morrey. Many important and natural problems do not, however, enjoy the strong ellipticity property (3): a fundamental example is that of the p -Laplace equation corresponding to $f(\xi) = |\xi|^p$. However, even in degenerate cases like the one of the p -Laplace equation, singularities in the Hessian of the integrand are localized at a single point, while instead answering question (Q) under the sole assumption (2) means having no (or, for partial results, as little as possible) control on the gradient of G (or the Hessian of f in the variational case). This naturally excludes most of the classical tools in regularity theory, such as integral estimates.

1. A quick overview of the literature. Our starting point is [1], where D. De Silva and O. Savin obtain regularity results in the absence of any quantitative assumptions on the strict monotonicity (2). For any strictly convex f , their results depend only on the closed sets $\mathcal{D}_-$ and $\mathcal{D}_+$ of values $\xi \in \mathbb{R}^2$ where ellipticity fails from below or from above:

$$\begin{aligned} \mathcal{D}_- &= \bigcap \left\{ X \subset \mathbb{R}^2 \text{ closed} : \exists \lambda > 0, D^2 f \geq \lambda \text{id in } \mathbb{R}^2 \setminus X \right\}, \\ \mathcal{D}_+ &= \bigcap \left\{ X \subset \mathbb{R}^2 \text{ closed} : \exists \Lambda > 0, D^2 f \leq \Lambda \text{id in } \mathbb{R}^2 \setminus X \right\}. \end{aligned}$$

The inequalities are in the sense of distributions with values in symmetric matrices. These definitions have natural extensions to any monotone field G . With this notation, the most general answer to (Q) known so far is, essentially, that

$$(4) \quad \text{Lipschitz solutions of (1) are } C^1 \text{ if } \mathcal{D} = \mathcal{D}_- \cap \mathcal{D}_+ \text{ is finite.}$$

Quite remarkably, this only involves the degeneracy set $\mathcal{D}$ where ellipticity fails *both* from above and below. This regularity result is proved in [4] by building on the arguments introduced in [1], where $\mathcal{D}$ was assumed empty, or $\mathcal{D}_-$ finite. One can actually replace ‘finite’ in (4) by ‘at most countable’, with the same proof. Moreover, an example from [4] demonstrates that, under the mere assumption (2), Lipschitz solutions of (1) may have point singularities, so one cannot in general expect better than partial regularity in (Q).

2. Main results. Our first main result is valid for *any* strictly monotone continuous field G . We denote by $\mathcal{H}^\alpha$ the α -dimensional Hausdorff measure.

Theorem A. *Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be continuous and strictly monotone (2), and $u: B_1 \rightarrow \mathbb{R}$ a Lipschitz solution to (1). Then the nondifferentiability set*

$$\mathcal{S}_u = \{x \in B_1: u \text{ is not differentiable at } x\}$$

satisfies $\mathcal{H}^1(\mathcal{S}_u) = 0$, and Du is continuous on $B_1 \setminus \mathcal{S}_u$.

The main point in the proof of Theorem A is an L^∞ to $W^{1,\infty}$ estimate inspired by the beautiful separation argument of [2].

Our second main theorem yields a precise partial regularity result under structural assumptions on $\mathcal{D}$.

Theorem B. *Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be continuous and strictly monotone (2). Then, for any Lipschitz solution u to (1), the singular set $\mathcal{S}_u \subset B_1$ is*

- *locally finite, if the graph of G over $\mathcal{D}$ is included in a finite disjoint union of C^1 curves in $\mathbb{R}^2 \times \mathbb{R}^2$;*
- *empty, if $\mathcal{D}$ is included in a countable disjoint union of boundaries of strictly convex sets.*

The result we actually prove is stronger than Theorem B, but we chose to present a weaker version for the sake of simplicity. This statement nevertheless conveys the main novelty of our methods, which is to deal with a large class of one-dimensional degeneracy sets $\mathcal{D}$, in contrast with the zero-dimensional assumption in (4). The second part of Theorem B applies to a wide class of radial fields $G = Df$ with f strictly convex and radial, a symmetry present in most physically relevant problems. More precisely, consider $f(\xi) = g(|\xi|)$ for some increasing and strictly convex C^1 function $g: [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = g'(0) = 0$. In this case, $\mathcal{D}$ is a union of the origin and circles, and, provided it does not contain open sets, the complete version of Theorem B shows that minimizers for the energy induced by f are C^1 regular. This is entirely new: in the context of radial functionals, the available techniques so far could only show such regularity under the assumption that $\mathcal{D}$ is precisely the origin.

The proof of Theorem B relies on studying Lipschitz blowups of u at any point $x_0 \in B_1$. In [3] it was shown that these are either affine, in which case a version of Theorem A shows that x_0 is a continuity point for Du , or their gradients need to take values in a single connected component of $\mathcal{D}$. In the second case, by our assumptions the blowups solve a Hamilton–Jacobi equation. Leveraging the hyperbolic nature of this problem we are able to obtain an ε -regularity result for such blowups. Further observations then allow us to show a quite rigid structure of blowups, which have at most two singular points. Finally, it can be shown that any solution u has a well-defined notion of Monge–Ampère measure, which for blowups is concentrated at most at one point: this yields that blowup limits are one-homogeneous, and standard arguments then imply that $\mathcal{S}_u$ is locally finite.

The fact that any blowup limit of solutions as in Theorem B is one-homogeneous is quite remarkable, since such homogeneity is usually obtained in elliptic problems through monotonicity formulas, which are unavailable in our context.

3. The differential inclusions approach. Ubiquitous in our approach is the correspondence between Lipschitz solutions of (1) and solutions to differential inclusions. Let us briefly explain how that works. We say that the closed set $K \subset \mathbb{R}^{2 \times 2}$ is *elliptic* if, given any equi-Lipschitz sequence $(w_k)_k$ on B_1 ,

$$\text{dist}(Dw_k, K) \rightarrow 0 \text{ in } \mathcal{D}'(B_1) \implies (w_k)_k \text{ is } W_{\text{loc}}^{1,1} \text{ strongly precompact.}$$

The question is whether Lipschitz solutions to the differential inclusion $Dw \in K$ enjoy better regularity properties, a question which entails understanding the connection between strong compactness properties and rigidity of exact solutions. It is well-known that a solution to (1) gives rise to a solution w to an elliptic differential inclusion. The converse is also, essentially, true, but requires exploiting several deep results from the literature. Indeed, the theory of 2×2 elliptic differential inclusions is extremely rich, and this wealth of results and techniques is crucial for our paper. Conversely, our results, especially Theorem A, represent an important step towards a complete understanding of regularity issues for elliptic differential inclusions in $\mathbb{R}^{2 \times 2}$.

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Wrinkling in the Lamé problem: a Γ -convergence approach

ROBERTA MARZIANI

Thin elastic sheets subjected to compressive strains tend to buckle out-of-plane rather than compress in-plane, giving rise to complex wrinkling patterns across multiple length scales. From a variational standpoint, these patterns emerge from the energetic competition between membrane stretching/compression and bending rigidity.

In this talk, we consider the classical Lamé problem, which is motivated by experiments on a thin elastic sheet floating on a water bath with a central water drop. Driven by capillary forces, the film exhibits a distinct two-phase behavior: a wrinkled inner region featuring radial oscillations around the drop, and an unrelaxed outer region under biaxial tension [4].

From a mathematical perspective, we model the sheet as a thin elastic annulus of cross-section $\Omega := \{x \in \mathbb{R}^2 : R_{\text{in}} < |x| < R_{\text{out}}\}$ and thickness $h \ll 1$, subject to radial dead loads $T_{\text{in}} > T_{\text{out}} > 0$ applied on the inner and outer boundaries, respectively. Stronger stretching at the inner boundary forces material circles towards the center, inducing azimuthal compression in an inner region $R_{\text{in}} < r < R_0$, while the outer region $R_0 < r < R_{\text{out}}$ remains under biaxial tension.

Within the Föppl–von Kármán theory, the elastic energy associated with an in-plane displacement $u = (u_1, u_2)$ and an out-of-plane displacement ξ is given by

$$E_h(u, \xi) := \frac{1}{2} \int_{\Omega} \left| e(u) + \frac{1}{2} \nabla \xi \otimes \nabla \xi \right|^2 dx + h^2 \int_{\Omega} |\nabla^2 \xi|^2 dx + B(u),$$

where $e(u)$ is the symmetric strain tensor and $B(u)$ incorporates the boundary radial loads.

In the asymptotic limit $h \rightarrow 0$, bending costs vanish, leading to a relaxed membrane problem E_{rel} with minimum energy $\mathcal{E}_0$. The relaxed minimizer u^* is planar, radially symmetric, and uniquely identifies a free boundary at radius R_0 separating the relaxed (compressed) region from the unrelaxed (tensile) one.

In a landmark paper, Bella and Kohn [2] proved the ansatz-free energy scaling law

$$\mathcal{E}_0 + C_0 h \leq \inf_{(u, \xi)} E_h(u, \xi) \leq \mathcal{E}_0 + C_1 h, \quad (0 < C_0 \leq C_1 < \infty).$$

An analysis of the relaxed problem reveals that the excess arclength to be accommodated in the inner region decays linearly with the distance from the free boundary (circle of radius R_0). Consequently, superimposing a single family of wrinkles forces the transition between the compressed and tensile regions to be energetically too costly, yielding a sub-optimal energy scaling of order $h(|\log h| + 1)$ [4]. To achieve the optimal linear scaling $O(h)$, Bella and Kohn [2] proposed a refined construction featuring a cascade of branching wrinkles near R_0 . However, since such branching patterns are rarely observed in physical experiments, a sharp discrepancy emerges: the single-frequency configuration matched experimentally fails to achieve the optimal energy scaling, whereas the energetically optimal branching mechanism is absent in real systems. This shows that scaling laws alone are insufficient to select the physically observed pattern.

The main goal of this talk is to present a Γ -convergence framework for the rescaled excess energy $(E_h - \mathcal{E}_0)/h$ as $h \rightarrow 0$ [5], building upon previous measure-theoretic ideas developed for a simplified toy model [1, 3].

Rescaling and Measure-Theoretic Setting. Passing to polar coordinates $(r, \theta) \in (R_{\text{in}}, R_{\text{out}}) \times [-\pi, \pi)$, we express the deformation in terms of its radial and tangential in-plane components (u_r, u_θ) and the out-of-plane displacement ξ . Balancing stretching and bending costs indicates that the characteristic wavelength of wrinkles is of order $\sqrt{h}$. Introducing the scaling parameter $L := h^{-1/2} \rightarrow \infty$, we rescale the angular coordinate θ so that the domain becomes $2\pi L$ -periodic in θ . After subtracting the relaxed energy $\mathcal{E}_0$, the rescaled excess functional $\mathcal{F}_L := L^2(E_h - \mathcal{E}_0)$ asymptotically approaches

$$\mathcal{F}_L(u_r, u_\theta, \xi) \sim \pi \int_{-\pi L}^{\pi L} \int_{R_{\text{in}}}^{R_0} \left(\dot{u}^*(r) (\partial_r \xi)^2 + \frac{(\partial_{\theta\theta} \xi)^2}{r^4} \right) r dr d\theta,$$

subject to the arclength constraint

$$\int_{-\pi L}^{\pi L} \frac{(\partial_{\theta\theta} \xi)^2}{2} d\theta \sim -r u^*(r) \quad \text{in } (R_{\text{in}}, R_0),$$

which encodes the geometric amount of arclength that must be released through out-of-plane oscillations.

As $L \rightarrow \infty$, the grid of Fourier frequencies $k \in \mathbb{Z}/L$ associated with the angular coordinate becomes dense. Denoting by $a_k(r)$ the k -th Fourier coefficient of $\xi(r, \cdot)$, we define the sequence of positive Radon measures

$$\mu^L(\xi) := \sum_{k \in \frac{\mathbb{Z}}{L}} k^2 a_k^2(r) \mathcal{L}^1 \llcorner (R_{\text{in}}, R_{\text{out}}) \times \delta_k \in \mathcal{M}_b^+((R_{\text{in}}, R_{\text{out}}) \times \mathbb{R}).$$

The sequence $(u_r^L, u_\theta^L, \xi^L)$ converges to a measure μ if $(\mu^L, \mu_{\cdot, r}^L)$ converges weak-* in the sense of measures to $(\mu, \mu_{\cdot, r})$, where $\mu_{\cdot, r}$ denotes the distributional derivative with respect to r .

The admissible class of limiting measures $\mathcal{M}_\infty$ consists of $\mu \in \mathcal{M}_b^+((R_{\text{in}}, R_0) \times \mathbb{R})$ satisfying $\mu_{\cdot, r} \ll \mu$ and the marginal constraint

$$\int_{(R_{\text{in}}, R_0) \times \mathbb{R}} \phi(r) d\mu(r, k) = - \int_{R_{\text{in}}}^{R_0} 2ru^*(r)\phi(r) dr \quad \forall \phi \in C_c^\infty(R_{\text{in}}, R_0).$$

Main Results. We present the Γ -limit of $\mathcal{F}_L$ and analyze the structure of its ground states [5]:

Theorem 1 ([5]). *Let $\mathcal{F}_L$ be the rescaled excess energy functional and let $\mathcal{F}_\infty : \mathcal{M}_\infty \rightarrow [0, \infty]$ be defined by*

$$\mathcal{F}_\infty(\mu) := \pi \int_{(R_{\text{in}}, R_0) \times \mathbb{R}} \left[\frac{\dot{u}^*(r)}{4k^2} \left(\frac{d\mu_{\cdot, r}}{d\mu}(r, k) \right)^2 + \frac{k^2}{r^4} \right] r d\mu(r, k).$$

Then, as $L \rightarrow \infty$ (i.e., $h \rightarrow 0$):

- (1) **Compactness & Γ -liminf:** *Any sequence $(u_r^L, u_\theta^L, \xi^L)$ with bounded energy $\mathcal{F}_L$ admits a convergent subsequence to some $\mu \in \mathcal{M}_\infty$, and*

$$\liminf_{L \rightarrow \infty} \mathcal{F}_L(u_r^L, u_\theta^L, \xi^L) \geq \mathcal{F}_\infty(\mu).$$

- (2) **Γ -limsup:** *For every $\mu \in \mathcal{M}_\infty$, there exists a recovery sequence $(u_r^L, u_\theta^L, \xi^L)$ converging to μ such that*

$$\limsup_{L \rightarrow \infty} \mathcal{F}_L(u_r^L, u_\theta^L, \xi^L) \leq \mathcal{F}_\infty(\mu).$$

The stretching term (i.e., the first term) in $\mathcal{F}_\infty$ can be recast as a Benamou–Brenier functional, making $\mathcal{F}_\infty$ convex, 1-homogeneous, and lower semicontinuous.

The proof of the Γ -limsup is particularly delicate due to the curvilinear geometry and the nonlinear arclength profile $-ru^*(r) = -T_{\text{in}} R_{\text{in}} r \log(r/R_0)$. The recovery sequence construction relies on:

- Discretizing μ in the frequency variable k over an intermediate scale L_0 ($1 \ll L_0 \ll L$) and constructing the corresponding Fourier coefficients;
- Regularizing the squared Fourier coefficients via convolution with an exponential kernel $\rho_\epsilon(r) = \frac{1}{2\epsilon} e^{-|r|/\epsilon}$ after a suitable extension outside (R_{in}, R_0) ;

- Correcting the profile of the out-of-plane displacement $\hat{\xi}^L$ with a factor $f^L(r) = \sqrt{-ru^*(r)/A^L(r)}$, where $A^L(r) := \int_{-\pi L}^{\pi L} \frac{(\partial_\theta \hat{\xi}^L)^2}{2} d\theta$, to satisfy the arclength constraint exactly;
- Correlating the in-plane components u_r and u_θ to suppress higher-order shear and stretching costs.

Theorem 2 ([5]). *The limiting functional $\mathcal{F}_\infty$ admits a minimizer $\hat{\mu} \in \mathcal{M}_\infty$. Furthermore, denoting by $\pi_2 : (R_{\text{in}}, R_0) \times \mathbb{R} \rightarrow \mathbb{R}$ the canonical projection onto the frequency variable k , and by $(\pi_2)_\# \hat{\mu}$ its push-forward measure (representing the marginal distribution of frequencies), any minimizer satisfies:*

- (1) **Equipartition of Energy:** *The stretching and bending contributions balance:*

$$\int_{(R_{\text{in}}, R_0) \times \mathbb{R}} \frac{\dot{u}^*(r)r}{4k^2} \left(\frac{d\hat{\mu}_r}{d\hat{\mu}} \right)^2 d\hat{\mu} = \int_{(R_{\text{in}}, R_0) \times \mathbb{R}} \frac{k^2}{r^3} d\hat{\mu}.$$

- (2) **Absence of Low Frequencies:** *There exists $C > 0$ such that the frequency marginal $(\pi_2)_\# \hat{\mu}$ carries no mass on small frequencies, i.e., $(\pi_2)_\# \hat{\mu}(\{|k| < C\}) = 0$.*

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The epsilon-regularity theorem for Brakke flows near triple junctions

SALVATORE STUWARD

(joint work with Yoshihiro Tonegawa)

Let $\mathbf{C} = \mathbf{Y}^1 \times \mathbb{R}^{k-1} \subset \mathbb{R}^n$ be the multiplicity-one stationary cone formed by three k -dimensional half-planes meeting at 120° along their common spine $\mathbf{S}$. Simon’s theorem [2] implies that a multiplicity-one stationary varifold sufficiently L^2 -close to $\mathbf{C}$ in a ball is a $C^{1,\alpha}$ deformation of $\mathbf{C}$ in a smaller ball. We present a parabolic counterpart, proved in [5], for possibly forced Brakke flows in arbitrary dimension and codimension. Previous results include the planar network theorem of Tonegawa–Wickramasekera [6] and the work of Schulze–White [1] in the classes of smooth flows with triple edges and mod 3 flows constructed by elliptic regularization.

The regularity theorem. Let $P_R = B_R(0) \times (-R^2, 0] \subset \mathbb{R}_X^n \times \mathbb{R}_t$ and consider an integral Brakke-like flow $\{V_t\}$ of dimension k in P_R , with locally square-integrable mean curvature h , uniform mass bounds $\|V_t\|(B_r(X)) \leq E_1 \omega_k r^k$, and with normal velocity $h + u^\perp$ in the sense of Brakke's inequality. Here, u is a given ambient forcing field satisfying $u \in L_t^q L_X^p(d\|V_t\|)$, where $2 \leq p < \infty$, $2 < q < \infty$, and $\alpha = 1 - k/p - 2/q > 0$. The relevant scale-invariant quantities to measure closeness to $\mathbf{C}$ are the L^2 -excess

$$\mu_R^2 = R^{-k-4} \int_{-R^2}^0 \int_{B_R} \text{dist}^2(X, \mathbf{C}) d\|V_t\|(X) dt,$$

and the $L^{p,q}$ -norm of u

$$F_R = R^\alpha \left(\int_{-R^2}^0 \left(\int_{B_R} |u|^p d\|V_t\| \right)^{q/p} dt \right)^{1/q}.$$

Theorem. Assume the structural condition (A6) described below. For fixed n, k, p, q, E_1 and constant c_0 in (A6), there is $\varepsilon > 0$ with the following property. If $\max\{\mu_R, F_R\} < \varepsilon$ and

$$\begin{aligned} \|V_{-9R^2/10}\|(B_{R/2}) &\leq 2\omega_k(R/2)^k, \\ \text{spt } \|V_0\| \cap B_{R/24}(Rw_i/2) &\neq \emptyset \quad (i = 1, 2, 3), \end{aligned}$$

where w_i are the unit directions of the three rays of $\mathbf{Y}^1$, then in $P_{R/2}$ the flow consists of three multiplicity-one sheets meeting at 120° along a common boundary. The sheets are normal graphs over domains in the planes of the three branches of $\mathbf{C}$, and their common boundary is a graph over the spine. These maps are of parabolic class $C^{1,\alpha}$, with scale-invariant estimates bounded by $c \max\{\mu_R, F_R\}$.

The mass and non-vanishing assumptions exclude higher multiplicity and loss of a branch. When $u = 0$, any $\alpha \in (0, 1)$ is admissible and the sheets are smooth in their interiors; whether the regularity at the junction improves from $C^{1,\alpha}$ to $C^{2,\alpha}$ and therefore to smooth is an open problem.

The structural condition and its role. Write $X = (x, y) \in \mathbf{S}^\perp \times \mathbf{S}$. In each sufficiently close cylinder P_r , after translating and rotating, the flow has unit density at almost every inner time. Let M_t^y be its one-dimensional slice at y . Condition (A6) requires, for almost every $y \in B_{r/2}^{k-1}$ and $t \in (-3r^2/4, 0)$,

$$\begin{aligned} r^{-1} \mathcal{H}^1(M_t^y \cap B_{r/2}) &\geq \frac{3}{2} - c_0 K^2, \\ r^{-3} \int_{M_t^y \cap B_{r/2}} |x|^2 d\mathcal{H}^1(x) &\geq \frac{1}{8} - c_0 K^2. \end{aligned}$$

Here K is the supremum over these slices and times of their Hausdorff distance from $\mathbf{Y}^1$ in $\{r/8 \leq |x| \leq r/2\}$, divided by r . The constants $3/2$ and $1/8$ are the corresponding normalized length and second moment of $\mathbf{Y}^1 \cap B_{r/2}$. The assumption converts Hausdorff control away from the spine into lower bounds for scale invariant length and second moment for the entire slice, with errors quadratic in the annular distance. This rules out certain topological degeneracies.

For a stationary varifold V with $\Theta_V(0) \geq \Theta_{\mathbf{C}}(0)$, the monotonicity formula gives

$$kr^{k-1} \int_{B_r} \frac{|X^\perp|^2}{|X|^{k+2}} d\|V\|(X) \leq \frac{d}{dr} (\|V\|(B_r) - \|\mathbf{C}\|(B_r)).$$

Here $X^\perp$ is the normal projection of X with respect to V . Simon uses this inequality to avoid certain derivatives of localization cut-offs. A directly usable counterpart is unavailable in our parabolic argument: Huisken's monotonicity formula is non-local in space, and its localization produces cut-off errors near the spine of $\mathbf{C}$, which therefore cannot be controlled by the known graphicality away from the spine. Condition (A6) supplies precisely the lower bounds needed to estimate these errors in terms of the L^2 excess and forcing. In turn, these bounds give quantitative control of the L^2 mean curvature and of the deviation from self-similarity measured by Huisken's monotonicity formula.

We prove that (A6) holds automatically for codimension-one flows with an underlying cluster structure, including the canonical BV–Brakke flows of [3]. It also holds, in arbitrary codimension, when for almost every t there is an integral cycle mod 3 whose mod 3 mass measure equals $\|V_t\|$. Consequently, no additional structural assumption is needed in these classes; in particular, the mod 3 result does not depend on the construction of the flow.

Non-concentration and excess decay. The proof follows Simon's blow-up strategy. End-time epsilon-regularity near planes [4] gives graphical parametrizations away from $\mathbf{S}$, with $C^{1,\alpha}$ estimates relative to excess and forcing. A contradiction argument reduces to sequences whose forcing is negligible relative to the excess. The excess-normalized graphing functions then have limits in $C^{1,\alpha}$ locally away from $\mathbf{S}$ on top of each branch of $\mathbf{C}$, and one can prove that such limits are caloric functions. Convergence away from $\mathbf{S}$ does not exclude concentration of the normalized L^2 excess in a shrinking neighborhood of the spine.

A no-hole property, proved with soft dimension reduction-type arguments without (A6), provides points of Gaussian density at least $3/2$ in every perpendicular slice at every inner time. Using these points as centers, and (A6) to control localization errors, we establish non-concentration estimates.

For example, after normalizing the larger cylinder to P_5 , for $0 < \kappa < 1/2$ and sufficiently small $\sigma > 0$ we obtain

$$\iint_{P_{1/2} \cap \{\text{dist}(X, \mathbf{S}) < \sigma\}} \text{dist}^2(X, \mathbf{C}) d\|V_t\|(X) dt \leq c_\kappa \sigma^{1-2\kappa} \max\{\mu_5, F_5\}^2,$$

provided the initial closeness is sufficiently small depending on σ . This prevents loss of excess at the spine in the blow-up limit. The limiting caloric functions then converge strongly in L^2 globally, and they satisfy compatibility conditions at the spine, so that one can prove that each branch is the restriction of a caloric function defined across the spine. One can then conclude quadratic decay of the functions to their first-order expansions. The corresponding constant and linear terms determine a translated and rotated triple junction. Relative to this new cone, the excess at a fixed smaller scale θR is at most $\theta^\alpha \max\{\mu_R, cF_R\}$. This

establishes excess decay to the class of triple junctions, and standard iteration yields the theorem.

Consequences. Under (A6), a backward-static multiplicity-one triple junction occurring as a tangent flow is unique, and parabolic blow-ups converge to it at rate $O(\lambda^\alpha)$. For unforced cluster or mod 3 flows $\mathcal{V} = \{V_t\}$, combining the theorem with White's stratification [7] also gives, locally where the Gaussian density is less than 2, $\text{Sing } \mathcal{V} = \mathcal{R} \cup \mathcal{S}$: $\mathcal{R}$ is the regular triple-junction locus and $\dim_{\mathcal{P}} \mathcal{S} \leq k$. For almost every t , $\mathcal{R}_t$ is a $(k-1)$ -dimensional $C^{1,\alpha}$ submanifold and $\dim_{\mathcal{H}} \mathcal{S}_t \leq k-2$.

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Quantitative profile decompositions for almost-critical points

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(joint work with André Guerra, Konstantinos Zemas)

In dimension $n \geq 3$, the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} dx, \quad u \in \dot{H}^1(\mathbb{R}^n), \quad 2^* = \frac{2n}{n-2},$$

is a classical example for Palais-Smale sequences $(E(u_j) \leq C, E'(u_j) \rightarrow 0 \text{ in } \dot{H}^{-1}(\mathbb{R}^n))$ failing to be compact. Indeed, a non trivial solution $\omega \in \dot{H}^1(\mathbb{R}^n)$ of $E'(\omega) = 0$ automatically generates a noncompact family of solutions

$$g_{a,\lambda}\omega = \lambda^{\frac{2-n}{2}} \omega\left(\frac{\cdot - a}{\lambda}\right), \quad a \in \mathbb{R}^n, \lambda > 0,$$

as the group $\{g_{a,\lambda}\}$ leaves both energy terms invariants and converges weakly to 0 as $\lambda + \lambda^{-1} + |a| \rightarrow \infty$. A seminal result of Struwe [10] asserts however that this mechanism is essentially the only way to lose compactness: this is a typical assertion of the theory of concentration-compactness developed in the 1980's. More precisely, given a Palais-Smale sequence $(u_j) \subset \dot{H}^1(\mathbb{R}^n)$, there exist an integer N , sequences of translations $a_{1,j}, \dots, a_{N,j} \in \mathbb{R}^n$ and scales $\lambda_{1,j}, \dots, \lambda_{N,j} > 0$,

and nontrivial critical profiles $\omega_1, \dots, \omega_N \in \dot{H}^1(\mathbb{R}^n)$, $E'(\omega_k) = 0$, such that (u_j) behaves like ω_k at scale $\lambda_{k,j}$ around $a_{k,j}$:

$$(1) \quad g_{a_{k,j}, \lambda_{k,j}}^{-1} u_j \rightharpoonup \omega_k \quad \text{weakly in } \dot{H}^1(\mathbb{R}^n),$$

and this accounts for all the lack of compactness in that there is no energy loss:

$$(2) \quad \int_{\mathbb{R}^n} |\nabla u_j|^2 dx \rightarrow \sum_{k=1}^N \int_{\mathbb{R}^n} |\nabla \omega_k|^2 dx, \quad E(u_j) \rightarrow \sum_{k=1}^N E(\omega_k).$$

Moreover, the rescaled profiles $g_{a_{k,j}, \lambda_{k,j}} \omega_k$ are weakly interacting:

$$I((a_{k,j}, \lambda_{k,j})_k) := \min_{k \neq \ell} \left(\frac{\lambda_{k,j}}{\lambda_{\ell,j}} + \frac{\lambda_{\ell,j}}{\lambda_{k,j}} + \frac{|a_{k,j} - a_{\ell,j}|^2}{\lambda_{k,j} \lambda_{\ell,j}} \right) \rightarrow +\infty,$$

which makes them asymptotically orthogonal and implies therefore

$$u_j - \sum_{k=1}^N g_{a_{k,j}, \lambda_{k,j}} \omega_k \rightarrow 0 \quad \text{strongly in } \dot{H}^1(\mathbb{R}^n).$$

In this last assertion, the choice of comparing (u_j) to the sum of rescaled profiles is, in some sense, arbitrary: any other sequence with the properties (1)-(2) will be strongly close to (u_j) . If one further assumes that $u_j \geq 0$, one can take $\omega_k = \omega$ for all k , where ω is the unique (modulo $G = \{g_{a,\lambda}\}$) nonnegative solution of $E'(\omega) = 0$. Recently, Figalli and Glaudo [5] proved a quantitative version of this compactness statement: for any u close enough to a sum of rescaled profiles $\sum g_{a_k, \lambda_k} \omega$ in $\dot{H}^1(\mathbb{R}^n)$ with $I((a_k, \lambda_k)) \gg 1$, we have

$$\inf_{(\tilde{a}_k, \tilde{\lambda}_k)} \left\| u - \sum_{k=1}^n g_{\tilde{a}_k, \tilde{\lambda}_k} \omega \right\|_{\dot{H}^1} \leq c_{n,N} \|E'(u)\|_{\dot{H}^{-1}} \quad \text{if } 3 \leq n \leq 5,$$

but this estimate cannot be true for $n \geq 6$. This surprising dependence on the dimension has been completely described by Deng, Sun and Wei [4], who proved

$$\inf_{(\tilde{a}_k, \tilde{\lambda}_k)} \left\| u - \sum_{k=1}^n g_{\tilde{a}_k, \tilde{\lambda}_k} \omega \right\|_{\dot{H}^1} \leq c_{n,N} \zeta_n(\|E'(u)\|_{H^{-1}}),$$

for an optimal rate function $\zeta_n(t) \gg t$ as $t \rightarrow 0$ for $n \geq 6$. Similar results have since then been obtained in related contexts (fractional Sobolev spaces, Sobolev spaces on the Heisenberg group, Yamabe functionals, etc), simplifying the original argument, see e.g. [1, 2, 3] and many more. Analogous estimates have also been established in some geometric problems (harmonic maps, parametric surfaces with prescribed curvature), see e.g. [7, 8, 9].

In [6], we develop an abstract Hilbert space framework to prove such estimates for energy functionals invariant under the action of a noncompact group of isometries, near weakly-interacting sums of critical profiles. Our abstract framework applies to the existing results, and motivates a new take on the surprising dimension-dependent behavior.

Applied to the above functional E , our results provide the existence of a *profile decomposition model* $\sigma((a_k, \lambda_k)) \in \dot{H}^1(\mathbb{R}^n)$, which is a C^1 function of the scales and translation parameters (a_k, λ_k) , such that

$$\inf_{(\tilde{a}_k, \tilde{\lambda}_k)} \|u - \sigma((\tilde{a}_k, \tilde{\lambda}_k))\|_{\dot{H}^1} \leq c_{n,N} \|E'(u)\|_{H^{-1}},$$

whenever u is close enough to $\sum g_{a_k, \lambda_k} \omega$ with $I((a_k, \lambda_k)) \gg 1$. The linear dependence on $\|E'(u)\|$ is optimal and independent of the dimension. This is completely analogous to auxiliary estimates already present in [4] and subsequent works, and in that sense, not new. We emphasize however that this dimension-independent quantification of Struwe's decomposition can be interpreted as more adapted to the geometry of E , than estimating the distance to sums of rescaled profiles.

This last claim can be illustrated on a very simple finite-dimensional example: consider $F(x) = e^{-x_1} + x_2^2$ on $\mathbb{R}^2$, then the analog of Struwe's qualitative result is that $|F'(x)| \ll 1$ implies $\|x - (\lambda, 0)\| \ll 1$ for some $\lambda \gg 1$. The same qualitative result is true for $F(x) = e^{-x_1} + (x_2 - f(x_1))^2$ and any $f(x_1) \rightarrow 0$ as $x_1 \rightarrow +\infty$ with $|f'| \leq 1/2$. It is however clear that a natural quantitative estimate should be on the distance of $\|x - (\lambda, f(\lambda))\|$ rather than $\|x - (\lambda, 0)\|$, which could be very badly estimated in terms of $|F'(x)|$. Among all points $z = (\lambda, t)$, the choice $t = f(\lambda)$ is the only one at which $F'(z)$ is parallel to the qualitative asymptotic manifold $\{(\lambda, 0)\}$: this criterion is precisely how we select the profile decomposition model σ in our abstract framework.

The energy convergence (2) is (unlike the closeness to sums of rescaled profiles) completely canonical in that it does not depend on an arbitrary choice of profile decomposition. Our abstract framework also deals with quantifying that energy convergence, via methods reminiscent of [7]. Again in the context of the above energy E , this provides the new estimate

$$|E(u) - NE(\omega)| \leq c_{n,N} \|E'(u)\|_{\dot{H}^{-1}},$$

near sums of N weakly interacting rescalings of the profile ω . The linear dependence on $\|E'(u)\|$ is optimal and dimension-independent. The analogous estimate for $\int_{\mathbb{R}^n} |\nabla u|^2 dx$ is also valid, as well as optimal estimates on the interaction $I((a_k, \lambda_k))$.

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