

Report No. 14/2026

DOI: 10.4171/OWR/2026/14

Arbeitsgemeinschaft: Arithmetic Holonomy Bounds and Applications to Irrationality

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29 March – 3 April 2026

ABSTRACT. The purpose of the Arbeitsgemeinschaft was to study the paper [2] of Calegari–Dimitrov–Tang on arithmetic holonomy bounds and their applications to irrationality, and most centrally the proof of the $\mathbb{Q}$ -linear independence of 1 , $\zeta(2)$, and $L(2, \chi_{-3})$.

Mathematics Subject Classification (2020): 11J72, 11M06, 11F11, 34M03, 34M35, 30C35, 33C20.

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Introduction by the Organizers

Apéry’s proof that $\zeta(3)$ is irrational introduced a new technique for proving the irrationality of interesting periods. As was soon realized by Beukers and Bombieri, Apéry’s construction is linked to G -functions and their arithmetic properties. The goal of the Arbeitsgemeinschaft was to understand the paper [2], which devised new methods to exploit the global arithmetic structure of the G -functions involved in Apéry style constructions, ultimately with the goal to prove the irrationality or linear independence of the Apéry limits involved. The workshop focused most particularly on the case of the $\mathbb{Q}$ -linear independence of 1 , $\zeta(2)$, and $L(2, \chi_{-3})$. Along with Apéry’s $\zeta(3)$, this is the only irrationality proof hitherto found for a non-critical special value of the L -function of a classical modular form.

The lectures were divided into four series A, B, C, and D, which ran in parallel rather than in sequence. The order of delivery of the lectures is shown in the following table.

	Schedule of the Arbeitsgemeinschaft				
	Mon	Tue	Wed	Thur	Fri
Morning I	A.1	C.2	C.4	B.5	C.7
Morning II	B.1	B.3	C.5	D.1	C.8
Afternoon I	C.1	B.4	Kirschtorte	C.6	Discussion
Afternoon II	B.2	C.3	Kirschtorte	A.2	Discussion

The thematic grouping of the four series was as follows.

- (§A) *Applications to irrationality.* Johannes Sprang opened up the first series by introducing the Apéry limits method through the standard proof of the irrationality of $\log 2$ using two holonomic functions with denominator control of type $[1, 2, \dots, n]$ and an overconvergence identity. The analogous example of $\log 3$ was then discussed: here the natural approximating forms are divergent, but an appropriate holonomy bound still gives an irrationality proof. The series was concluded by Paul Brommer-Wierig with the applications of the arithmetic holonomy bounds to give arithmetic characterization of the functions $\log(1-x)$ and $\log^2(1-x)$, as well as to the irrationality of the two-logarithms products $\log(1+1/m)\log(1+1/n)$ for all positive integer pairs m, n with $|1 - m/n| < 10^{-6}$.
- (§B) *G-functions, differential equations, and modular periods.* This series supplied the geometric and differential-equation background. Kenz Kallal's talk reviewed local systems with flat (integrable) connections, and explained in detail the basic but fundamental example of the Picard–Fuchs equation of the Legendre family $y^2 = x(x-1)(x-\lambda)$, whose periods are complete elliptic integrals, and whose regular solution at the origin is the hypergeometric G -function ${}_2F_1(1/2, 1/2; 1; \lambda)$. Michael Barz's talk introduced the geometric theory of G -functions and the fundamental notions of p -curvatures and the Chudnovskys' theorem on global nilpotence. Javier Fresán's talk explained the differential algebra and Picard–Vessiot ideas behind the functional bad approximability theorems, going back to Hermite's ideas in his famous memoir on the exponential function, and all the way to the Chudnovsky–Osgood theorem on functional bad approximability. Peter Jossen computed the monodromies of the various G -functions involved in [2], and showed in turn how that computation gave all the requisite functional linear independencies. In the Apéry limits method, the Dirichlet L -values such as $\zeta(2)$ and $L(2, \chi_{-3})$ arise from Eichler integrals of Eisenstein series via the coefficients of the period polynomials, and this was explained in detail in David Savitt's talk.
- (§C) *Arithmetic holonomy bounds.* This was the main technical series with regard to the new developments in [2]. It began with Louise Nataf's talk that explained Siegel's lemma and its use in constructing auxiliary functions, such as in Roth's theorem. Li Lai gave the first holonomy bounds as an upper estimate of the dimension over $\mathbb{Q}(x)$ of spaces of holonomic functions

whose Taylor coefficients satisfy denominator conditions and whose pull-backs by a holomorphic map are meromorphic. Philipp Habegger’s talk explained the concentration of measure phenomenon in high dimensions, and how it combines with the Chudnovsky–Osgood bad approximability theorem to give an enhanced auxiliary function construction. Gabriele Bogo’s talk used the enhanced auxiliary construction with a numerical integration method to prove a refined holonomy bound in the integral coefficients case. Kalyani Kansal’s talk then explained how to handle denominators and additional integrations. The proof of the irrationality of $L(2, \chi_{-3})$ was concluded in Fangu Chen’s talk. Luigi Martinelli and Thomas Mordant’s talks explained Bost’s method of slopes and outlined more conceptual geometric extensions of the holonomy bounds (in particular, over curves of arbitrary genus), continuing the arithmetic intersection theory line of the recent work [1] of Bost and Charles. Their entry in these proceedings announces general holonomy bounds for *eleutherian systems* over an arbitrary algebraic curve, to be treated in detail in a paper in preparation by Bost, Martinelli, and Mordant.

- (§D) *Complex analysis and conformal maps.* The sole talk in this series was delivered by Pablo Chenal. It provided an introduction to potential theory, particularly in anticipation of the final lectures by Martinelli and Mordant, and in order to supply the analytic input for the archimedean estimates in [2]. In the application one chooses a map $\varphi : \overline{D(0,1)} \rightarrow \mathbb{C} \setminus \{1\}$, with $\varphi(0) = 0$ and large enough $|\varphi'(0)|$, and optimizes the archimedean functionals appearing in the holonomy bounds. The lecture explained the contour used in [2] for the $L(2, \chi_{-3})$ problem.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, “US Junior Oberwolfach Fellows”.

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Arbeitsgemeinschaft: Arithmetic Holonomy Bounds and Applications to Irrationality

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Abstracts

A.1: Proving Irrationality: Introduction to the method

JOHANNES SPRANG

1. INTRODUCTION

Euler’s celebrated explicit formula for the values of the Riemann zeta function at the positive even integers raises immediately the question about the values of the Riemann zeta function at the odd positive integers. To this day, very little is known about this question. The first result in this direction is:

Theorem 1.1 (Apéry [1]). $\zeta(3) \notin \mathbb{Q}$.

The next breakthrough was given by Rivoal and Ball–Rivoal:

Theorem 1.2 (Rivoal [9], Ball–Rivoal [2]). *For $\epsilon > 0$ and any sufficiently large positive odd integer s , we have*

$$\dim_{\mathbb{Q}}(\mathbb{Q} + \zeta(3)\mathbb{Q} + \zeta(5)\mathbb{Q} + \cdots + \zeta(s)\mathbb{Q}) \geq (1 - \epsilon) \frac{\log(s)}{1 + \log(2)}.$$

The lower bound on the number of irrational odd zeta values has been improved considerably over the last decade by work of Fischler, Fischler–Sprang–Zudilin, Lai–Yu, Lai, see [4, 5, 6, 7]. Nevertheless, the only particular odd zeta value, for which we can prove irrationality, remains $\zeta(3)$.

Theorem 1.3 (Zudilin, [10]). *At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.*

The following remarkable result gives the first example of a special value of a Dirichlet L -function at a non-critical integer > 1 since Apéry’s result:

Theorem 1.4 (Calegari–Dimitrov–Tang [3]). *The numbers $1, \zeta(2), L(2, \chi_{-3})$ are linearly independent over $\mathbb{Q}$.*

The goal of this *Arbeitsgemeinschaft* is to understand the proof of Theorem 1.4. In this talk, we will explain the basic method at a very simple toy example.

2. “A PROOF THAT EULER MISSED” – VAN DER POORTEN

We write $[1, \dots, m]$ for the least common multiple of the integers $1, 2, \dots, m$. Let us recall the following basic irrationality criterion:

Lemma 2.1. *Let $f(x) = \sum_{n \geq 0} \alpha_n x^n \in \mathbb{Q}[[x]]$. Suppose that:*

- (1) *There exist $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$ such that $[1, \dots, nd_1] \cdots [1, \dots, nd_r] \alpha_n \in \mathbb{Z}$.*
- (2) *$f(x)$ has radius of convergence $\geq R$.*

If $R > \exp(d_1 + \cdots + d_r)$ then $f(x) \in \mathbb{Q}[x]$.

This lemma is usually applied to prove the irrationality of a real number η by applying it to $f(x) = B(x) - \eta A(x)$. Let us briefly sketch the proof of Apéry's theorem.

Sketch of proof of Apéry's theorem: Define

$$A(x) := \sum_{n \geq 0} a_n x^n, \quad \text{with } a_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$B(x) := \sum_{n \geq 0} b_n x^n, \quad \text{with } b_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^k \frac{1}{m^3}.$$

Both power series are solutions of the same ODE with regular singular points at $\{0, (\sqrt{2} \pm 1)^4, \infty\}$, have radius of convergence $(\sqrt{2} - 1)^4$, and the coefficients satisfy $a_n, [1, \dots, n]^3 b_n \in \mathbb{Z}$. One can check that $B(x) - \zeta(3)A(x)$ has radius of convergence exactly $(\sqrt{2} + 1)^4 > \exp(3)$. If $\zeta(3)$ were rational, we would get a contradiction to the Lemma, since $B(x) - \zeta(3)A(x)$ cannot be a polynomial. $\square$

Before proceeding, let us state the following special case of Lemma 2.1:

Corollary 2.2. *Every $f(x) \in \mathbb{Z}[[x]]$ of radius of convergence > 1 is a polynomial.*

This has been generalized by Borel and Pólya:

Theorem 2.3 (Borel, Pólya). *If $f(x) \in \mathbb{Z}[[x]]$ is the germ of a holomorphic function on a simply connected domain Ω with $0 \in \Omega$ of conformal mapping radius $\rho(\Omega, 0) > 1$, then $f(x) \in \mathbb{Q}(x)$.*

This naturally raises the question of a generalization of Lemma 2.1 replacing condition 2 by a condition on a conformal mapping radius. We will see such a generalization at the end of the talk.

3. A PROOF THAT VAN DER POORTEN MISSED

Fix an integer $N \geq 2$ and consider

$$P(x) := x^2 - 2Nx + 1$$

with roots $\alpha = N - \sqrt{N^2 - 1}$ and $\beta = N + \sqrt{N^2 - 1}$. The case $N = 3$ of this example has been discussed by van der Poorten in [8]. We will see that a systematic investigation of this example can guide us in a natural way towards a version of Lemma 2.1 with denominators. The functions

$$A(x) := \frac{1}{\sqrt{P(x)}}$$

$$B(x) := A(x) \cdot \int_0^x A(t) dt$$

are solutions of the same ODE with regular singular points at $\{\alpha, \beta, \infty\}$ and both have radius of convergence equal to α , however, one can prove that $B(x) - \eta A(x)$ for $\eta = \frac{1}{2} \log \left(\frac{N+1}{N-1} \right)$ has radius of convergence β . On the other hand, it is not

difficult to show the following integrality properties for the coefficients of the Taylor expansions of $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$:

$$a_n, [1, \dots, n]b_n \in \begin{cases} \mathbb{Z} & \text{if } N \text{ is odd,} \\ 2^{-n}\mathbb{Z} & \text{if } N \text{ is even.} \end{cases}$$

It turns out that Lemma 2.1 applied to $f(x) = B(x) - \eta A(x)$ suffices to prove irrationality of $\eta = \frac{1}{2} \log \left(\frac{N+1}{N-1} \right)$ in all cases except for $N = 2$. However, a refined irrationality criterion by Zudilin (see [11]) allows one to use $B(x) - \eta A(x)$ to prove irrationality of $\eta = \frac{1}{2} \log \left(\frac{N+1}{N-1} \right)$ for all values of N . Roughly speaking, this theorem can be formulated as follows: Suppose that $f(x) = \sum_{n \geq 0} \alpha_n x^n \in \mathbb{Q}[[x]]$ is the germ of a holomorphic function on $\Omega = \mathbb{C} \setminus [R, \infty)$ and satisfies the integrality assumption (1) of Lemma 2.1. If $4R > \exp \left(\frac{3}{2}(d_1 + \dots + d_r) \right)$ then $f \in \mathbb{Q}(x)$. Since $4R = \rho(\Omega, 0)$ is exactly the conformal mapping radius of $\Omega = \mathbb{C} \setminus [R, \infty)$, this indicates that there should be a generalization of the Theorem of Borel and Pólya to power series with denominators. Such a result has been proven by Calegari–Dimitrov–Tang in [3, Theorem 2.7.10] as a special case of much more general holonomy bounds:

Theorem 3.1 (Calegari–Dimitrov–Tang [3]). *Let $f(x) = \sum_{n \geq 0} \alpha_n x^n \in \mathbb{Q}[[x]]$. Suppose that*

- (1) *There exist $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$ such that $[1, \dots, nd_1] \cdots [1, \dots, nd_r] \alpha_n \in \mathbb{Z}$.*
- (2) *$f(x)$ is the germ of a meromorphic function on a simply connected domain $\Omega \ni 0$ of conformal mapping radius $\rho(\Omega, 0)$.*

If $\rho(\Omega, 0) > \exp \left(\frac{3}{2}(d_1 + \dots + d_r) \right)$ then $f(x) \in \mathbb{Q}[x]$.

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A.2: Applications to functions on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and irrationality of products of logarithms

PAUL BROMMER-WIERIG

1. APPLICATIONS TO FUNCTIONS ON $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

We use the notations $m, r, \mathbf{b}, \mathbf{e}, \sigma_m, \tau^b(\mathbf{b}), \tau^\sharp(\mathbf{e}), \tau(\mathbf{b}, \mathbf{e})$ as in the earlier talks. Our goal is to apply the following holonomy bound, to be proved in lectures C.7 and C.8, to functions on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$:

Theorem 1.1 (Theorem 7.0.1 of [2]). *Let $\varphi: (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic with conformal radius*

$$\log(|\varphi'(0)|) > \max\{\sigma_m, \tau(\mathbf{b}, \mathbf{e})\}.$$

Assume there exist $f_1, \dots, f_m \in \mathbb{Q}[[x]]$ linearly independent over $\mathbb{Q}(x)$ of denominator types

$$f_i(x) = a_{i,0} + \sum_{n=1}^{\infty} \frac{a_{i,n}x^n}{n^{e_i}[1, \dots, b_{i,1}n] \cdots [1, \dots, b_{i,r}n]}$$

with $a_{i,n} \in \mathbb{Z}$. If $f_i(\varphi(z)) \in \mathbb{C}[[z]]$ is the germ of a meromorphic function on $|z| < 1$ for all $1 \leq i \leq m$, then

$$m \leq \frac{1}{\log(|\varphi'(0)|) - \tau(\mathbf{b}, \mathbf{e})} \iint_{\mathbb{T}^2} \log(|\varphi(z) - \varphi(w)|) \mu(z) \mu(w).$$

If moreover all the f_i are holonomic then the inequality $\log(|\varphi'(0)|) > \tau(\mathbf{b}, \mathbf{e})$ suffices.

This bound includes the basic form proved in C.5 as the special case $\mathbf{e} = 0$. If $\mathbf{e} = 0$ and φ is the uniformization of a contractible domain $\Omega \subsetneq \mathbb{C}$ containing 0, we obtain the so-called *univalent holonomy bound*:

$$(1) \quad m \leq \frac{\log(\rho(\Omega, 0))}{\log(\rho(\Omega, 0) - \tau(\mathbf{b}))}.$$

From (1) we get the following arithmetic characterization of the logarithm:

Theorem 1.2 (Theorem 2.7.2 of [2]). *Suppose $f \in \mathbb{Q}[[x]]$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$ and of denominator type $[1, \dots, n]$. Then there exist $Q_0, Q_1 \in \mathbb{Q}(x)$ such that*

$$f(x) = Q_0(x) + Q_1(x) \log(1 - x).$$

Proof. We apply the univalent bound to the uniformization of $\Omega = \mathbb{C} \setminus [1, \infty)$ which is the Koebe map $\varphi: \mathbb{D} \rightarrow \Omega$ given by $\varphi(z) = \frac{4z}{(1+z)^2}$. The conformal radius of φ is $\varphi'(0) = 4$. Then for $\mathbf{b} = (0 \ 1 \ 1)^t$ we get $\tau(\mathbf{b}) = 8/9$ and hence

$$m \leq \frac{\log(4)}{\log(4) - \frac{8}{9}} = 2,787 \dots < 3.$$

□

Remark 1.3. Using additional inputs from transcendence theory one can even show that $Q_0, Q_1 \in \mathbb{Q}[x, \frac{1}{1-x}]$ in the conclusion of Theorem 1.2, see [2, pp. 20–21].

The next goal is to state a conjecture that generalizes Theorem 1.2 to denominator types $[1, \dots, n]^2$. Let us say that a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Q}[[x]]$ convergent on $|x| < 1$ satisfies *Condition M* if:

- (1) There exists a positive integer M such that $[1, \dots, n]^2 a_n \in M^{-1}\mathbb{Z}$ for all $n \geq 0$.
- (2) f admits an analytic continuation to a holomorphic function on $\mathbb{C} \setminus [1, \infty)$.
- (3) f admits an analytic continuation to a meromorphic function along all paths in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

In order to apply Theorem 1.1 to the space $\mathcal{H}$ of all functions satisfying Condition M, we need an auxiliary function $\varphi: \mathbb{D} \rightarrow \mathbb{C} \setminus \{1\}$ with $\varphi^{-1}(0) = \{0\}$. By [3, bottom of p. 198] the modular map λ maximizes the conformal radius among all such functions. Moreover, André’s holonomicity criterion [1, § VIII 1.6] (which can also be reproved via Theorem 1.1) implies that any $f(x) \in \mathbb{Q}[[x]]$ of denominator type $[1, \dots, n]^2$, with the property that $f(\lambda(z))$ is the germ of a meromorphic function on $\mathbb{D}$, is automatically holonomic. Since $\lambda'(0) = 16 > e^2$, Theorem 1.1 implies that $\mathcal{H}$ is finite-dimensional.

Define $A_0(x) = 1$, $A_1(x) = \log(1-x)$, $A_2(x) = \log^2(1-x)$, $A_3(x) = \text{Li}_2(x)$ and $A_4(x) = \frac{1}{\sqrt{1-x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-t}} dt$. The following conjecture is about classifying all functions in $\mathcal{H}$:

Conjecture 1 (Conjecture 2.8.1 of [2]). Let $f(x) \in \mathbb{Q}[[x]]$ be convergent on $|x| < 1$. Then the following are equivalent:

- (1) f satisfies Condition M.
- (2) There exist $Q_0, \dots, Q_4 \in \mathbb{Q}(x)$ such that

$$f(x) = \sum_{i=0}^4 Q_i(x) A_i(x).$$

Remark 1.4. As before, it is even conjectured that $Q_i(x) \in \mathbb{Q}[x, \frac{1}{1-x}]$. This refinement is especially important for applications to actual periods. For example the refined conjecture implies that the special values $\text{Li}_2(1/n)$ for $n \in \mathbb{Z} \setminus \{0, 1\}$ are irrational.

Unfortunately Conjecture 1 withstands the application of Theorem 1.1, as the best bound one can get so far is $\dim(\mathcal{H}) \leq 9$ using the same contour as in the proof of the irrationality of $L(2, \chi_{-3})$. On the other hand, it is possible to completely classify functions of denominator type $[1, \dots, n][1, \dots, \frac{n}{2}]$ subject to a related condition, see [2, Thm. 2.8.4].

2. IRRATIONALITY OF PRODUCTS OF LOGARITHMS

In this section we sketch the proof of the following theorem:

Theorem 2.1 (Theorem 14.0.1 of [2]). *Let $m, n \in \mathbb{Z} \setminus \{-1, 0\}$ be such that $|\frac{m}{n} - 1| < 10^{-6}$. Then*

$$1, \log\left(1 + \frac{1}{m}\right), \log\left(1 + \frac{1}{n}\right) \text{ and } \log\left(1 + \frac{1}{m}\right) \log\left(1 + \frac{1}{n}\right)$$

are linearly independent over $\mathbb{Q}$.

We may assume that $m \neq n$, otherwise the statement of Theorem 2.1 follows from Baker's theorem. Let $a = 2m + 1$ and $b = 2n + 1$ and rewrite

$$\log\left(1 + \frac{1}{m}\right) = \log\left(\frac{a+1}{a-1}\right), \quad \log\left(1 + \frac{1}{n}\right) = \log\left(\frac{b+1}{b-1}\right).$$

Since a is odd, the function $A(a, x) = \frac{1}{\sqrt{1-2ax+x^2}}$ is in $\mathbb{Z}[[x]]$. Furthermore, $A(a, x)$ satisfies the following ODE:

$$(2) \quad (1 - 2ax + x^2)y' + (x - a)y = 0.$$

Consider the Hadamard product $P_A(x) = A(a, x) \star A(b, x)$. It satisfies an order 2 homogeneous ODE $\mathcal{M}$ with apparent singularities at ± 1 and essential singularities at 0 and ∞ . The singularities $(a \pm \sqrt{a^2 - 1})$ of (2) (and similarly those of the ODE for $A(b, x)$) contribute the four singularities $(a \pm \sqrt{a^2 - 1})(b \pm \sqrt{b^2 - 1})$ of $\mathcal{M}$. We denote the singularity $(a - \sqrt{a^2 - 1})(b - \sqrt{b^2 - 1})$, which is close to 0, by α .

Remark 2.2. The assumption $|\frac{m}{n} - 1| < 10^{-6}$ in Theorem 2.1 is not arbitrary but stems from the following phenomenon: Ultimately we construct inhomogeneous ODEs in $\mathcal{M}$ whose solutions overconverge beyond α and to which we apply Theorem 1.1. In order to execute this strategy we are forced to choose an auxiliary function $\varphi: \mathbb{D} \rightarrow \mathbb{C} \setminus \{1\}$ which avoids values within ε of 1 (in particular the singularities $(a - \sqrt{a^2 - 1})(b + \sqrt{b^2 - 1})$ and $(a + \sqrt{a^2 - 1})(b - \sqrt{b^2 - 1})$). In the end our φ will be more or less of the form $\varphi = \psi \circ \lambda$ where $\psi: \overline{\mathbb{D}} \rightarrow \mathbb{D}$ is the uniformization of a suitably chosen domain. But if ψ hits e.g. $3/4$ then φ hits $\lambda(3/4) = 0,9999999999999798332\dots$. This is why we have to be so careful concerning the numerics.

We now consider the inhomogeneous ODE

$$(3) \quad (1 - 2ax + x^2)y' + (x - a)y = 1.$$

Then (3) has a unique $\mathbb{Q}[[x]]$ solution vanishing at 0, namely

$$H(a, x) = A(a, x) \int_0^x A(a, t) dt.$$

Since a is odd, $H(a, x)$ has denominator type $[1, \dots, n]$. Define

$$\eta_a = \frac{1}{2} \log\left(\frac{a+1}{a-1}\right), \quad \eta_b = \frac{1}{2} \log\left(\frac{b+1}{b-1}\right), \quad \eta_{ab} = \eta_a \eta_b$$

as well as

$$\begin{aligned} P_a(x) &= (H(a, x) - \eta_a A(a, x)) \star A(b, x), \\ P_b(x) &= A(a, x) \star (H(b, x) - \eta_b A(b, x)), \\ P_{ab}(x) &= H(a, x) \star H(b, x) - \eta_{ab} A(a, x) \star A(b, x). \end{aligned}$$

Then P_a, P_b, P_{ab} all overconverge beyond α . Assuming now that there exist $r_0, r_a, r_b, r_{ab} \in \mathbb{Z}$ not all zero such that

$$(4) \quad r_a \eta_a + r_b \eta_b + r_{ab} \eta_{ab} = r_0$$

then the function

$$P = r_a P_a + r_b P_b + r_{ab} P_{ab}$$

is a power series in $\mathbb{Q}[[x]]$. It is of denominator type $[1, \dots, n]^2$ and satisfies an inhomogeneous ODE $\mathcal{M}P \in \mathbb{Q}[x]$. We then take the $+$ -symmetrizations $G_A(y) = \text{Sym}^+ P_A(x)$ and $G(y) = \text{Sym}^+ P(x)$. Then G_A still has integer coefficients, whereas G is of denominator type $[1, \dots, 2n]^2$. Denote by $y_{a^\pm, b^\pm}$ the images of $(a \pm \sqrt{a^2 - 1})(b \pm \sqrt{b^2 - 1})$ under $\pi: Y(2) \rightarrow Y_0(2), x \mapsto y = x + \frac{x}{x-1}$. Notice that there exists a non-zero differential operator $\mathcal{L}$ on $Y_0(2)$ with the property $\mathcal{L}G_A(y) = 0$ and $\mathcal{L}G(y) \in \mathbb{Q}[y]$ and such that the singularities of $\mathcal{L}$ are contained in $\{0, 4, y_{a^\pm, b^\pm}, \infty\}$. We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume there exists a relation (4). Thus there exists a power series $G(y)$ with $\mathbb{Q}$ -coefficients of denominator type $[1, \dots, 2n]^2$ to which the previous analysis applies. There are 17 functions linearly independent over $\mathbb{C}(y)$, one of which is $G(y)$, whose denominator types are given by the following denominator matrix

$$\mathbf{b} = \begin{pmatrix} 0 & 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 0 & 2 & \cdots & 2 \end{pmatrix}^t \in \mathbb{R}^{17 \times 2}$$

and the following integration vector

$$\mathbf{e} = (0 \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad 1 \quad \cdots \quad 1)$$

where the first $\cdots$ refers to a total of six 0s and the second $\cdots$ to a total of eight 1s. (For details, we refer to [2, Lem. 14.3.1].) Let $h = \lambda + \frac{\lambda}{\lambda-1}$ and

$$h(z, c) = \frac{z(1+c^2) - 1 - c^2 + \sqrt{(1+c^2)^2(1+z)^2 - 16c^2z}}{2(c^2 - 1)}.$$

Then for $\Sigma_{Y_0(2)}^0 = \{0, y_{a^-, b^-}\}$ and $\Sigma_{Y_0(2)}^1 = \{y_{a^+, b^-}, y_{a^-, b^+}, y_{a^+, b^+}, \infty\}$ the map $\varphi: \overline{\mathbb{D}} \rightarrow \mathbb{C}, \varphi = h \circ \psi$ where $\psi = -\frac{3}{4}h(z, \frac{23}{10})$ has a univalent leaf over $U_{Y_0(2)} = D(0, \frac{1}{100})$. Now we can apply Theorem 1.1 to get

$$\begin{aligned} m &\leq \frac{1}{\log(256 \cdot \frac{1287}{2516}) - \frac{1032659}{242760}} \iint_{\mathbb{T}^2} \log(|\varphi(z) - \varphi(w)|) \mu(z) \mu(w) \\ &< \frac{1}{\log(256 \cdot \frac{1287}{2516}) - \frac{1032659}{242760}} \cdot 10 < 17, \end{aligned}$$

arriving at a contradiction. $\square$

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B.1: Arithmetic of ordinary differential equations of geometric origin

KENZ KALLAL

The purpose of this lecture is to introduce two ideas:

- (I) Modular forms, and more generally periods of de Rham cohomology classes, satisfy *ODEs of geometric origin* (henceforth referred to as ODEGOs for short). These ODEs have already been seen to be relevant in lecture A.1: they enforce recurrence relations on coefficients of modular forms as power series (suitably interpreted). They will come back into play in lecture B.3.
- (II) There is a general relationship between ODEGOs and *G-Functions*, which are power series satisfying an ODE whose coefficients have special arithmetic properties; this fits into a general theory that is not yet completely proven — the missing link is the Bombieri–Dwork conjecture. This will set the stage for lecture B.2.

For the basic algebraic theory of ODEs in characteristic zero, we consider a *flat connection* on a vector bundle V (a map of vector bundles $\nabla : V \rightarrow V \otimes \Omega^1$ satisfying the Leibniz rule and having local bases of solutions) to be the basic definition. We note that for dimension reasons, given a local vector field X , local sections of V always satisfy an ODE which is a polynomial in the operator $\nabla_X = \langle \nabla(-), X \rangle$. A standard reference for the basic theory is [1].

ODEGOs are “of geometric origin.” That means that they are built from ODEs derived from a smooth proper map $\pi : X \rightarrow \operatorname{Spec} \overline{\mathbf{Q}}(\lambda)$. A classic example of such a map is the generic fiber of the Legendre family of elliptic curves

$$\mathcal{E} = \{y^2 = x(x-1)(x-\lambda)\} \rightarrow S = \mathbf{P}_{\overline{\mathbf{Q}}}^1 - \{0, 1, \infty\}$$

where the coordinate on $\mathbf{P}^1$ is λ . In fact this is the universal family on $Y(2)_{\overline{\mathbf{Q}}}$. That there is an isomorphism $Y(2)_{\overline{\mathbf{Q}}} \cong \mathbf{P}_{\overline{\mathbf{Q}}}^1 - \{0, 1, \infty\}$ can be seen using the fact that $X(2)$ has genus zero. The isomorphism can be constructed complex-analytically by the hauptmodul

$$\begin{aligned} \lambda : Y(2)_{\mathbf{C}} &\rightarrow \mathbf{P}^1 - \{0, 1, \infty\}, \\ \lambda(\tau) &= \left(\frac{\sum_{n \in \mathbf{Z}} q^{(n+1/2)^2}}{\sum_{n \in \mathbf{Z}} q^{n^2}} \right)^4 \end{aligned}$$

where $q = \exp(\pi i \tau)$. To get an ODE, we need to get a vector bundle on S with flat connection ∇ , and we need a choice of section of that vector bundle. Our starting data will be as follows:

- (1) The vector bundle will be the relative (algebraic) de Rham cohomology $\mathcal{V} := H_{dR}^1(\mathcal{E}/S)$, which is a rank-2 vector bundle on S .
- (2) The section of $\mathcal{V}$ will be defined by considering the “obvious” section of the Hodge bundle $\omega := \pi_* \Omega_{\mathcal{E}/S}^1$ given by $\eta = dx/y$. This includes into $\mathcal{V}$ via the Hodge filtration (in this context it is just considering dx/y as a family of differentials of the second kind).
- (3) The connection ∇ on $\mathcal{V}$ will be the Gauss–Manin connection, which is defined by saying that the periods of $\nabla_X v$ are the directional derivatives along X of the periods of v .

If the “ODE” coming from this data is

$$a_2(\lambda) \nabla_{\frac{\partial}{\partial \lambda}}^2(\eta) + a_1(\lambda) \nabla_{\frac{\partial}{\partial \lambda}}^1(\eta) + a_0(\lambda) \eta = 0,$$

then since the Gauss–Manin connection is given by differentiating periods, we really conclude that the periods

$$(1) \quad 2 \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, 2 \int_0^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

satisfy the literal ODE

$$(2) \quad a_2(\lambda) \frac{\partial^2}{\partial \lambda^2} + a_1(\lambda) \frac{\partial}{\partial \lambda} + a_0(\lambda) = 0.$$

Since the Gauss–Manin connection has an algebraic definition defined over $\overline{\mathbf{Q}}$, we know abstractly that the $a_2(\lambda)$ can further be chosen to be in $\overline{\mathbf{Q}}(\lambda)$. In fact, we compute explicitly that they can be taken to be

$$a_0(\lambda) = 1, a_1(\lambda) = 8\lambda - 1, a_2(\lambda) = 4\lambda(\lambda - 1).$$

We also remark that if we rewrite the elliptic curves in the standard coordinate z for $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$, then $\eta = f(\tau)dz$ for $f(\tau)$ a modular form of weight 1, which turns out to be $2\pi(1 + 2q + 2q^4 + 2q^9 + \cdots)^2$. The two periods above are then obtained by integrating z from 0 to 1 and from 0 to τ , and so are given by $f(\tau)$ and $\tau f(\tau)$, which means that we also conclude that $f(\tau)$ and $\tau f(\tau)$ are equal to the two elliptic integrals (1), and satisfy the ODEGO (2) as well. In general by taking k -th symmetric powers one obtains an order- $(k+1)$ ODEGO satisfied by modular forms of weight k .

Back to our specific ODEGO for the Legendre family: we see from our computation of the a_i that it has regular singularities at $\lambda = 0, 1, \infty$ and in fact it is a hypergeometric ODE with parameters $1/2, 1/2; 1$. Those parameters tell you exactly the eigenvalues of the local monodromies at the singularities, and the fact that the parameters are rational tell you that the eigenvalues are roots of unity (i.e. the local monodromies are quasiunipotent). See [2] for the full detail of the geometric theory of hypergeometric equations. The solutions can therefore be computed as power series in λ as

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right] (\lambda) = \sum_n \frac{(1/2)_n^2}{(n!)^2} \lambda^n = 1 + \frac{1}{2^2} \lambda + \frac{9}{2^6} \lambda^2 + \frac{25}{2^8} \lambda^3 + \frac{1225}{2^{14}} \lambda^4 + \frac{3969}{2^{16}} \lambda^5 + \cdots$$

and

$$-2 \log \left(\frac{\lambda}{16} \right) {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right] (\lambda) - 8 \sum_n \frac{(1/2)_n^2}{(n!)^2} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right) \lambda^n.$$

The first solution is distinguished by the fact that it stays bounded and in fact approaches 1 at the cusp $\tau = i\infty$ (i.e. $\lambda = 0$). Since this is the case for $f(\tau)$, which approaches 2π , we conclude that

$$f(\tau) = 2\pi \cdot {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right] (\lambda).$$

After possible dividing by 2π , we also observe that both solutions involve power series in λ with the property that the LCM of the first N denominators grows at most exponentially in N , but the coefficients themselves also grow at most exponentially. This makes those series G -functions, a definition which will be fully introduced in lecture B.2.

For every positive integer i and every proper smooth map of schemes $\pi : X \rightarrow \operatorname{Spec} \overline{\mathbf{Q}}(\lambda)$, one obtains a Gauss–Manin connection on $H_{dR}^i(X/\overline{\mathbf{Q}}(\lambda))$ and therefore for each element $v \neq 0$ of this finite-dimensional $\overline{\mathbf{Q}}(\lambda)$ -vector space one obtains an ODE of order at most $\operatorname{rk} H_{dR}^i(X/\overline{\mathbf{Q}}(\lambda))$. In particular, one obtains an element

$$D_{\pi,v,i} \in \overline{\mathbf{Q}} \left[\lambda, \frac{d}{d\lambda} \right].$$

The actual definition of an ODEGO is that it is an element of $\overline{\mathbf{Q}} \left[\lambda, \frac{d}{d\lambda} \right]$ given by a product of factors of various $D_{\pi,v,i}$.

In the extended example above, we calculated the ODEGO $D_{\pi:\mathcal{E} \rightarrow S, dx/y, 1}$ — in fact we found it to be a hypergeometric ODE with rational parameters, which therefore has regular singularities with quasiunipotent local monodromies. It is a theorem (see [3]) that $D_{\pi,\eta,i}$ always has regular singularities with quasiunipotent local monodromies. We also solved this ODEGO with a specific hypergeometric series in λ , an element of $\overline{\mathbf{Q}}[[\lambda]]$ which happened to be a G -function. It is a theorem of André [4] that solutions to ODEGOs can be taken to be G -functions in some generalized sense. The converse is the yet-unproved Bombieri–Dwork conjecture. The Chudnovsky–André–Katz theorem [5] is a partial converse, which asserts that every G -function in $\overline{\mathbf{Q}}[[\lambda]]$ satisfies an ODE with regular singularities and quasiunipotent local monodromies. This picture will be filled in some more in lecture B.2.

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B.2: Introduction to G -functions

MICHAEL BARZ

In his talk, Kenz explained the theory of Picard–Fuchs equations to us, and in particular showed us the example of the formal power series

$$(1) \quad 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \cdots,$$

which was the solution to a certain linear ODE of geometric origin. It is interesting to observe that the only prime number which appears in the denominators of the coefficients in Equation 1 is 2, and that the 2-adic valuation of the denominators grows linearly.

It was eventually observed that *all* formal power series solutions to ODEs coming from geometry have similar arithmetic properties, and it is conjectured that the converse should be true as well. We now make this notion more precise. A G -function is a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \bar{\mathbb{Q}}[[x]]$ such that

- (1) $f(x)$ obeys *some* linear ODE in $\bar{\mathbb{Q}}[x]$,
- (2) there is a constant C such that the least common denominator of $a_0, a_1, \dots$, a_n is at most C^n ,
- (3) and there is a constant D such that, for all n , we have $|\alpha_n| \leq D^n$, where α_n is any Galois conjugate of a_n .

G -functions were introduced by Siegel [1], although for a different reason: he suspected one could prove general theorems on the irrationality of special values of G -functions. Perhaps surprisingly, it turns out that this story is closely related to the relationship of G -functions and geometry, as we reveal in a moment.

The key to this relationship is the concept of p -curvature. A formal power series $f(x) \in \mathbb{F}_p[[x]]$ always has the property that $(\frac{d}{dx})^p f(x) = 0$, as can be seen by differentiating x^n . In particular, if f obeys the ODE

$$(2) \quad \frac{df}{dx} = a(x)f(x),$$

then we must have

$$0 = \frac{d^p}{dx^p} f(x) = (a(x)^p + a^{(p-1)}(x))f(x).$$

We call the quantity $a(x)^p + a^{(p-1)}(x)$ the p -curvature of Equation 2, and its nonvanishing can obstruct the existence of solutions to Equation 2. Generally, for an $n \times n$ system of first order ODEs with coefficients in $\mathbb{F}_p(x)$, the p -curvature will be an $n \times n$ matrix with entries in $\mathbb{F}_p(x)$. The p -curvature of the mod p reduction of an ODE is closely related to the p -adic radius of convergence of the solutions to an ODE.

Katz [2] proved that any linear ODE of geometric origin had *nilpotent p -curvature* for almost every prime p . This implies (by the Katz–Honda theorem)

that linear ODEs of geometric origin have quasi-unipotent local monodromies and regular singularities; in particular, by studying the arithmetic properties of ODEs of geometric origin, Katz proved a rather nice theorem about their solutions over the complex numbers.

The Chudnovsky brothers [3] proved that G -functions also had nilpotent p -curvature for almost every primes p . This result was actually a byproduct of their proof of a theorem ensuring the irrationality of special values of G -functions; to prove the irrationality theorem required them to understand the p -adic radius of convergence of solutions to ODEs, which as a side effect told them something about p -curvature.

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B.3: Eichler integrals and Apéry limits

DAVID SAVITT

Our goal in this talk is to explain Beukers’s interpretation [1] of Apéry’s proof of the irrationality of $\zeta(3)$. Recall the basic irrationality criterion that we hope to use.

Lemma 1. *Suppose $f_1, \dots, f_m \in \mathbb{Q}[[x]]$ are power series such that the coefficient of x^n in each f_i has denominator dividing $[1, \dots, n]^r$. If $f = \sum_i \beta_i f_i$ has radius of convergence greater than e^r , and is not a polynomial, then some β_i is irrational.*

The miracle is that it is sometimes possible to apply Lemma 1 in practice.

1. EICHLER INTEGRALS

Let $\mathcal{C}_k = \mathcal{C}^\infty(\mathcal{H})_k$ denote the space of smooth functions on the upper half plane $\mathcal{H}$, endowed with the weight k slash action of $\mathrm{GL}_2^+(\mathbb{R})$:

$$\phi|_k \gamma(\tau) = (\det \gamma)^{k/2} (c\tau + d)^{-k} \phi(\gamma\tau).$$

Write $\mathcal{O}_k \subset \mathcal{C}_k$ for the subspace of holomorphic functions. As usual write $q = e^{2\pi i\tau}$, so that $\theta := q \frac{d}{dq} = (2\pi i)^{-1} \frac{d}{d\tau}$.

Theorem 2 (Bol’s theorem). *Suppose $k \geq 1$. The operator θ^{k-1} defines a map $\mathcal{O}_{2-k} \rightarrow \mathcal{O}_k$; that is, it intertwines the $\mathrm{GL}_2^+(\mathbb{R})$ -actions.*

To see this, let e_k, f_k be the operators $2i \frac{\partial}{\partial \tau} + \frac{k}{y}$ and $-2iy^2 \frac{\partial}{\partial \bar{\tau}}$ respectively. Note that $\ker f_k = \mathcal{O}_k$. One can compute by hand that $e_k : \mathcal{C}_k \rightarrow \mathcal{C}_{k+2}$, $f_k : \mathcal{C}_k \rightarrow \mathcal{C}_{k-2}$, and

$$e_{k-2} f_k - f_{k+2} e_k = k.$$

Package these maps together by defining $\mathcal{C}_{\mathbb{Z}} = \oplus_{k \in \mathbb{Z}} \mathcal{C}_k$ and writing $e = \oplus e_k : \mathcal{C}_{\mathbb{Z}} \rightarrow \mathcal{C}_{\mathbb{Z}}$, and similarly $f = \oplus f_k$. Then

$$[e, f] = h$$

where h is multiplication by k on the k th graded factor of $\mathcal{C}_{\mathbb{Z}}$. One computes $[h, e] = 2e$ and $[h, f] = -2f$ as well, thus proving:

Proposition 3. *The maps e, f, h define an action of $\mathfrak{sl}_2$ on $\mathcal{C}_{\mathbb{Z}}$.*

Proof of Theorem 2. Let V be any $\mathfrak{sl}_2$ -module, and $v \in V$ a vector with $fv = 0$ and $hv = \lambda v$. One checks by induction on m that $fe^m v = -m(\lambda + m - 1)e^{m-1}v$.

An element $v \in \mathcal{O}_{2-k}$ (regarded as an element of $\mathcal{C}_{\mathbb{Z}}$ supported in degree $2 - k$) satisfies the hypotheses of the preceding observation, with $\lambda = 2 - k$. It follows that $fe^{k-1}v = 0$, or equivalently that $e^{k-1}v \in \mathcal{O}_k$ is holomorphic. Now $e^{k-1}v$ has the form $(2i)^{k-1} \left(\frac{\partial}{\partial \tau}\right)^{k-1} v + \sum_{i=1}^{k-1} y^{-i} v_i$ with each v_i holomorphic. The only way this can be holomorphic is if $v_i = 0$ for all i . Thus e^{k-1} and $\left(\frac{\partial}{\partial \tau}\right)^{k-1}$ acting on $\mathcal{O}_{2-k}$ differ by a scalar multiple, hence also θ^{k-1} . The theorem follows, since e^{k-1} intertwines the actions, and θ^{k-1} lands in $\mathcal{O}_k$. $\square$

Let $F = \sum_{n \geq 1} a_n q^n \in M_k(\Gamma, \chi)$ be a holomorphic modular form of weight $k \geq 2$ for some group Γ . The function $\tilde{F} = \sum_{n \geq 1} a_n n^{1-k} q^n$ is evidently a holomorphic preimage of F under θ^{k-1} , called the *Eichler integral* of F . It follows immediately from Theorem 2 that $c(F, \gamma) := \chi(\gamma)^{-1} \tilde{F}|_{2-k} \gamma - \tilde{F}$ is annihilated by θ^{k-1} for all $\gamma \in \Gamma$, and therefore is a polynomial in τ of degree at most $k - 2$.

The coefficients of these polynomials are periods of F , and it is sometimes possible to compute them explicitly. Suppose for example that F is an eigenvector for the Fricke involution $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, so that $F|_k w_N = \varepsilon F$ with $\varepsilon^2 = (-1)^k$. Here is a precise statement when there is no central L -value $L(F, k/2)$.

Proposition 4 ([1, Prop 1.2]). *In the setting above, suppose either k is odd, or else k is even and $\varepsilon = (-1)^{(k-2)/2}$, and write*

$$h(\tau) = \tilde{F}(\tau) - \sum_{r=0}^{\lfloor (k-3)/2 \rfloor} \frac{L(F, k-r-1)}{r!} (2\pi i \tau)^r.$$

Then $h|_{2-k} w_N = \varepsilon \cdot h$.

When $k = 3$, or $k = 4$ and $\varepsilon = -1$, we see that $h(\tau)$ is just $\tilde{F}(\tau) - L(F, k-1)$.

Only $\lfloor (k-1)/2 \rfloor$ different L -values appear here because $L(F, s)$ and $L(F, k-s)$ are related by the functional equation.

2. DIFFERENTIAL EQUATIONS

To see how products of the form $E\tilde{F}$ enter the picture, suppose that $E \in M_{k-2}(\Gamma, \chi^{-1})$ is a modular form of weight $k - 2$. We saw in earlier lectures that there is a linear differential operator $\mathcal{L}$ of order $k - 1$ on $\Gamma \backslash \mathcal{H}$ whose space of solutions is precisely $V = E \cdot \mathbb{C}[\tau]_{\deg \leq k-2}$. Conceptually, the existence of $\mathcal{L}$ with this solution space reflects the observation that V is a representation of Γ

under the Möbius action. The local system $\Gamma \backslash (\mathcal{H} \times V)$ on $\Gamma \backslash \mathcal{H}$ yields a flat connection on a rank $k-1$ vector bundle under the Riemann–Hilbert correspondence. Choosing a cyclic vector gives the operator $\mathcal{L}$.

Proposition 5. *Let $F(\tau) = \sum_{n \geq 1} a_n q^n \in M_k(\Gamma, \chi)$ be a modular form with Eichler integral $\tilde{F}$. Then $\mathcal{L}(E\tilde{F})$ is a modular function for Γ .*

Proof. For $\gamma \in \Gamma$ we compute $(E\tilde{F})|_0\gamma = E|_{k-2}\gamma \cdot \tilde{F}|_{2-k}\gamma = E(\tilde{F} + c(F, \gamma))$. Since $c(F, \gamma)$ is a polynomial of degree at most $k-2$, the term $E \cdot c(F, \gamma)$ is annihilated by $\mathcal{L}$. Since $\mathcal{L}$ is an operator on $\Gamma \backslash \mathcal{H}$ we deduce that

$$\mathcal{L}(E\tilde{F})|_0\gamma = \mathcal{L}((E\tilde{F})|_0\gamma) = \mathcal{L}(E\tilde{F}).$$

Equivalently, $\mathcal{L}(E\tilde{F})$ is a modular function. $\square$

Suppose that $X(\Gamma)$ has genus 0, and let $x : \Gamma \backslash \mathcal{H} \rightarrow \mathbb{C}$ be a Hauptmodul. Then the differential operator $\mathcal{L}$ has coefficients that are rational functions of x ; so that, after clearing denominators, we see that E and $E\tilde{F}$ are solutions to the homogeneous and inhomogeneous linear differential equations $\mathcal{L}(u) = 0$ and $\mathcal{L}(u) = y(x)$ with the same polynomial coefficients.

3. THE APÉRY–BEUKERS ARGUMENT

Consider the following situation. Choose Γ such that $X(\Gamma)$ has genus 0, and suppose $x : \Gamma \backslash \mathcal{H} \rightarrow \mathbb{C}$ is a Hauptmodul whose q -expansion lies in $q + q^2\mathbb{Z}[[q]]$. Assume that $x(\tau_0)$ is the nearest singularity to 0 of the local inverse function x^{-1} to x at 0, so that $|x(\tau_0)|$ is the radius of convergence of x^{-1} , and suppose $\gamma_0 \in \Gamma$ generates the stabilizer of τ_0 . The prototypical example to keep in mind is $\Gamma = \langle \Gamma_1(N), w_N \rangle$, $\gamma_0 = w_N$, and $\tau_0 = i/\sqrt{N}$.

Let $F \in M_k(\Gamma, \chi, \mathbb{Z})$ be a modular form of weight k with $a_0 = 0$ and $F|_k\gamma_0 = \varepsilon F$, and such that $h(\tau) := \tilde{F}(\tau) - L(F, k-1)$ satisfies $h|_{2-k}\gamma_0 = \varepsilon h$. In the prototypical example, we take $k \in \{3, 4\}$ and $\varepsilon = -1$ when $k = 4$, and the property of h follows by Proposition 4.

Finally suppose we have $E \in M_{k-2}(\Gamma, \chi^{-1}, \mathbb{Z})$.

Consider the product $E(\tau)h(\tau) = E(\tau)(\tilde{F}(\tau) - L(F, k-1))$, which by construction is fixed by γ_0 . We may express $E(\tau)h(\tau)$ locally near $q = 0$ as a function of x , or in other words we can expand $E(\tau)$ and $E(\tau)\tilde{F}(\tau)$ in terms of x . In doing so, the coefficients of $E(\tau)$ remain in $\mathbb{Z}$, while since the n th Fourier coefficient of $\tilde{F}(\tau)$ has denominator n^{k-1} we can at least say that the n th coefficient of $E(\tau)\tilde{F}(\tau)$ when expressed in terms of x lies in $\frac{1}{[1, 2, \dots, n]^{k-1}}\mathbb{Z}$.

A generic function $f(\tau)$ convergent on $|q| < 1$, expressed in terms of x , will become branched at $x(\tau_0)$, and will have radius of convergence $|x(\tau_0)|$. But the fact that $E(\tau)h(\tau)$ is γ_0 -invariant implies that $E(\tau)h(\tau)$ descends holomorphically through that branch value. Thus the radius of convergence of $E(\tau)h(\tau)$, expanded in terms of x , is at least the absolute value of the *second* singular value of x .

If one is very lucky, this is greater than e^{k-1} . If $E(\tau)(\tilde{F} - L(F, k-1))$ is not a polynomial in x then it follows by Lemma 1, applied to $E(\tau)(\tilde{F} - L(F, k-1))$ expressed as power series in x , that $L(F, k-1)$ is irrational.

Finally, we note that the coefficients of E and of $E\tilde{F}$, expressed in terms of x , are precisely the sequences a_n and b_n that occur in an Apéry-type proof that $b_n/a_n \rightarrow L(F, k-1)$. The recursive formulas satisfied by a_n and b_n are nothing more than the differential equations $\mathcal{L}(E) = 0$ and $\mathcal{L}(E\tilde{F}) = y(x)$ (for some $y(x)$ in the function field of $X(\Gamma)$).

The irrationality of $\zeta(3)$ follows by an application of this machine with group $\Gamma = \langle \Gamma_1(6), w_6 \rangle$, which has Hauptmodul $x(\tau) = \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(3\tau)\eta(2\tau)} \right)^{12}$. It can be checked that $x(\tau) : \Gamma \backslash \mathcal{H} \rightarrow \mathbb{C}$ has a fundamental domain with cusps $i\infty$ and $\frac{1}{2}$ and corners $\frac{i}{\sqrt{6}}$ and $\frac{2}{5} + \frac{i}{5\sqrt{6}}$. We have $x(i\infty) = 0$, $x(\frac{1}{2}) = \infty$, and

$$x\left(\frac{i}{\sqrt{6}}\right) = (\sqrt{2} - 1)^4, \quad x\left(\frac{2}{5} + \frac{i}{5\sqrt{6}}\right) = (\sqrt{2} + 1)^4.$$

We leave it to the reader to find the modular form $F(\tau) \in M_4(\Gamma_1(6), \mathbb{Z})$ with $a_0 = 0$, $F|_4 w_6 = -F$, and $L(F, 3) = \zeta(3)$ to which the argument applies, and concludes with the observation that $(\sqrt{2} + 1)^4 > e^3$.

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B.4: Functional bad approximability

JAVIER FRESÁN

This talk explained an analogue for holonomic power series of the theorems by Liouville, Roth, and Schmidt stating that algebraic numbers of degree at least 2 are badly approximable by rational numbers. It began with the case of exponentials.

THEOREM 1. Let $\alpha_1, \dots, \alpha_m$ be distinct complex numbers.

- (1) The power series $e^{\alpha_1 x}, \dots, e^{\alpha_m x} \in \mathbb{C}[[x]]$ are linearly independent over $\mathbb{C}[x]$.
- (2) For $P_1, \dots, P_m \in \mathbb{C}[x]$ not all equal to zero,

$$\text{ord}_{x=0}(P_1(x)e^{\alpha_1 x} + \dots + P_m(x)e^{\alpha_m x}) \leq \sum_{i=1}^m (\deg P_i + 1) - 1.$$

- (3) There is a unique tuple up to scalar multiplication for which equality holds.
- (4) The corresponding linear combination is given by

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{e^{zx}}{(z - \alpha_1)^{\deg P_1 + 1} \dots (z - \alpha_m)^{\deg P_m + 1}} dz.$$

The last statement is due to Hermite [5]. In particular, the inequality

$$(1) \quad \text{ord}_{x=0}(e^x - P(x)/Q(x)) \leq \deg P + \deg Q + 1$$

holds for all rational functions $P/Q \in \mathbb{C}(x)$. It is in this sense that the exponential function is badly approximable by rational functions. We call the *Hermite–Padé approximant* of order $[m/n]$ the unique rational function P/Q with $\deg P = m$ and $\deg Q = n$ for which equality holds in (1). Explicitly, we can choose

$$P(x) = \sum_{k=0}^m \binom{m}{k} \frac{(m+n-k)!}{(m+n)!} x^k = {}_1F_1 \left[\begin{matrix} -m \\ -m-n \end{matrix} \middle| x \right]$$

and, similarly, $Q(x) = {}_1F_1 \left[\begin{matrix} -n \\ -m-n \end{matrix} \middle| -x \right]$. A way to reformulate the second statement in the spirit of Bost’s slopes method is that the *evaluation map*

$$\begin{aligned} \mathbb{C}[x]_{\leq d_1} \times \cdots \times \mathbb{C}[x]_{\leq d_m} &\longrightarrow \mathbb{C}[[x]]/(x^{d_1+\cdots+d_m+m}) \\ (P_1, \dots, P_m) &\longmapsto \sum_{i=1}^m P_i(x) e^{\alpha_i x} \end{aligned}$$

is injective (and hence a bijection) for all integers $d_1, \dots, d_m \geq 0$.

After the proof of Theorem 1, the talk discussed Hermite’s use of explicit Hermite–Padé approximants to prove that e is transcendental, and then turned to a question of Kolchin, who after proving an analogue of Liouville’s inequality for holonomic power series [4] asked the following:

KOLCHIN’S PROBLEM. Given a holonomic power series $f \in \mathbb{C}[[x]] \setminus \mathbb{C}(x)$ and $\varepsilon > 0$, prove that there exists a real number $C(\varepsilon, f)$ such that the inequality

$$\text{ord}_{x=0}(f(x) - P(x)/Q(x)) \leq (2 + \varepsilon) \max(\deg P, \deg Q) + C(\varepsilon, f)$$

holds for all rational functions $P/Q \in \mathbb{C}(x)$.

More generally, one could ask for analogues of Schmidt’s subspace theorem about simultaneous approximation of algebraic numbers. The following theorem, subsuming work of Shidlovsky [7], the Chudnovskys [3], Bertrand–Beukers [1], and Osgood [6], contains a general answer to this question.

THEOREM 2. Let $f_1, \dots, f_m \in \mathbb{C}[[x]]$ be $\mathbb{C}(x)$ -linearly independent power series satisfying the linear system of differential equations

$$(2) \quad \frac{d}{dx} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \quad A \in M_{m \times m}(\mathbb{C}(x)).$$

Then there exist real numbers h and c depending only on A such that, for all polynomials $P_1, \dots, P_m \in \mathbb{C}[x]$ not all equal to zero, the inequality

$$\text{ord}_{x=0}(P_1(x)f_1(x) + \cdots + P_m(x)f_m(x)) \leq \mu \max_{1 \leq i \leq m} \deg P_i + \mu h + c$$

holds, where μ is the minimal order of a nonzero differential operator annihilating the linear combination $\sum_{i=1}^m P_i(x)f_i(x)$.

Note that $\mu \leq m$, since the $\mathbb{C}(x)$ -linear span of $f_1, \dots, f_m$ is stable under derivation. If A has simple poles at $S \subset \mathbb{P}^1$, then around each $s \in S$ there is a fundamental matrix of solutions of the form $Y_s \cdot (x-s)^{B_s}$, where $Y_s \in \mathrm{GL}_m(\mathbb{C}[[x-s]])$ and $B_s \in M_m(\mathbb{C})$, and one can take

$$h = - \sum_{s \in S} (\text{smallest real part of an eigenvalue of } B_s),$$

$$c = (|S| - 2) \frac{\mu(\mu - 1)}{2}.$$

This applies, for example, to the situation when $f_1, \dots, f_m$ are G -functions, in which case the eigenvalues of the matrices B_s are rational numbers. The upper bound given by Theorem 2 will be used to create a tension with Siegel's lemma, which provides us with linear combinations with high vanishing order at $x = 0$.

The proof was not discussed in detail, but one useful ingredient from differential Galois theory was highlighted (see [1, Proposition 3]): the differential operator L of minimal order μ that annihilates the linear combination $F(x) = \sum_{i=1}^m P_i(x) f_i(x)$ is obtained as follows. Let G be the differential Galois group of the system (2) and pick a basis $F_1, F_2, \dots, F_\mu$ of the orbit of F under G ; then

$$Ly = \frac{1}{W(F_1, \dots, F_\mu)} \begin{pmatrix} y & y' & \dots & y^{(\mu)} \\ F_1 & F_1' & \dots & F_1^{(\mu)} \\ \dots & \dots & \dots & \dots \\ F_\mu & F_\mu' & \dots & F_\mu^{(\mu)} \end{pmatrix},$$

where $W(F_1, \dots, F_\mu)$ denotes the wronskian. This allows us to write down a particular basis of solutions of L around each point of $\mathbb{P}^1$ involving solutions of the original differential system (2), from which we get a handle on the singularities of L . The inequality of the theorem then comes from an inspection of the generalised orders at $s \in S$ of the wronskian of these bases.

For later use, the talk concluded with the statement [2, Lemma 3.2.14], which is deduced from Theorem 2 by applying the “tensor power trick”. For an integer $d \geq 1$, set

$$\underline{x} = (x_1, \dots, x_d) \quad \text{and} \quad f_{\underline{i}}(\underline{x}) = \prod_{s=1}^d f_{i_s}(x_s) \quad \text{for} \quad \underline{i} = (i_1, \dots, i_d) \in \{1, \dots, m\}^d.$$

Because of the separate variables, the assumption that $f_1, \dots, f_m$ are $\mathbb{C}(x)$ -linearly independent implies that the $f_{\underline{i}}$'s are $\mathbb{C}(\underline{x})$ -linearly independent.

THEOREM 3. Let $f_1, \dots, f_m \in \mathbb{C}[[x]]$ be $\mathbb{C}(x)$ -linearly independent holonomic power series. For each $\varepsilon > 0$, there exists a real number $C(\varepsilon, f_1, \dots, f_m)$ with the following property: let $D \geq 1$ be an integer, and let

$$Q_{\underline{i}}(\underline{x}) \in \mathbb{C}[x_1, \dots, x_d] \quad \text{with} \quad \deg_{x_j} Q_{\underline{i}} < D \quad \text{for all} \quad j \in \{1, \dots, d\}$$

be polynomials, not all equal to zero, indexed by $\{1, \dots, m\}^d$. Then in

$$F(\underline{x}) = \sum_{i \in \{1, \dots, m\}^d} Q_{\underline{i}}(\underline{x}) f_{\underline{i}}(\underline{x}) \in \mathbb{C}[[x_1, \dots, x_d]] \setminus \{0\}$$

every lowest-order nonzero monomial $\beta x_1^{n_1} \cdots x_d^{n_d}$ satisfies

$$n_j \leq (m + \varepsilon)D + C(\varepsilon, f_1, \dots, f_m).$$

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B.5: Linear independence of certain G -functions

PETER JOSSEN

This contribution covers the material of the talk and some additional details. The references [1] and [3] are general resources for G -functions.

1. SINGULARITIES AND DENOMINATORS

We begin by defining sets of G -functions with prescribed singularities and prescribed denominators. For simplicity, we work throughout with G -functions with rational coefficients.

– **1.1.** Singularities of G -functions can and should be understood in terms of singularities of differential operators. Let us recall that a nonzero differential operator

$$L = P_n(x)\partial^n + \cdots + P_1(x)\partial + P_0(x)$$

with $P_0, \dots, P_n \in \mathbb{Q}[x]$ is said to be of *type G* if its size, or equivalently its generic radius of convergence is finite. Equivalently, the operator L is of type G if and only if it admits a full system of solutions of the form

$$(1) \quad \sum_{j=1}^m x^{a_j} \log(x)^{b_j} f_j(x)$$

where $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Z}_{\geq 0}$ and where the $f_j \in \mathbb{Q}[[x]]$ are G -functions. Of course, the monodromy terms $x^{a_j} \log(x)^{b_j}$ are only necessary if $x = 0$ is a singularity of L ,

and we may deduce from their shape that $x = 0$ is at worst a regular singularity with rational local exponents.

In general, by theorems of André and Katz, all singularities of a differential operator of type G are regular with rational local exponents. Factors and products of operators of type G are again of type G . Picard-Fuchs differential operators are of type G , and a conjecture attributed to Bombieri and Dwork predicts that every differential operator of type G is a product of factors of Picard-Fuchs operators. Put more succinctly, the conjecture predicts that every differential operator of type G comes from geometry.

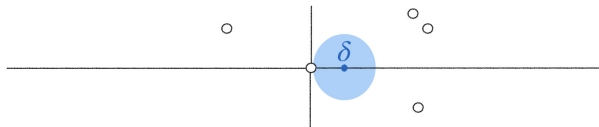
– **1.2.** A theorem of Chudnovski states that every G -function and indeed any expression of the form (1) is annihilated by a differential operator of type G , in fact, any differential operator of minimal degree annihilating a G -function is of type G . Given a finite subset $S \subseteq \mathbb{P}^1(\mathbb{Q})$, we say that a G -function has singularities in S if it is a solution of a nonzero differential operator $L \in \mathbb{Q}[x, \partial]$ with singularities contained in S . Such a G -function f admits analytic continuation along any path in $\mathbb{C} \setminus S$ originating in the disc of convergence of f .

– **1.3.** Let $S \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$ be a finite subset containing $\{0, \infty\}$. The set $\mathcal{G}_S$ of G -functions with singularities in S is a differential subring of $\mathbb{Q}[[x]]$, stable under taking primitives. It contains rational functions with poles contained in the set $S \setminus \{0\}$, and more generally algebraic power series associated with a finite étale covering of $\mathbb{P}^1 \setminus S$. The only G -functions with singularities in $\{0, \infty\}$ are polynomials. Given a second finite set of singularities T , the Hadamard product defines a $\mathbb{Q}$ -linear map

$$\mathcal{G}_S \otimes_{\mathbb{Q}} \mathcal{G}_T \rightarrow \mathcal{G}_{ST}$$

where ST is the set of all products of an element of S and an element of T . In particular, the algebra $\mathcal{G}_S$ is stable under Hadamard products with elements of $\mathcal{G}_{\{0,1,\infty\}}$. The algebra $\mathcal{G}_{\{0,1,\infty\}}$ is quite rich, it contains all hypergeometric G -functions.

– **1.4.** Let S be a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$ containing $\{0, \infty\}$, and let L be a differential operator of type G , say of degree $n > 0$ with singularities contained in S . Solutions of L of the form (1) form a $\mathbb{Q}$ -vector space¹ of dimension n , call it V . These solutions can be interpreted as holomorphic functions on the open disc $U(\delta) = \{z \in \mathbb{C} \mid |z - \delta| < \delta\}$,



¹We stick to rational coefficients, in general this would be a k -vector space for some number field k .

using the standard determination of the logarithm, where of course $\delta > 0$ is small enough so that $U(\delta) \cap S = \emptyset$. Let us denote by $\pi = \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, 0_+)$ the fundamental group of the topological space $\mathbb{P}^1(\mathbb{C}) \setminus S$ with respect to a base point in $U(\delta)$. The group π acts on $V \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{C}}$ by analytic continuation of solutions along paths. This action

$$(2) \quad \rho : \pi \rightarrow \mathrm{GL}_V(\mathbb{C})$$

is the monodromy representation. Since L is regular singular, the Zariski closure $\overline{\rho(\pi)} \subseteq \mathrm{GL}_{V, \mathbb{C}}$ of the image of ρ is equal to the differential Galois group of L . From differential Galois theory we know that the transcendence degree of the differential $\mathbb{C}(x)$ -algebra generated by V is equal to the dimension of the Galois group.

– **1.5.** The field of definition of the monodromy representation (2) is typically a transcendental extension of $\mathbb{Q}$, unless it is a representation with finite image in which case L is solved by algebraic functions. We may think in this situation of V as a rational structure on the $\mathbb{C}$ -linear space of solutions $V_{\mathbb{C}}$ of L . This is the *de Rham* rational structure. If L comes from geometry, then there is a second rational structure on $V_{\mathbb{C}}$, called the *Betti structure*. With respect to this rational structure, the monodromy representation is defined over $\mathbb{Q}$, and even preserves a $\mathbb{Z}$ -lattice.

Definition 1.6. Let $f(x) = a_0 + a_1x + a_2x^2 + \dots$ be a power series with rational coefficients. A denominator for f is any sequence of positive integers $d = (d_j)_{j=0}^{\infty}$ such that $d_0|d_1|d_2|\dots$ and such that

$$d_j a_k \in \frac{1}{N} \mathbb{Z}$$

holds for some fixed integer $N \geq 1$ and for all $j \geq k \geq 0$.

– **1.7.** Every G -function f , by definition, admits a denominator d with the property that

$$(3) \quad \limsup_{n \rightarrow \infty} \frac{\log(d_n)}{n}$$

is finite². It is conjectured that every G -function f admits a denominator of the form

$$d_n = b_0^n \prod_{j=1}^m [1, 2, \dots, b_j n]$$

where $b_0, b_1, \dots, b_m$ are positive integers and $[1, 2, \dots, b_j n]$ stands for the lowest common multiple of all integers $1, 2, \dots, b_j n$. Fischler and Rivoal have shown in [4] that f admits such a denominator if it is annihilated by a differential operator of type G which comes from geometry. Thus, this conjecture about denominators

²Siegel, in his original definition, asks for the weaker condition that $\lim_{n \rightarrow \infty} \frac{\log(d_n)}{n \log(n)} = 0$ holds. It is conjectured that if a holonomic power series f admits a denominator satisfying Siegel's condition, then this denominator automatically satisfies the stronger condition (3).

is a consequence of the conjecture of Bombieri and Dwork on the geometric origin of differential operators of type G .

Question 1.8. Let $\mathcal{G}_S(d)$ denote the set of G -functions with singularities in $S \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$ and denominator d . What is the dimension of the $\mathbb{Q}(x)$ -vector space generated by $\mathcal{G}_S(d)$? Is it finite, and if so, is there a reasonable upper bound?

Question 1.9. Let $\mathcal{L}$ denote the smallest differential $\mathbb{Q}[x, x^{-1}, (1-x)^{-1}]$ -algebra which contains the logarithmic series

$$\mathrm{Li}_1(x) = -\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

and which is stable under Hadamard product. Since Hadamard powers of Li_1 are the polylogarithm series $\mathrm{Li}_2, \mathrm{Li}_3, \dots$, the algebra $\mathcal{L}$ contains all those. What sort of denominators do elements of $\mathcal{L}$ have, and given a denominator d , for example $d_n = [1, 2, \dots, n]^k$ for some $k \geq 1$, can we find a basis for the space of elements of $\mathcal{L}$ with denominator d ? How fast does the dimension grow with k ? And how to explain relations like

$$(\mathrm{Li}_0 \mathrm{Li}_1) \odot (\mathrm{Li}_0 \mathrm{Li}_1) = \mathrm{Li}_0 (\mathrm{Li}_1^2 - \mathrm{Li}_2)$$

in $\mathcal{L}$, or, more optimistically, how to classify them?

2. THE MAP $Y(2) \rightarrow Y_0(2)$

In this section we study a map between modular curves $p: Y(2) \rightarrow Y_0(2)$. Both of these curves are rational, and p is a rational function. Our aim is to understand how p acts on G -functions.

– **2.1.** Recall that $Y(2)$ is the modular curve parametrising elliptic curves together with an ordered pair of two distinct points of exact order 2, and that $Y_0(2)$ is the modular curve parametrising elliptic curves together with one point of exact order 2. There is therefore a map

$$p: \begin{array}{ccc} Y(2) & \longrightarrow & Y_0(2) \\ (E; P, Q) & \mapsto & (E; P) \end{array}$$

given by discarding the second point of order 2. It is a finite map of degree 2. The automorphism of $Y(2)$ over $Y_0(2)$ is the map

$$w: \begin{array}{ccc} Y(2) & \longrightarrow & Y(2) \\ (E; P, Q) & \mapsto & (E; P, P+Q) \end{array}$$

satisfying $w^2 = \mathrm{id}$.

– **2.2.** The curve $Y(2)$ can be identified with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ via the map that sends $\lambda \notin \{0, 1, \infty\}$ to the elliptic curve in Legendre form

$$E: y^2 = x(x-1)(x-\lambda)$$

marked at the two points $P = (1, 0)$ and $Q = (0, 0)$. Via this identification, the automorphism w of $Y(2)$ translates (rechristening x the parameter λ) to the map

$$w : \begin{array}{ccc} \mathbb{P}^1 \setminus \{0, 1, \infty\} & \longrightarrow & \mathbb{P}^1 \setminus \{0, 1, \infty\} \\ x & \mapsto & \frac{x}{x-1} \end{array}$$

whose extension to $\mathbb{P}^1$ exchanges 1 and ∞ and fixes 0. The map w has one other fixed point, namely $x = 2$. The modular curve $Y_0(2)$ can be identified with the quotient of $Y(2)$ modulo $x \sim w(x)$. We find $Y_0(2) = \mathbb{P}^1 \setminus \{0, \infty\}$ via the map

$$(4) \quad p : \begin{array}{ccc} \mathbb{P}^1 \setminus \{0, 1, \infty\} & \longrightarrow & \mathbb{P}^1 \setminus \{0, \infty\} \\ x & \mapsto & x + w(x) = \frac{x^2}{x-1} \end{array}$$

which is a finite étale covering of degree two outside the only ramification point $w(2) = 4$. The elliptic curve $y^2 = x(x-1)(x-2)$ has complex multiplication by $\mathbb{Z}[i]$ where i acts as $i(x, y) = (2-x, iy)$. In particular, the automorphism i leaves $P = (1, 0)$ fixed, and exchanges the other two 2-torsion points $Q = (0, 0)$ and $P+Q = (2, 0)$.

– **2.3.** From now on, we forget about the modular interpretation and simply regard p as the rational map given in (4). We reserve the letter x for the coordinate in the domain $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $y = p(x)$ for the coordinate in the codomain $\mathbb{P}^1 \setminus \{0, \infty\}$ of p . On function fields, the map p corresponds to the quadratic extension

$$\mathbb{Q}(y) \xrightarrow{\subseteq} \mathbb{Q}(x)$$

identifying y with $\frac{x^2}{x-1}$. We can use p to push forward or to pull back various types of sheaves, including $\mathcal{D}$ -modules. In elementary terms, the push forward

$$p_* : \text{Differential } \mathbb{Q}(x)\text{-modules} \rightarrow \text{Differential } \mathbb{Q}(y)\text{-modules}$$

is just the change of structure via the field extension $\mathbb{Q}(y) \subseteq \mathbb{Q}(x)$. Ranks double by this operation. For every solution f of a differential $\mathbb{Q}(x)$ -module M , one gets two solutions

$$\text{Sym}^+(f)(y) = f(x) + f(w(x)) \quad \text{Sym}^-(f)(y) = (x - w(x))(f(x) - f(w(x)))$$

for p_*M . On the other hand, the pullback

$$p^* : \text{Differential } \mathbb{Q}(y)\text{-modules} \rightarrow \text{Differential } \mathbb{Q}(x)\text{-modules}$$

is extension of scalars, sending a module M to $p^*M = \mathbb{Q}(x) \otimes_{\mathbb{Q}(y)} M$. Ranks are preserved by this operation. For every solution $f(y)$ of a differential $\mathbb{Q}(y)$ -module M , one gets the solution $f(x + w(x))$ for p^*M .

– **2.4.** Differential modules of type G , that is to say those corresponding to differential operators of type G , are preserved under six operations, so in particular the two functors p_* and p^* send differential modules of type G to differential modules of type G .

Proposition 2.5. *Let $f(x) \in \mathbb{Q}[[x]]$ be a power series, and let $d = (d_n)_{n=0}^\infty$ be a sequence of positive integers.*

- (1) *The sequence d is a denominator for $f(x)$ if and only if it is a denominator for $f(w(x))$.*
- (2) *Assume $f(x) = f(w(x))$, so there exists a unique $g \in \mathbb{Q}[[x]]$ with $f(x) = g(p(x))$. The sequence d is a denominator for g if and only if $(d_{2n})_{n=0}^\infty$ is a denominator for f .*

Proof. Statement (1) is close to obvious, so we only prove (2). Let us write

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \quad f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n p(x)^n$$

for the given power series. From $p(x) = -x^2 - x^3 - \dots$ we gather that the power series $p(x)^n$ reads

$$p(x)^n = (-1)^n x^{2n} + (-1)^n x^{2n+1} + \dots$$

for any $n \geq 1$. This allows us to express the coefficients of f in terms of the coefficients of g and vice versa as

$$\begin{aligned} a_{2n} &= \pm b_n + [\text{a } \mathbb{Z}\text{-linear combination of } b_0, \dots, b_{n-1}] \\ a_{2n+1} &= \pm n b_n + [\text{a } \mathbb{Z}\text{-linear combination of } b_0, \dots, b_{n-1}] \\ b_n &= \pm a_{2n} + [\text{a } \mathbb{Z}\text{-linear combination of } a_0, a_2, \dots, a_{2n-2}] \end{aligned}$$

which proves the proposition. $\square$

– **2.6.** For ease of mind, here are the first few expressions for the coefficients a_n in the previous proof:

$$\begin{array}{ll} a_0 &= b_0 \\ a_1 &= 0 \\ a_2 &= -b_1 \\ a_3 &= -b_1 \end{array} \quad \begin{array}{ll} a_4 &= b_2 - b_1 \\ a_5 &= 2b_2 - b_1 \\ a_6 &= -b_3 + 3b_2 - b_1 \\ a_7 &= -3b_3 + 4b_2 - b_1 \end{array}$$

In particular, we see from this proof that if $f(x) = f(w(x))$ holds, then the coefficients a_{2n+1} is a $\mathbb{Z}$ -linear combination of the preceding even indexed coefficients $a_0, a_2, \dots, a_{2n}$.

3. FIVE SPECIAL FUNCTIONS

We now turn our attention to five particular G -functions with singularities in the set $S = \{0, 1, \infty\}$ and small denominators, namely

$$\begin{aligned} A_1(x) &= \text{Li}_0(x) \\ A_2(x) &= \text{Li}_1(x) \\ A_3(x) &= \text{Li}_2(x) \\ A_4(x) &= \frac{1}{2} \text{Li}_1(x)^2 \\ A_5(x) &= J(x) \end{aligned}$$

with J to be defined. Our goal is to understand the differential $\mathbb{Q}(x)$ -module generated by these functions in as much detail as possible. We start with some comments on the polylogarithm functions, and by defining the function J . We will use the notation

$$F\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} x^n \quad b_j \notin \mathbb{Z}_{\leq 0}$$

for hypergeometric series, where $(a)_n$ denotes the rising Pochhammer symbol. Such a series is a G -function if $p = q$ and all parameters $a_1, \dots, a_p, b_1, \dots, b_q$ are rational. Galoshkin's criterion states that this is essentially the only way that a hypergeometric series can be a G -function.

Exercise 3.1. Show that the hypergeometric series

$$F\left(\begin{matrix} 1, 1 \\ i, -i \end{matrix} \middle| x\right) \in \mathbb{Q}[[x]]$$

does not admit a denominator bounded by (3), hence is not a G -function. Show that its p -adic radius of convergence is ≥ 1 for all primes p .

– **3.2.** The polylogarithm function $\text{Li}_j(x)$, for an integer parameter j , is the power series

$$\text{Li}_j(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^j} = x F\left(\begin{matrix} 1, \dots, 1 \\ \underbrace{2, \dots, 2}_j \end{matrix} \middle| x\right)$$

which converges on the open unit disc regardless of the parameter j . For $j \leq 0$, the function $\text{Li}_j(x)$ is a rational function with a pole of order $1 - j$ at $x = 1$, in particular $\text{Li}_0(x) = x(1 - x)^{-1}$. The function $\text{Li}_1(x)$ is related to the standard logarithm by

$$\text{Li}_1(x) = -\log(1 - x)$$

and the higher polylogarithm functions may be constructed inductively from $\text{Li}_1(x)$ by

$$(5) \quad \text{Li}_j(x) = \int \text{Li}_{j-1}(x) \frac{dx}{x},$$

with integration constant $\text{Li}_j(0) = 0$, or alternatively using Hadamard products $\text{Li}_j(x) \odot \text{Li}_k(x) = \text{Li}_{j+k}(x)$.

– **3.3.** Comparing (5) with the following definition³, the function $J(x)$ may be viewed as a twist of $\text{Li}_2(x)$ by $\sqrt{1 - x}$, namely

$$(6) \quad J(x) = \sqrt{1 - x} \int \frac{\text{Li}_1(x)}{\sqrt{1 - x}} \frac{dx}{x}.$$

³This notation differs somewhat from [2]: the function denoted $J(x)$ here is $-(1 - x)$ times the function $J(x)$ defined in *loc. cit.*, equation (10.7.1).

In this definition, the chosen sign of the square root is inconsequential as long as we choose twice the same sign. Let us agree on choosing

$$\frac{1}{\sqrt{1-x}} = F\left(\begin{matrix} \frac{1}{2} \\ 1 \end{matrix} \middle| -x\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n!)}{2^n (n!)^2} x^n$$

for future calculations. The primitive in the definition is supposed to be the power series with constant term 0, so the constant term of J is zero as well. The function J is closely related to the hypergeometric series

$$(7) \quad F(x) = F\left(\begin{matrix} \frac{1}{2}, & 1 \\ \frac{3}{2}, & \frac{3}{2} \end{matrix} \middle| x\right)$$

which is a known culprit. It is related to Catalan's constant $G = L(2, \chi_{-4})$ by the identity $F(1) = 2G$ due to Nielsen in 1909. Evaluating the hypergeometric series $F(x)$ at $x = 1$ leads to a diverging series, and $F(1)$ must be interpreted as its Abel summation.

Proposition 3.4. *The equality $J(x) = xF(\frac{x^2}{4(x-1)})$ holds.*

Proof. Set $H(x) = xF(\frac{x^2}{4(x-1)})$. The functions $J(x)$ and $H(x)$ both are solutions of the inhomogeneous differential equation

$$Y'(x) = \frac{-1}{2(1-x)}Y(x) + \frac{\text{Li}_1(x)}{x}$$

by direct computation for J and by equation (10.1.6) in [2] for $H(x)$. The difference $D = J - H$ therefore satisfies the first order homogeneous equation

$$D'(x) = \frac{-1}{2(1-x)}D(x)$$

hence $D(x) = \lambda\sqrt{1-x}$ for some $\lambda \in \mathbb{Q}$. Since $J(0) = H(0) = 0$ and $D(0) = 1$ we find $\lambda = 0$, hence the claim. $\square$

Proposition 3.5. *A denominator $(d_n)_{n=0}^{\infty}$ for the power series $J(x)$ is given by $d_n = [1, 2, \dots, n]^2$.*

Proof. We start by finding a denominator for the hypergeometric series $F(\frac{y}{4})$, which is

$$F\left(\frac{y}{4}\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (1)_n}{(\frac{3}{2})_n (\frac{3}{2})_n} \frac{y^n}{4^n} = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)(2n+1)!} y^n = \sum_{n=0}^{\infty} b_n y^n$$

For a prime number p , the p -adic valuation of the coefficient b_n is bounded below by

$$v_p(b_n) = -v_p(2n+1) - \underbrace{\sum_{j=1}^{\infty} \left\lfloor \frac{2n+1}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor}_{(*)} \geq -2 \lfloor \log_p(2n+1) \rfloor$$

given that the quantity $(*)$ is either 0 or 1, and is 0 for $j \geq \lfloor \log_p(2n+1) \rfloor$. It follows that $([1, 2, \dots, 2n+1]^2)_{n=0}^\infty$ is a denominator for $F(\frac{y}{4})$, and the proposition follows from statement (2) in Proposition 2.5. $\square$

– **3.6.** The functions $A_1, \dots, A_5$ form a solution vector of a system of differential equations as follows:

$$\partial \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} = \underbrace{\frac{1}{x(1-x)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-x & 0 & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 & -\frac{1}{2}x \end{pmatrix}}_{G(x)} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix}$$

The associated differential module M is the $\mathbb{Q}(x)$ -vector space $\mathbb{Q}(x)^5$ on with the derivation given by multiplication with the matrix $G(x)$ on standard basis vectors. There are several things we can read from this system. From the fact that G has only simple poles and zeros, we see that M has regular singularities at $\{0, 1, \infty\}$. From the fact that G is triangular, we see that M is a repeated extension of one dimensional modules, four trivial ones and one, unsurprisingly, associated with the differential equation $y' = -\frac{1}{2}(1-x)^{-1}y$ with solution $\sqrt{1-x}$.

– **3.7.** A fundamental matrix of solutions for the differential system $Y' = GY$ is given by

$$Y = \begin{pmatrix} \text{Li}_0(x) & 0 & 0 & 0 & 0 \\ \text{Li}_1(x) & 2\pi i & 0 & 0 & 0 \\ \text{Li}_2(x) & 2\pi i \log(x) & -4\pi^2 & 0 & 0 \\ \frac{1}{2} \text{Li}_1(x)^2 & 2\pi i \text{Li}_1(x) & 0 & -2\pi^2 & 0 \\ J(x) & 2\pi i K(x) & 0 & 0 & -4\pi^4 \sqrt{1-x} \end{pmatrix}$$

with

$$K(x) = \sqrt{1-x} \int \frac{1}{\sqrt{1-x}} \frac{dx}{x} = (1-x) \sum_{n=0}^{\infty} \frac{(1-x)^n}{n + \frac{1}{2}}.$$

Denote by $Y_1, Y_2, \dots, Y_5$ the columns of Y . They are clearly $\mathbb{C}$ -linearly independent, hence span the complex vector space of solutions of the differential system $Y' = GY$. The factors $(2\pi i)$ are unnecessary for that, and without them the power series in Y would have rational coefficients. With these factors however, it turns out that the monodromy representation is integral. This phenomenon is an instance of a Betti and a de Rham rational structure on a space of solutions as explained in 1.5.

– **3.8.** Denote by $\mathcal{A}$ the differential $\mathbb{Q}(x)$ -algebra generated by the coefficients of the matrix Y . We regard $\mathcal{A}$ as a subalgebra of the differential algebra of holomorphic functions on the open set $U(\delta)$ that we have introduced in 1.4. The

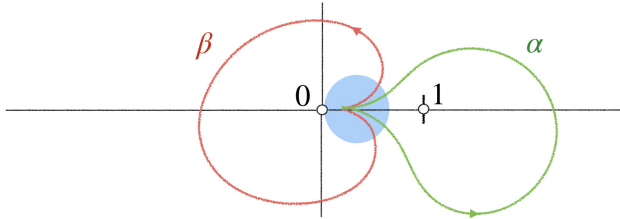
fundamental group $\pi = \pi_1(\mathbb{P}^1\mathbb{C} \setminus \{0, 1, \infty\}, \delta)$ acts on $\mathcal{A}$ by automorphisms of differential $\mathbb{Q}(x)$ -algebras. We denote this action by

$$f \mapsto f^\sigma$$

for $f \in \mathcal{A}$ and $\sigma \in \pi$, where of course $f^\sigma \in \mathcal{A}$ is the function obtained from f by analytic continuation along σ . The monodromy representation associated with the differential equation $Y' = GY$ is the representation

$$\rho : \pi \rightarrow \mathrm{GL}_5(\mathbb{C})$$

characterised by $Y^\sigma(x) = Y(x)\rho(\sigma)$. The group π is the free group generated by two simple loops α and β as follows,



hence to describe the monodromy representation ρ , it suffices to write down the two matrices $\rho(\alpha)$ and $\rho(\beta) \in \mathrm{GL}_5(\mathbb{C})$. The monodromy action on polylogarithms and algebraic functions is well understood, for the record:

$$\begin{aligned} \mathrm{Li}_1^\alpha(x) &= \mathrm{Li}_1(x) - 2\pi i & \mathrm{Li}_1^\beta(x) &= \mathrm{Li}_1(x) \\ \mathrm{Li}_2^\alpha(x) &= \mathrm{Li}_2(x) - 2\pi i \log(x) & \mathrm{Li}_2^\beta(x) &= \mathrm{Li}_2(x) \\ \log^\alpha(x) &= \log(x) & \log^\beta(x) &= \log(x) + 2\pi i \\ (\sqrt{1-x})^\alpha &= -\sqrt{1-x} & (\sqrt{1-x})^\beta &= \sqrt{1-x} \end{aligned}$$

Since J is holomorphic at $x = 0$ and K is holomorphic at $x = 1$ we have $J^\beta(x) = J(x)$ and $K^\alpha(x) = K(x)$. Further, we have $K^\beta(x) = K(x) + 2\pi i\sqrt{1-x}$ because in the expression

$$K(x) = \sqrt{1-x} \log(x) + \sqrt{1-x} \int \left(\frac{1}{\sqrt{1-x}} - 1 \right) \frac{1}{x} dx$$

the second term is holomorphic at $x = 0$. Lastly, $J^\alpha(x) = J(x) - 2\pi i K(x)$ holds because in the expression

$$J(x) = \sqrt{1-x} \int \frac{\mathrm{Li}_1(x)}{\sqrt{1-x}} \frac{dx}{x} = \mathrm{Li}_1(x) K(x) + \sqrt{1-x} \int \frac{1}{1-x} \left(\int \frac{dx}{x\sqrt{1-x}} \right) dx$$

both, K and the second term, are holomorphic at $x = 1$. This concludes all necessary computations to establish

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \rho(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.9. *The five functions $A_1, \dots, A_5$ are $\mathbb{Q}(x)$ -linearly independent.*

Proof. Any $\mathbb{Q}(x)$ -linear relation $f_1 A_1 + \dots + f_5 A_5 = 0$ yields a $\mathbb{Q}(x)$ -linear relation $f_1 A_1^\sigma + \dots + f_5 A_5^\sigma = 0$ for every $\sigma \in \pi$. The solution vector Y_1 is cyclic for the monodromy representation, in the sense that the orbit $\{Y_1^\sigma \mid \sigma \in \pi\}$ spans the whole solution space $\langle Y_1, \dots, Y_5 \rangle$. It follows that any $\mathbb{Q}(x)$ -linear relation as above yields a relation between the rows of the solution matrix Y , which must be trivial. $\square$

Corollary 3.10. *The following seven functions are linearly independent over $\mathbb{Q}(y)$:*

$$\begin{aligned} B_1(y) &= 1 \\ B_2(y) &= \sqrt{y(4-y)} \arcsin(\sqrt{y}/2) \\ B_3(y) &= \arcsin(\sqrt{y}/2) \\ B_4(y) &= (\text{Sym}^- \text{Li}_2)(y) \\ B_5(y) &= {}_yF\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{y}{4}\right) \\ B_6(y) &= \int \frac{B_3(y)}{y} dy \\ B_7(y) &= \int \frac{B_4(y)}{y} dy \end{aligned}$$

As power series, they have denominators at worst $[1, 2, \dots, 2n]n$.

Proof. The functions $B_1, \dots, B_5$ are essentially symmetrisations of $A_1, \dots, A_5$ as in 2.3, the independence of B_6 and B_7 can again be shown by a monodromy argument. The statement about denominators follows from Proposition 2.5. $\square$

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C.1: Siegel's lemma and an application to Roth's theorem

LOUISE NATAF

In the course of his work on diophantine approximation, Carl Ludwig Siegel stated a version of the following result: if A is an $M \times N$ matrix with rational integer coefficients, and if $N > M$, then there exists a $1 \times N$ column vector X with rational integer coefficients such that $AX = 0$ and

$$\|X\|_\infty \leq (N\|A\|_\infty)^{M/(N-M)},$$

where $||\cdot||_\infty$ denotes the maximum of the standard absolute values of coefficients.

This statement, called Siegel's lemma, is a consequence of Dirichlet's pigeon-hole principle. Indeed, for k a positive integer, let T_k be the set of column vectors X with coefficients in $\{0, \dots, k\}$. Then T_k has cardinal $(k+1)^N$ and its image AT_k by A has cardinal at most $(Nk||A||_\infty + 1)^M$. Choosing k to be the integer part of $(N||A||_\infty)^{M/(N-M)}$, we get

$$(k+1)^N > (Nk||A||_\infty + 1)^M,$$

hence by Dirichlet's pigeon-hole principle there are two column vectors in T_k whose image by A is the same; taking the difference of those two vectors, one gets a vector X with rational integer coefficients such that $AX = 0$ and $||X||_\infty \leq k$, which concludes the proof.

There have since been many improvements on Siegel's lemma, which have also been called Siegel's lemma. Namely, using geometry of numbers over the adèles, one can get a homogeneous and relative version over any number field K of degree d . The statement goes as follows: if A is an $M \times N$ matrix with coefficients in K , and if $N > dM$, then there exists a $1 \times N$ column vector X with coefficients in $\mathbb{Z}$ such that $AX = 0$ and

$$H(X) \leq (\sqrt{N}H(A))^{dM/(N-dM)},$$

where $H(\cdot)$ denotes the homogeneous multiplicative height of our matrices X and A seen respectively as an element of $\mathbb{P}_K^{N-1}$ and as an element of $\mathbb{P}_K^{MN-1}$, and where the multiplicative height of a point $x \in \mathbb{P}_K^{n-1}$ with homogeneous coordinates $x = (x_1 : \dots : x_n)$ is given by $H(x) = \prod_{v \in M_K} \max_j |x_j|_v$ with M_K the set of places of K and $|\cdot|_v$ the normalised absolute value corresponding to the place v .

In order to give an idea of how Siegel's lemma can be used, we will sketch the proof of Roth's theorem, whose statement is the following: let $\alpha \in \mathbb{R} - \mathbb{Q}$ be algebraic over $\mathbb{Q}$. Let $\kappa > 2$. Then there are only finitely many rational approximations p/q to α such that $|\alpha - p/q| \leq q^{-\kappa}$.

To prove this result, Roth proceeds by contradiction by assuming there are infinitely many such rational approximations. Then we can construct a family of approximations whose denominators grow rapidly. Namely, for m a positive integer and M, L large constants, one can find m approximations, the p_j/q_j for $j \in \{1, \dots, m\}$, such that $\log q_1 > L$ and $\log q_{j+1} > M \log q_j$ for $j \in \{1, \dots, m\}$. Such a family of approximations will be said to be (L, M) -independent.

Let D, t be large real numbers. The heart of the "easy" part of the proof, due to Siegel and Schneider (see [2]) resides in finding a nonzero polynomial $P(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ with partial degree at most $d_j := D/\log q_j$ in the variable x_j and with reasonable coefficients, that vanishes to order t at $(\alpha, \dots, \alpha)$, which means that for all $(i_1, \dots, i_m)$ with $i_1/d_1 + \dots + i_m/d_m < t$,

$$\partial_{x_1}^{i_1} \dots \partial_{x_m}^{i_m} P(\alpha, \dots, \alpha) = 0.$$

Writing

$$P(x_1, \dots, x_m) = \sum_{\substack{0 \leq k_1 \leq d_1 \\ \vdots \\ 0 \leq k_m \leq d_m}} a_{k_1, \dots, k_m} x_1^{k_1} \dots x_m^{k_m},$$

the “vanishing to high order” condition becomes a system of equations in the unknowns $a_{k_1, \dots, k_m} \in \mathbb{Z}$, with coefficients in $\mathbb{Q}(\alpha)$. The number of equations is asymptotic to $V_m(t)d_1 \dots d_m$ when m tends to infinity, where $V_m(t)$ is the volume of

$$\mathcal{V}_m(t) = \{(x_1, \dots, x_m) \in [0, 1]^m \mid x_1 + \dots + x_m \leq t\}.$$

The number of coefficients is asymptotic to $d_1 \dots d_m$ when m tends to infinity. Hence if $[\mathbb{Q}(\alpha) : \mathbb{Q}]V_m(t) < 1$ (say $[\mathbb{Q}(\alpha) : \mathbb{Q}]V_m(t) \leq 1 - \delta$) and m is large, we can find (this is when we use Siegel’s lemma) such a polynomial P with integer coefficients bounded by $C_1^{d_1 + \dots + d_m}$, where C_1 is a constant depending only on α , m and δ .

Now the difficult step that is due to Roth and earned him a Fields medal (and that we will not be getting into) is to prove that P can be chosen to not vanish at the rational point $(p_1/q_1, \dots, p_m/q_m)$ (we will call this the nonvanishing condition). This is where the (L, M) -independence comes into play and L, M are made to be sufficiently large. It was only done for $m = 1$ or 2 before Roth, which is not enough since m will have to tend to infinity in the end.

On one hand, the “vanishing to high order” condition yields, by Taylor expansion of P at the point $(\alpha, \dots, \alpha)$, that

$$|P(p_1/q_1, \dots, p_m/q_m)| \leq C_2^{d_1 + \dots + d_m} \max_j q_j^{-\kappa t d_j},$$

where C_2 is a constant depending only on α , m and δ .

On the other hand, the nonvanishing condition yields, since P has integer coefficients, the following lower bound:

$$q_1^{-d_1} \dots q_m^{-d_m} \leq |P(p_1/q_1, \dots, p_m/q_m)|.$$

Comparison of the upper and lower bounds shows that

$$\kappa t \leq m + \frac{C_3}{M},$$

where C_3 is again a constant depending only on α , m and δ . Thus as M tends to infinity, we find $\kappa \leq m/t$ as long as $[\mathbb{Q}(\alpha) : \mathbb{Q}]V_m(t) \leq 1 - \delta$. The last step is the probability estimate that for $0 \leq \varepsilon \leq 1/2$, $V_m((1/2 - \varepsilon)m) \leq e^{-6m\varepsilon^2}$, hence when m tends to infinity we get $\kappa \leq 1/(1/2 - \varepsilon)$, hence a contradiction for ε small enough, which concludes the proof of Roth’s theorem.

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C.2: Basic holonomy bounds

LI LAI

In this talk, we present a proof of [1, Eq. (2.2.3)], which provides the simplest arithmetic holonomy bounds. This proof is a minor modification of the argument in the proof of [2, Lemma 2.0.8]:

- We first employ Siegel's lemma to construct multi-variable Padé approximations;
- Subsequently, we compare the Liouville lower bound with the Cauchy upper bound for a leading coefficient;
- Finally, we adjust the parameters to optimize the resulting bound.

The multi-variable setting is essential to this argument, which can be viewed as a tensor power trick.

To be more precise, denote by $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ the closed unit disc and by $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ the unit circle in $\mathbb{C}$. Let $\varphi : \overline{\mathbb{D}} \rightarrow \mathbb{C}$ be a holomorphic function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Let $b \in \mathbb{Q}_{>0}$, $\sigma \in \mathbb{Z}_{\geq 0}$, and $\tau = b\sigma$. Define the following subset $\mathcal{G}(\varphi; b, \sigma)$ of the formal power series ring $\mathbb{Q}[[x]]$ in one indeterminate x :

$$\mathcal{G}(\varphi; b, \sigma) = \left\{ f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{[1, \dots, bn]^{\sigma}} \in \mathbb{Q}[[x]] \mid \begin{array}{l} a_n \in \mathbb{Z} \text{ for all } n \geq 0 \text{ and } \varphi^* f \text{ is the} \\ \text{germ of a meromorphic function on } \overline{\mathbb{D}} \end{array} \right\},$$

where $[1, \dots, bn]$ is the least common multiple of $1, 2, \dots, [bn]$. Define the $\mathbb{Q}(x)$ -linear space

$$\mathcal{H}(\varphi; b, \sigma) = \text{Span}_{\mathbb{Q}(x)} \mathcal{G}(\varphi; b, \sigma).$$

The goal of this talk is to prove that if $\log |\varphi'(0)| > \tau = b\sigma$, then

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi; b, \sigma) \leq \frac{\log \|\varphi\|_{\mathbb{T}}}{\log |\varphi'(0)| - \tau} \quad \text{and}$$

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi; b, \sigma) \leq e \cdot \frac{\int_{\mathbb{T}} \log^+ |\varphi| d\mu_{\text{Haar}}}{\log |\varphi'(0)| - \tau}.$$

As a corollary, every $f \in \mathcal{G}(\varphi; b, \sigma)$ is holonomic; that is, there exists a nonzero differential operator $\mathcal{L} \in \mathbb{Q}(x) \left[\frac{d}{dx} \right]$ annihilating f . (Hence, f is a G -function.) This corollary explains the terminology “arithmetic holonomy bounds”.

To sketch a proof, let $f_1, \dots, f_m \in \mathcal{G}(\varphi; b, \sigma)$ be linearly independent over $\mathbb{Q}(x)$. We aim to prove a bound for m . Let $d \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{Z}_{\geq 1}$, and $\kappa \in (0, 1)$ be parameters. The linear independence of $f_1, \dots, f_m$ over $\mathbb{Q}(x)$ implies the linear independence over $\mathbb{Q}(x_1, \dots, x_d)$ of the following m^d formal power series in d indeterminates:

$$f_{\mathbf{i}}(\mathbf{x}) := f_{i_1}(x_1) f_{i_2}(x_2) \cdots f_{i_d}(x_d), \quad \text{where } \mathbf{i} = (i_1, \dots, i_d) \in \{1, \dots, m\}^d.$$

By Siegel's lemma, there exists a nonzero d -variate formal series $F(x_1, \dots, x_d)$ of the form

$$F(\mathbf{x}) = \sum_{\mathbf{i} \in \{1, \dots, m\}^d} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{\geq 0}^d \\ |\mathbf{n}| \leq D}} A_{\mathbf{i}, \mathbf{n}} \mathbf{x}^{\mathbf{n}} f_{\mathbf{i}}(\mathbf{x})$$

satisfying the following properties as d, κ fixed and $\alpha \rightarrow \infty$:

- $\text{ord}_{\mathbf{x}=\mathbf{0}} F(\mathbf{x}) > \alpha$;
- $D \sim \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{\frac{1}{d}} \alpha$;
- all $A_{\mathbf{i}, \mathbf{n}}$ are integers with $|A_{\mathbf{i}, \mathbf{n}}| \leq \exp(\kappa C \alpha + o(\alpha))$, where C is a constant depending only on φ, b, σ .

Take a fixed holomorphic function $h_0(z)$ on $\overline{\mathbb{D}}$ with $h_0(0) = 1$ such that $h_0(z) \cdot f_i(\varphi(z))$ is the germ of a holomorphic function on $\overline{\mathbb{D}}$ for each $i = 1, \dots, m$. Then the function

$$H(z_1, \dots, z_d) := h_0(z_1) \cdots h_0(z_d) F(\varphi(z_1), \dots, \varphi(z_d))$$

can be extended to a d -variate holomorphic function on the polydisc $\overline{\mathbb{D}}^d$, and we again denote it by $H(\mathbf{z})$.

Let $\beta > \alpha$ be the exact order of vanishing of $F(\mathbf{x}) \in \mathbb{Q}[[\mathbf{x}]] \setminus \{0\}$ at $\mathbf{x} = \mathbf{0}$, and consider any nonzero monomial $c_\beta \mathbf{x}^\beta$ of that lowest order $\beta = |\beta|$. Then, we have the Liouville lower bound

$$|c_\beta| \geq \prod_{j=1}^d \frac{1}{[1, \dots, b\beta_j]^\sigma}.$$

On the other hand, $c_\beta \varphi'(0)^\beta \mathbf{z}^\beta$ is a lowest-order monomial in $H(\mathbf{z}) \in \mathbb{C}[[\mathbf{z}]]$. We have the Cauchy upper bound

$$|c_\beta| \cdot |\varphi'(0)|^\beta \leq \|H\|_{\mathbb{T}^d}.$$

By comparing the Liouville lower bound with the Cauchy upper bound, and fixing d and κ while letting $\alpha \rightarrow \infty$, we obtain

$$m \leq \frac{\left(1 + \frac{1}{\kappa}\right)^{\frac{1}{d}} \log \|\varphi\|_{\mathbb{T}}}{\log |\varphi'(0)| - \tau - \kappa C}, \quad \forall d \in \mathbb{Z}_{\geq 1}, \quad \forall \kappa \in \left(0, \frac{\log |\varphi'(0)| - \tau}{|C|}\right).$$

Letting first $d \rightarrow \infty$ and then $\kappa \rightarrow 0$, we obtain the arithmetic holonomy bound

$$m \leq \frac{\log \|\varphi\|_{\mathbb{T}}}{\log |\varphi'(0)| - \tau}.$$

To prove the other arithmetic holonomy bound

$$m \leq e \cdot \frac{\int_{\mathbb{T}} \log^+ |\varphi| \mu_{\text{Haar}}}{\log |\varphi'(0)| - \tau},$$

we merely make minor changes to the arguments above. Replace $F(\mathbf{x})$ by

$$\tilde{F}(\mathbf{x}) = \sum_{\mathbf{i} \in \{1, \dots, m\}^d} \sum_{\mathbf{n} \in \{1, \dots, \tilde{D}\}^d} \tilde{A}_{\mathbf{i}, \mathbf{n}} \mathbf{x}^{\mathbf{n}} f_{\mathbf{i}}(\mathbf{x}),$$

where $\tilde{D} \sim \frac{1}{(d!)^{\frac{1}{d}}} \cdot \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{\frac{1}{d}} \alpha$. Replace $H(\mathbf{z})$ by

$$\tilde{H}(\mathbf{z}) = h(z_1)^{\tilde{D}} \cdots h(z_d)^{\tilde{D}} h_0(z_1) \cdots h_0(z_d) \tilde{F}(\varphi(z_1), \dots, \varphi(z_d)),$$

where h is any holomorphic function on $\overline{\mathbb{D}}$ with $h(0) = 1$. Similar arguments yield

$$m \leq e \cdot \frac{\max\{\log \|h\|_{\mathbb{T}}, \log \|h \cdot \varphi\|_{\mathbb{T}}\}}{\log |\varphi'(0)| - \tau}.$$

Using a classical lemma of Nevanlinna (see [2, Lemma 2.3.1]), the infimum of the above bound over all holomorphic functions $h : (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 1)$ is exactly

$$e \cdot \frac{\int_{\mathbb{T}} \log^+ |\varphi| \mu_{\text{Haar}}}{\log |\varphi'(0)| - \tau}.$$

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C.3: Concentration of mass and a refined auxiliary function

PHILIPP HABEGGER

This talk, part of the *Arbeitsgemeinschaft* 2614, covered Calegari, Dimitrov, and Tang’s construction [3] of an auxiliary function that is used in their proof of the refined holonomy bound. The construction is given in Lemma 6.2.6 and some important properties are collected in Corollary 6.2.11, both in [3].

Before reviewing the basic structure of this auxiliary function, let us introduce the classical discrepancy function D . It plays a key role in the construction of the auxiliary function and further estimates that lead up to the refined holonomy bound. For $\mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d$ we set

$$D(\mathbf{t}) = \sup_{0 \leq \alpha < \beta < 1} \left| \frac{\#\{i : t_i \in [\alpha, \beta]\}}{d} - (\beta - \alpha) \right|.$$

If the components of $\mathbf{t}$ are equidistributed in the interval $[0, 1]$, then we expect the proportion of entries in a particular subinterval to be about the length of this subinterval. Therefore, small discrepancy signifies equidistribution. For example,

$$(1) \quad D\left(\frac{0}{d}, \frac{1}{d}, \frac{2}{d}, \dots, \frac{d-1}{d}\right) = \frac{1}{d}.$$

This is the minimal possible discrepancy for a d -tuple since

$$\frac{1}{d} \leq D(\mathbf{t}) \leq 1$$

for all $\mathbf{t} \in [0, 1]^d$.

Clearly, $D(\mathbf{t})$ is invariant under permuting the coordinates of $\mathbf{t}$. Given $\mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d$ with $D(\mathbf{t}) < \epsilon$, there is a permutation η of $\{1, \dots, d\}$ such that

$$(2) \quad \left| t_{\eta(j)} - \frac{j-1}{d} \right| \leq \epsilon.$$

In other words, up to reordering, a low-discrepancy tuple is almost the standard tuple appearing in (1).

The **Koksma–Hlawka Inequality** states that a low-discrepancy tuple is useful for numerical integration. Indeed, if $f: [0, 1] \rightarrow \mathbf{R}$ is a function of bounded variation $V(f)$, then

$$\left| \int_0^1 f(x) dx - \frac{1}{d} \sum_{i=1}^d f(t_i) \right| \leq V(f) D(t_1, \dots, t_d)$$

for all $(t_1, \dots, t_d) \in [0, 1]^d$.

We take the point of view of [2] and consider $D: [0, 1]^d \rightarrow \mathbf{R}$ as a function. It is a continuous and real semi-algebraic map. Classical lattice point counting results give an asymptotic count for low-discrepancy tuples in a lattice. For example,

$$(3) \quad \#\{\mathbf{k} \in \mathbf{Z}^d : \mathbf{k}/D \in [0, 1]^d \text{ and } D(\mathbf{k}/D) < \epsilon\} = (\text{vol}(P_\epsilon^d) + o_{d,\epsilon}(1)) D^d,$$

where d, ϵ are fixed, $D \rightarrow \infty$, and where $\text{vol}(P_\epsilon^d)$ denotes the volume of the semi-algebraic set $P_\epsilon^d = \{\mathbf{t} \in [0, 1]^d : D(\mathbf{t}) < \epsilon\}$.

These estimates are only useful when d is large in terms of ϵ , as no d -tuple can have discrepancy less than $1/d$. Călegari, Dimitrov, and Tang prove the following statement. The probability that a random tuple in $[0, 1]^d$ has discrepancy $\geq \epsilon$ decays exponentially in the dimension d . Indeed, their Theorem 4.2.1 [3] implies

$$(4) \quad 1 - \text{vol}(P_\epsilon^d) = \text{vol}(\{\mathbf{t} \in [0, 1]^d : D(\mathbf{t}) \geq \epsilon\}) \leq c_1 e^{-c_2 \epsilon^4 d}$$

for all $d \gg_\epsilon 1$; here $c_1 > 0$ and $c_2 > 0$ are absolute and explicit constants. This theorem is an application of the classical **Erdős–Turán Theorem** which bounds the discrepancy in terms of exponential sums. It is a quantitative form of **Weyl’s Equidistribution Theorem**. Bounding the resulting exponential sums requires **Hoeffding’s Inequality** from probability theory. A related volume estimate, albeit in a different setting, appears in Bombieri and Gubler’s exposition of **Roth’s Theorem** from diophantine approximation, see Lemma 6.3.5 [1].

Let us now give a rough overview of the conclusion of Lemma 6.2.6 [3] where the auxiliary function is constructed.

Suppose $f_1, \dots, f_m \in \mathbf{Q}[[x]]$ are holonomic, see Definition 2.0.4 [2], and $\mathbf{Q}(x)$ -linearly independent. Let $A \geq 1, B \geq 0, \sigma \geq 0$ be integers, to be thought of as fixed parameters. Suppose that the degree n term of f_i has the form

$$(5) \quad \frac{a_{in}}{A^{n+1}[1, \dots, Bn]^\sigma}$$

where a_{in} is an integer and $[1, \dots, Bn]$ denotes the least common multiple of $1, \dots, Bn$. We also suppose that each f_i converges on $\{z \in \mathbf{C} : |z| < \rho\}$. Let us write $\mathbf{x}$ for the unknowns $(x_1, \dots, x_d)$.

Let $\epsilon \in (0, 1)$ be a further parameter. Then there is $\delta \in (0, \epsilon)$ such that the following holds true. For all $d \gg_{A,B,\sigma,\rho} 1$ with $d \in m\mathbf{Z}$ and all $D \gg_d 1$ there

exists a non-zero

$$F = \sum_{\substack{\mathbf{i} \in V_m^d \\ \mathbf{k} \in \mathbf{Z}^d \cap [0, D]^d \\ \mathbf{k}/D \in P_\epsilon^d}} c_{\mathbf{i}, \mathbf{k}} \mathbf{x}^{\mathbf{k}} \prod_{s=1}^d f_{i_s}(\mathbf{x}_s) \in \mathbf{Q}[[\mathbf{x}]]$$

where

- all $c_{\mathbf{i}, \mathbf{k}}$ are integers with $|c_{\mathbf{i}, \mathbf{k}}| < e^{\epsilon d D}$,
- we have $V_m^d \subset \{1, \dots, m\}^d$ and in each tuple, an index in $\{1, \dots, m\}$ is hit precisely d/m times,
- the minimal non-zero monomial (with respect to a specified term ordering) $\beta_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ of F satisfies

$$(6) \quad D \left(\frac{\mathbf{n}}{(m + \delta)D} \right) < \epsilon.$$

The bound (6), by virtue of (2), suggests that the sum of the entries of $\mathbf{n}/((m + \delta)D)$ should be close to $\sum_{j=1}^d \frac{j-1}{d} = \frac{d-1}{2} = \frac{d}{2} + O(1)$. Indeed, using $\delta \leq \epsilon$ we find that the ℓ^1 -norm of $\mathbf{n}$ satisfies

$$|\mathbf{n}|_{\ell^1} = \frac{mdD}{2} (1 + O(\epsilon + d^{-1})) = \frac{mdD}{2} (1 + O(\epsilon))$$

if $1/d \leq \epsilon$.

The existence of F ultimately follows from **Siegel's Lemma**, see Lemma 2.9.1 [1]. Indeed, one can recast the properties stated above as a system of linear equations, to be solved in the unknowns $c_{\mathbf{i}, \mathbf{k}}$. The number of unknowns is

$$(7) \quad N = \#V_m^d \cdot \#\{\mathbf{k} \in \mathbf{Z}^d : \mathbf{k}/D \in [0, 1]^d \text{ and } \mathbf{k}/D \in P_\epsilon^d\}.$$

The equations arise from imposing the vanishing of sufficiently many $\beta_{\mathbf{n}}$ to achieve the desired high zero multiplicity.

In order to apply Siegel's Lemma, one needs to establish $N > M$. To bound N from below, a simple combinatorial estimate on $\#V_m^d$ suffices. But bounding the number of $\mathbf{k}$ in (7) from below is more challenging. For this set, Calegari, Dimitrov, and Tang use the lattice point counting estimate (3) combined with the volume decay (4). The number M of equations is bounded from above similarly. The Dirichlet exponent $M/(N - M)$ turns out to be asymptotically small as $d \rightarrow \infty$.

Finally, the height of the system is controlled using the structure of the coefficients (5) together with

$$\log[1, \dots, n] = n(1 + o(1)) \quad \text{for } n \rightarrow \infty$$

which follows from the **Prime Number Theorem**.

The final bullet point is a consequence of the **Chudnovsky–Osgood Theorem** presented in another talk.

One feature of this auxiliary function is that the number of variables d is asymptotically large. This is reminiscent of Roth's work on diophantine approximation. Large values of d enable the vector $\frac{\mathbf{n}}{(m + \delta)D} \in [0, 1]^d$ to have small discrepancy.

Later in the proof of the refined holonomy bound, this allows for applications of the Koksma–Hlawka Inequality to bound the absolute value $|F|$ from above.

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C.4: The fine holonomy bound (integral case)

GABRIELE BOGO

The goal of the talk was to state and sketch the proof of a simplified version of Theorem 6.0.2 of [2]. The theorem gives a bound on the number of $\mathbb{Q}(x)$ -linearly independent power series $f(x) \in \mathbb{Q}[[x]]$ whose coefficients have denominators of a prescribed type and satisfy certain analytic conditions (made precise in the statements below). The simplification adopted in the talk consisted in considering only power series with integral coefficients (the case $\mathbf{b} = \mathbf{e} = 0$ in the language of [2]).

Similar but weaker bounds (stated in Equation (1)) were proved in the earlier paper [1] by the same authors. One key new ingredient is the use of the nondecreasing rearrangement of a measurable function.

For a measurable function $g: [0, 1] \rightarrow \mathbb{R}$, the *nondecreasing rearrangement* of g is the unique (up to functions vanishing outside a set of measure zero) nondecreasing measurable function $g^*: [0, 1] \rightarrow \mathbb{R}$ that has the same distribution as g . A way to construct g^* from a given g is to consider, for $n \geq 1$, the piecewise constant step function g_n which takes the value $g(k/n)$ on the interval $I_k = [(k-1)/n, k/n)$ for $k = 1, \dots, n$ (extending I_n to contain 1). Let $g_n^*(t)$ denote the step function which is also constant in the intervals $I_1, \dots, I_n$ and takes the respective values $\{g(1/n), \dots, g(n/n)\}$ rearranged in nondecreasing order (see Figure 1 for the case $n = 4$). Then $g_n^*(t)$ is the nondecreasing rearrangement of g_n and g_n^* converges uniformly to g^* .

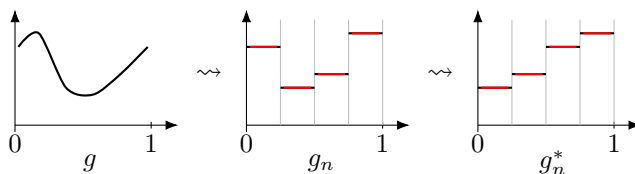


FIGURE 1. Approximation of a function by step functions and rearrangement of the values in nondecreasing order for $n = 4$.

A basic version of (the integral version of) Theorem 6.0.2 of [2] is the following.

Theorem 0.1. *Let $\varphi: (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic map satisfying $|\varphi'(0)| > 1$. Let $f_1, \dots, f_m \in \mathbb{Z}[[x]]$ be $\mathbb{Q}(x)$ -linearly independent and such that $f_i(\varphi(z)) \in \mathbb{C}[[z]]$ is the germ of a meromorphic function on $|z| < 1$ for every $i = 1, \dots, m$. Then*

$$m \leq \frac{\int_0^1 2t \cdot (\log |\varphi(e^{2\pi it})|)^* dt}{\log |\varphi'(0)|}.$$

A first bound of this kind was given in Theorem 2.0.1 of [1]. In the case of integral power series, that bound read

$$(1) \quad m \leq e \cdot \frac{\int_0^1 \max(\log |\varphi(e^{2\pi it})|, 0) dt}{\log |\varphi'(0)|}.$$

Theorem 1, together with the trivial inequality

$$\int_0^1 2t \cdot g^*(t) dt \leq 2 \int_0^1 \max(g(t), 0) dt,$$

which holds for every measurable function g (applied here to $g = \log |\varphi|$), improves the bound (1) by lowering the numerical constant from e to 2.

Idea of the proof. We briefly indicate the main ideas in the proof of Theorem 1, referring to Sections 6.5 and 6.7 of [2] for details. As explained in Section 2.12 of [2], the proof of the holonomy bound can be divided into three steps. The first two involve the construction of an auxiliary function $F(x_1, \dots, x_d)$ in d variables ($d \rightarrow \infty$ eventually), built out of the series f_i , with maximal possible vanishing order at $\underline{x} = 0$. This is done by using a Siegel-lemma type argument (Lemma 6.2.6 of [2], discussed in the previous lecture). The last step involves a Diophantine analysis of a lowest-order nonzero coefficient β , attached to the monomial $x_1^{k_1} \cdots x_d^{k_d}$, of the auxiliary function $F(\underline{x})$. This step is further divided into finding an upper (Cauchy) bound and a lower (Liouville) bound for $|\beta|$. The integrality assumption of Theorem 1 gives the lower bound for free, since $\beta \in \mathbb{Z} \setminus \{0\}$, hence $|\beta| \geq 1$.

To obtain the upper bound, one uses the fact that the functions $f_i(\varphi(z))$ are meromorphic in the unit disk. Indeed, by substituting $x_j = \varphi(z_j)$ for $j = 1, \dots, d$, the auxiliary function $F(\varphi(z_1), \dots, \varphi(z_d))$ is meromorphic on a polydisk and becomes holomorphic after multiplication by a suitable holomorphic factor $G(\underline{z})$. One can then express β by a Cauchy integral. Thus the problem is reduced to bounding the growth, on the boundary of a polydisk, of $|F(\varphi(z_1), \dots, \varphi(z_d))G(\underline{z})|$.

Intuitively, as $d \rightarrow \infty$, the exponent vector $(k_1, \dots, k_d)$ of a minimal term monomial can be regarded as close to some ordering of the set $\{jD/d : j = 0, \dots, d-1\}$ (by Theorem 4.2.1 of [2]). By noting that, among all such orderings, the largest value of $|\varphi(z_1)^{k_1} \cdots \varphi(z_d)^{k_d}|$ is obtained when the vector $(k_1, \dots, k_d)$ is ordered in the same way as $(|\varphi(z_1)|, \dots, |\varphi(z_d)|)$, one is naturally led to the rearrangement integral

$$\int_0^1 2t \cdot (\log |\varphi(e^{2\pi it})|)^* dt.$$

There is one further technical point. The Cauchy estimates described above can be made rigorous, with the help of Koksma's inequality (see Section 2.5.1 of [1]),

when the points $z_1, \dots, z_d$ are well distributed on the torus (locus of small discrepancy). On the exceptional set of poorly distributed points (big discrepancy locus), one multiplies by an auxiliary Vandermonian factor which is very small there, by Bilu's equidistribution theorem (see Lemma 2.5.9 of [1]) but whose growth elsewhere is controlled by Fekete's theorem (see Lemma 2.5.3 of [1]). This allows one to obtain a uniform boundary estimate without losing the main rearrangement term.

Comparing the lower bound for the first nonzero coefficient with the upper bound obtained in this way, and then letting the auxiliary parameters tend to their limiting values in the correct order, gives

$$m \log |\varphi'(0)| \leq \int_0^1 2t (\log |\varphi(e^{2\pi it})|)^* dt,$$

which is equivalent to the claimed bound.

The notion of nondecreasing rearrangement is only one of the new ingredients of Theorem 6.0.2 of [2]. A second one is the use of several auxiliary holomorphic functions $\varphi_0, \dots, \varphi_l: (\mathbb{D}, 0) \rightarrow (\mathbb{C}, 0)$. We do not discuss this case here, but we record the full version of Theorem 6.0.2 in the case of power series with integral coefficients.

Theorem 0.2. *Let $\varphi_0, \dots, \varphi_l: (\mathbb{D}, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic mappings satisfying $0 < |\varphi'_0(0)| < \dots < |\varphi'_l(0)|$ and $|\varphi'_l(0)| > 1$. Let m be a positive integer and consider a partition of the interval $[0, m]$*

$$0 = \gamma_0 < \gamma_1 < \dots < \gamma_l < \gamma_{l+1} = m$$

and use these choices to define an L^1 function $g_{\varphi, \gamma}: [0, 1] \rightarrow \mathbb{R} \cup \{-\infty\}$ by piecewise patching the functions $\log |\varphi_k|$ on the circle $\mathbf{T}$ as follows

$$g_{\varphi, \gamma}(t) := \log \left| \varphi_k \left(e^{2\pi i \frac{mt - \gamma_k}{\gamma_{k+1} - \gamma_k}} \right) \right| \quad \text{if } t \in \left[\frac{\gamma_k}{m}, \frac{\gamma_{k+1}}{m} \right), \quad k = 0, \dots, l.$$

Let $f_1, \dots, f_m \in \mathbb{Z}[[x]]$ be $\mathbb{Q}(x)$ -linearly independent and such that $f_i(\varphi_k(z)) \in \mathbb{C}[[z]]$ is the germ of a meromorphic function on $|z| < 1$ for every $i = 1, \dots, m$ and $k = 0, \dots, l$. Then

$$m \leq \frac{\int_0^1 2t \cdot g_{\varphi, \gamma}^*(t) dt + \frac{1}{m} \sum_{k=1}^l \gamma_k^2 \log \left(\frac{|\varphi'_k(0)|}{|\varphi'_{k-1}(0)|} \right)}{\log |\varphi'_l(0)|}.$$

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C.5: Arithmetic sharpening and summary of the holonomy bounds

KALYANI KANSAL

Setup. Let $f_1, \dots, f_m \in \mathbb{Q}[[x]]$ be $\mathbb{Q}(x)$ -linearly independent. Assume

$$f_i(x) = a_{i,0} + \sum_{n=1}^{\infty} \frac{a_{i,n} x^n}{n^{e_i} [1, \dots, b_{i,1} n] \cdots [1, \dots, b_{i,r} n]}, \quad a_{i,n} \in \mathbb{Z}.$$

Each column of $(b_{i,j})$ has the shape

$$0 = b_{1,j} = \cdots = b_{u_j,j} < b_{u_j+1,j} = \cdots = b_{m,j} =: b_j.$$

So denominators are getting worse as i increases.

Let

$$\phi_0, \dots, \phi_\ell : (D, 0) \rightarrow (\mathbb{C}, 0)$$

with

$$|\phi'_0(0)| < |\phi'_1(0)| < \cdots < |\phi'_\ell(0)|, \text{ and } |\phi'_\ell(0)| > e^{b_1 + \cdots + b_r}.$$

So far we have seen:

$$F(x) = \sum_{i=(i_1, \dots, i_d) \in \{1, \dots, m\}^d} \sum_{k \in [0, D]^d} c_{i,k} x^k \prod_{s=1}^d f_{i_s}(x_s) = \sum_n \beta_n x^n,$$

and we look at $\log |\beta|$, where β is the coefficient of x^n with $|n|$ minimal.

We had used the power series

$$G(z) = h(z_1) \cdots h(z_d) F(\phi(z)),$$

holomorphic on a neighborhood of $\overline{D}^d$, where h is holomorphic on D with $h(0) = 1$. Hence, β can be extracted from the n -th coefficient of $G(z)$. The bound on β had been obtained using the Cauchy integral formula and estimates on distributions of both k and n .

The new point is that the logs of denominators should give a lower bound

$$-\alpha \tau(b, e) \leq \log |\beta|, \quad \tau(b, e) = \tau^b(b) + \tau^\sharp(e),$$

so that the computation in the previous talk giving $\log |\beta| \leq \frac{\alpha}{m} (*) + \alpha \log |\phi'_\ell(0)|$ for a suitable $(*)$ should now give

$$\log |\phi'_\ell(0)| - \tau(b, e) \leq \frac{1}{m} (*).$$

Hence, if $|\phi'_\ell(0)| \geq e^{\tau(b, e)}$,

$$m \leq \frac{(*)}{\log |\phi'_\ell(0)| - \tau(b, e)}.$$

In this talk, we will compute $\tau^b(b)$ and $\tau^\sharp(e)$. This will only use estimates on n , namely

$$n_j = \frac{mjD}{d} + O(\varepsilon D),$$

after relabeling $x_1, \dots, x_d$ if necessary so that $n_1 \leq \cdots \leq n_d$.

1. The $\tau^b(b)$ term. Assume $e_i = 0$ for all $i \in \{1, \dots, m\}$.

The denominator of β divides the lcm of

$$\left\{ \prod_{j=1}^d \prod_{h=1}^r [1, \dots, b_{i_j, h} n_j] \right\}_{(i_1, \dots, i_d)},$$

where each i occurs d/m times in this tuple. Because of the shape of $(b_{i, j})$, the $(i_1, \dots, i_d)$ -th element in the set above is biggest possible when n_j increases as b_{i_j} increases (or does not decrease), and this biggest element gives the lcm.

Thus, we may assume

$$(i_1, \dots, i_d) = (\underbrace{1, \dots, 1}_{d/m}, \underbrace{2, \dots, 2}_{d/m}, \dots, \underbrace{m, \dots, m}_{d/m}).$$

Write

$$j = (i-1)\frac{d}{m} + s, \quad i \in \{1, \dots, m\}, \quad s \in \{1, \dots, d/m\}.$$

Then

$$b_{i_j, h} = b_{i, h}, \quad n_j = (i-1)D + \frac{smD}{d} + O(\varepsilon D).$$

Hence the denominator divides

$$\prod_{i=1}^m \prod_{s=1}^{d/m} \prod_{h=1}^r [1, \dots, b_{i, h} n_{(i-1)d/m+s}],$$

and by the prime number theorem this is

$$\exp \left(\sum_{i=1}^m \sum_{s=1}^{d/m} \sum_{h=1}^r b_{i, h} \left((i-1)D + \frac{smD}{d} \right) + O(\varepsilon Dd) \right).$$

Let

$$\sigma_i = \sum_{h=1}^r b_{i, h}, \quad t_{i, s} = \frac{i-1}{m} + \frac{s}{d}.$$

Then the exponent is

$$\alpha \sum_{i=1}^m \sigma_i \sum_{s=1}^{d/m} 2t_{i, s} \frac{1}{d} + O(\varepsilon \alpha).$$

Now

$$\sum_{s=1}^{d/m} 2t_{i, s} \frac{1}{d} = \int_{(i-1)/m}^{i/m} 2t \, dt + o_{d \rightarrow \infty}(1),$$

so this becomes

$$\exp \left(\alpha \sum_{i=1}^m \sigma_i \frac{2i-1}{m^2} + O(\varepsilon \alpha) \right).$$

Therefore, setting

$$\tau^b(b) = \frac{1}{m^2} \sum_{i=1}^m (2i-1)\sigma_i,$$

the log of the denominator coming from the b -part is asymptotically bounded by $\alpha\tau^b(b)$.

2. The $\tau^\sharp(e)$ term. Assume $b_{i,h} = 0$ for all i, h .

The denominator of β comes from sums of terms of the form

$$x_1^{k_1} \cdots x_d^{k_d} \frac{x_1^{n_1-k_1}}{(n_1-k_1)^{e_{i_1}}} \cdots \frac{x_d^{n_d-k_d}}{(n_d-k_d)^{e_{i_d}}}.$$

Recall

$$k_j \in [0, D], \quad n_j \sim \frac{mjD}{d}, \quad n_j - k_j \in [n_j - D, n_j] \sim \left[\frac{mjD}{d} - D, \frac{mjD}{d} \right].$$

Fix $\xi \in [0, m]$. We consider primes $\leq \xi D$ and $> \xi D$ dividing the denominator separately.

Consider

$$[1, \dots, \xi D]^{(\sum_{i=1}^m e_i)d/m} x_1^{k_1} \cdots x_d^{k_d} \frac{x_1^{n_1-k_1}}{(n_1-k_1)^{e_{i_1}}} \cdots \frac{x_d^{n_d-k_d}}{(n_d-k_d)^{e_{i_d}}}.$$

The denominator of the above divides

$$(**) \quad \prod_{j=1}^d \prod_{\substack{p > \xi D \\ p \text{ divides some integer} \\ \text{in } [mjD/d - D, mjD/d]}} p^{\max(e_i)}$$

upto the following sources of error: Firstly, $n_j = \frac{mjD}{d} + O(\varepsilon D)$. Secondly, some primes $\leq \xi D$ may survive in the denominator or some primes $> \xi D$ may occur in bigger powers. This error can be made precise using the prime number theorem. Both the errors account for a factor given by $\exp(o_{d \rightarrow \infty, \epsilon \rightarrow 0}(\alpha))$. Thus

$$\log \text{denom}(\beta) \leq \min_{\xi \in [0, m]} \left\{ \log[1, \dots, \xi D]^{(\sum e_i)d/m} + \log(**) + o(\alpha) \right\}$$

For the first term,

$$\log[1, \dots, \xi D]^{(\sum e_i)d/m} = \left(\sum_{i=1}^m e_i \right) \frac{d}{m} \xi D + o(\alpha) = \frac{2\alpha}{m^2} \xi \sum_{i=1}^m e_i + o(\alpha).$$

The second term is asymptotically given by $(\max e_i)D \sum_{j=1}^d g_\xi(mj/d)$ for some function $g_\xi(t)$ calculated in Section 5 of [1]. As $d \rightarrow \infty$, this is given by

$$\max(e_i) \frac{dD}{m} \int_0^m g_\xi(t) dt = \max(e_i) \frac{2\alpha}{m^2} I_m^\xi(\xi),$$

where $I_m^\xi(\xi) := \int_0^m g_\xi(t) dt$.

Therefore

$$\tau^\sharp(e) = \min_{\xi \in [0, m]} \frac{2}{m^2} \left(\xi \sum_{i=1}^m e_i + \max(e_i) I_m^\xi(\xi) \right).$$

3. Other bounds.

Theorem ([1, Thm. 2.5.1]). Consider two positive integers $m, r \in \mathbb{N}_{>0}$ and an $m \times r$ rectangular array of nonnegative real numbers

$$b = (b_{i,j})_{1 \leq i \leq m, 1 \leq j \leq r},$$

all of whose columns are of the form

$$0 = b_{1,j} = \cdots = b_{u_j,j} < b_{u_j+1,j} = \cdots = b_{m,j} =: b_j, \quad j = 1, \dots, r.$$

Let

$$\sigma_i = b_{i,1} + \cdots + b_{i,r}, \quad i = 1, \dots, m,$$

and define

$$\tau(b) = \frac{1}{m^2} \sum_{i=1}^m (2i-1) \sigma_i.$$

Let $\phi : (D, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic with

$$|\phi'(0)| > e^{\sigma_m}.$$

Suppose there exists an m -tuple $f_1, \dots, f_m \in \mathbb{Q}[[x]]$ of $\mathbb{Q}(x)$ -linearly independent formal functions with denominator types

$$f_i(x) = \sum_{n=0}^{\infty} \frac{a_{i,n} x^n}{[1, \dots, b_{i,1}n] \cdots [1, \dots, b_{i,r}n]}, \quad a_{i,n} \in \mathbb{Z},$$

such that $f_i(\phi(z)) \in \mathbb{C}[[z]]$ is the germ of a meromorphic function on $|z| < 1$ for all $i = 1, \dots, m$. Then

$$m \leq \frac{\iint_{\mathbb{T}^2} \log |\phi(z) - \phi(w)| \mu(z) \mu(w)}{\log |\phi'(0)| - \tau(b)}.$$

Moreover, if the f_i are assumed to be holonomic, then the condition $|\phi'(0)| > e^{\sigma_m}$ can be relaxed to

$$|\phi'(0)| > e^{\tau(b)}.$$

Corollary ([1, Thm. 2.7.1]). If $\Omega \subset \mathbb{C}$ is contractible with $0 \in \Omega$ and conformal mapping radius $\rho(\Omega, 0) > e^{\tau(b)}$, and if $f_1, \dots, f_m$ are meromorphic in Ω , then

$$m \leq \frac{\log \rho(\Omega, 0)}{\log \rho(\Omega, 0) - \tau(b)}.$$

Note that the Riemann mapping theorem supplies a biholomorphism

$$\varphi : D \rightarrow \Omega$$

to plug into the previous theorem.

[1, Thm. 2.7.10]. Let Ω be as above, and let $m = 1$, $f = f_1$. Suppose

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{[1, \dots, b_1 n] \cdots [1, \dots, b_r n]}, \quad a_n \in \mathbb{Z},$$

is the Taylor expansion of a meromorphic function on Ω . Assume either

- (1) $\Omega = \{|x| < R\}$ with $R > e^{b_1 + \cdots + b_r}$, or
- (2) the conformal mapping radius satisfies

$$\rho(\Omega, 0) > \exp\left(\frac{3}{2}(b_1 + \cdots + b_r)\right).$$

Then $f(x)$ is the Taylor expansion of a rational function in $\mathbb{Q}(x)$.

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C.6: Application of the holonomy bound to the irrationality of

$$L(2, \chi_{-3})$$

FANGU CHEN

This talk is concerned with the application of the arithmetic holonomy bound ([1, Theorem 6.0.2]) to the proof of irrationality of $L(2, \chi_{-3})$. Assume for contradiction that $L(2, \chi_{-3}) \in \mathbb{Q}$. Then Zagier's local system yields a fictive G -function $H(x) \in \mathbb{Q}[[x]]$ with denominator type $[1, \dots, n]^2$ and overconvergence at $\{0, 1/9\}$. The symmetrization $G(y)$ of $H(x)$ and its derivatives and integrals, together with known genuine G -functions $B_1(y), \dots, B_7(y) \in \mathbb{Q}[[y]]$ whose existence is unconditional, span a 14-dimensional $\mathbb{Q}(y)$ -vector space. These functions have controlled denominator types and certain analytic properties, and the holonomy bound dictates that the dimension of this space must be $\leq 13.730\dots < 14$, leading to contradiction.

1. HOLONOMIC FUNCTIONS ASSOCIATED TO ZAGIER'S SEQUENCE

Consider the modular curve $Y_0(6) = \Gamma_0(6) \backslash \mathbf{H} \simeq \mathbb{P}^1 \backslash \{0, 1/9, 1, \infty\}$ with the hauptmodul

$$(1) \quad x = q \prod_{n=1}^{\infty} \frac{(1 - q^n)^4 (1 - q^{6n})^8}{(1 - q^{2n})^8 (1 - q^{3n})^4}, \quad q = e^{2\pi i \tau}.$$

Zagier's sequences ([2, Case C]) give rise to multivalued holonomic functions $H_A(x) = \sum a_n x^n$ and $H_B(x) = \sum b_n x^n$ on $\mathbb{P}^1 \backslash \{0, 1/9, 1, \infty\}$, with $a_n \in \mathbb{Z}$ and $[1, 2, \dots, n]^2 b_n \in \mathbb{Z}$ such that

$$H(x) := H_B(x) - \frac{L(2, \chi_{-3})}{2} H_A(x)$$

has radius of convergence $R = 1$.

The power series $H_A(x)$ and $H_B(x)$ can be defined via (1) by

$$H_A(x) = A(q), \quad \frac{H_B(x)}{H_A(x)} = B(q),$$

where $A \in M_1(\Gamma_0(6), \chi_{-3})$ is a weight one Eisenstein series, and B is an Eichler integral of a weight three Eisenstein series in $M_3(\Gamma_0(6), \chi_{-3})$ satisfying

$$\lim_{q \rightarrow 1} B = \frac{L(2, \chi_{-3})}{2}.$$

The form $A(B - \frac{1}{2}L(2, \chi_{-3}))$ has a weight 0 symmetry defined by the Fricke involution $w_6 : \tau \mapsto -1/(6\tau)$, which leads to overconvergence of $H(x)$ at $x = 1/9$.

Consider the branched double covering $\mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{P}^1 \setminus \{0, \infty\}$ defined by

$$x \mapsto y = x + \frac{x}{x-1}.$$

The symmetrization

$$G(y) := \text{Sym}^+ H(x) = H(x) + H\left(\frac{x}{x-1}\right) \in \mathbb{C}[[y]]$$

is holomorphic on a contractible neighborhood of 0 containing $-1/72$. It continues analytically as a multivalued holomorphic function along all paths in $\mathbb{P}^1 \setminus \{0, -1/72, 4, \infty\}$, and has finite local monodromy of order dividing 2 around the branch point $y = 4$.

2. COMPUTATION OF INVARIANTS ON DENOMINATOR TYPE

Under the assumption that $L(2, \chi_{-3})$ is rational, the function $G(y) \in \mathbb{Q}[[y]]$. The denominator type of $G(y)$ and its derivatives $G'(y)$, $G''(y)$, $G'''(y)$ is $[1, \dots, 2n]^2$. In addition, the integrals $\int G(y) dy$, $\int \frac{G(y)-G(0)}{y} dy$, and $\int \frac{G(y)-G(0)-G'(0)y}{y^2} dy$ have denominator type $n[1, \dots, 2n]^2$.

The seven genuine functions $B_1(y), \dots, B_7(y) \in \mathbb{Q}[[y]]$ and their respective denominator types are summarized in Table 1.

The fourteen functions can be ordered so that the denominator type array

$$\mathbf{b} = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}^t$$

and the integration vector $\mathbf{e} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$, from which we compute the denominator rates

$$\tau^{\flat}(\mathbf{b}) = \frac{1 \cdot 0 + (3+5) \cdot 2 + (7+9+11+13+\dots+27) \cdot 4}{14^2} = \frac{191}{49}.$$

and

$$\tau^{\sharp}(\mathbf{e}) = \frac{2}{14^2} \min_{\xi \in [0, 14]} \left\{ \xi \sum_{i=1}^{14} e_i + \left(\max_i e_i \right) I_{\xi}^{14}(\xi) \right\} = \min_{\xi \in [0, 14]} \left\{ \frac{6\xi + I_{\xi}^{14}(\xi)}{98} \right\} = \frac{27}{80}.$$

<i>G</i> -function	Denominator type
$B_1(y) \in \mathbb{Q}(y)$	1
$B_2(y) = \sum_{n=2}^{\infty} \frac{(n-2)!n!}{(2n)!} 2y^n$	$[1, \dots, 2n]$
$B_3(y) = \sum_{n=1}^{\infty} \frac{((n-1)!)^2}{(2n)!} y^n$	$n[1, \dots, 2n]$
$B_4(y) = \text{Sym}^- \text{Li}_2(y)$	$[1, \dots, 2n]^2$
$B_5(y) = \sum_{n=1}^{\infty} \frac{((n-1)!)^2 y^n}{(2n-1)!(2n-1)}$	$(2n-1)[1, \dots, 2n-1]$, in particular $[1, \dots, 2n]^2$
$B_6(y) = \int \frac{B_3(y)}{y} dy$	$n^2[1, \dots, 2n]$, in particular $n[1, \dots, 2n]^2$
$B_7(y) = \int \frac{B_4(y)}{y} dy$	$n[1, \dots, 2n]^2$

TABLE 1. The 7 genuine functions and their denominator types.

3. OVERCONVERGENCE AND UNIVALENT LEAF

Let $f \in \mathbb{C}[[x]]$ be a holomorphic function germ which extends analytically as a multivalued holomorphic function along all paths in $\mathbb{P}^1 \setminus \Sigma$, for some finite set of punctures Σ with a disjoint partition $\Sigma = \Sigma^0 \sqcup \Sigma^1$. Suppose f is overconvergent at Σ^0 and $\varphi : (\mathbf{D}, 0) \rightarrow (\mathbb{P}^1 \setminus \Sigma^1, 0)$ is a holomorphic map. In general, there is no relationship between the overconvergence sets for f and φ^*f .

If f is holomorphic on a contractible open neighborhood $0 \in U \subset \mathbf{D}$, such that φ has a univalent leaf $\varphi^{-1}(U)_0$ over U at $0 \in \mathbf{D}$ (i.e., the restriction $\varphi|_{\varphi^{-1}(U)_0} : \varphi^{-1}(U)_0 \xrightarrow{\sim} U$ is a conformal isomorphism from the connected component of $\varphi^{-1}(U)$ containing 0 to U) containing $\varphi^{-1}(\Sigma^0)$, then the pullback $\varphi^*f \in \mathbb{C}[[z]]$ converges on the full disc $\mathbf{D}$.

We apply a stacky version of this principle. Consider a holomorphic function germ $F \in \mathbb{C}[[y]]$ that continues analytically along all paths in $\mathbb{P}^1 \setminus \{0, -1/72, 4, \infty\}$ and has finite local monodromy of order dividing 2 around $y = 4$. The holomorphic mapping φ constructed in [1, Appendix A] factors as $\varphi = h \circ \psi$, where

$$(2) \quad h = -256q \prod_{n=1}^{\infty} (1 + q^n)^{24}, \quad q = e^{2\pi i \tau}$$

is a hauptmodul for $Y_0(2) \simeq \mathbb{P}^1 \setminus \{0, \infty\}$, and $\psi : \mathbf{D} \rightarrow \Omega \subset \mathbf{D}$ is a biholomorphism with $\psi(0) = 0$. This map ψ is chosen such that its image Ω inside $\mathbf{D}$ avoids all preimages of $-1/72$ under h , except for the one closest to 0. Given that F is overconvergent at $\{0, -1/72\}$, the pullback φ^*F is holomorphic on $\mathbf{D}$.

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C.7 and C.8: One-variable proofs of the holonomy bounds via the method of slopes

LUIGI MARTINELLI, THOMAS MORDANT

The aim of these talks was to present one-variable proofs of the holonomy bounds (notably of [4, Theorem 7.0.1]) based on the content of [4, §7] and on the relevant material in [2] and [3], while putting emphasis on the geometry underlying these bounds.

We like to think of these proofs as instances of “arithmetic Italian geometry:” they involve inequalities relating intersection numbers attached to projective surfaces, now in the arithmetic context of Arakelov geometry, where classical intersection numbers are replaced by real numbers defined in terms of heights and integrals of Green functions.

The use of auxiliary functions and Siegel’s lemma in classical Diophantine approximation proofs is incorporated into the slopes method: the main objects of interest become the evaluation morphisms from spaces of sections of ample line bundles — a.k.a. auxiliary polynomials — to spaces of formal sections of those. The holonomy bounds appear as (limit cases of) estimates relating the heights of these evaluation morphisms, and the slopes of their source and target spaces.

In this contribution, we focus on geometric versions of the holonomy bounds in [4], which notably cover [4, Theorem 7.0.1] in the special case where $\mathbf{e} = 0$. The level of generality of these versions has been chosen to clarify the geometric meaning of the data and of the hypotheses in these holonomy bounds, and to indicate how a geometric approach leads to holonomy bounds concerning formal functions on arbitrary projective curves.

These geometric holonomy bounds were presented in the second half of Lecture C.8, by Luigi Martinelli, after the necessary material from Arakelov geometry had been introduced in Lecture C.7 by Thomas Mordant, and used in the first half of Lecture C.8 to present the slopes method. Further details will appear in a forthcoming joint paper with Jean-Benoît Bost.

1. A general geometric framework

Let us describe the framework of our versions of the holonomy bounds.

We consider an arbitrary smooth projective curve C over $\mathbb{Q}$, equipped with some $\mathbb{Q}$ -rational point P , and the holonomy bounds will provide upper bounds on the size m of some m -tuple $(f_1, \dots, f_m)$ of elements of the completion of the local ring $\mathcal{O}_{C,P}$, when these elements are $\mathbb{Q}(C)$ -linearly independent and holonomic, and satisfy suitable arithmetic and analytic conditions.

We might in fact consider C to be a smooth projective curve over an arbitrary number field K , and P a closed point of C . However, for simplicity’s sake, we discuss only the situation where K is $\mathbb{Q}$ and P belongs to $C(\mathbb{Q})$.

1.1. As above, let C be an irreducible smooth projective curve over $\mathbb{Q}$, equipped with a point $P \in C(\mathbb{Q})$. As is customary, we denote by $\widehat{C}_P$ the formal completion of C at P . The choice of a uniformizer of C at P provides an isomorphism from $\widehat{C}_P$ to $\mathrm{Spf} \mathbb{Q}[[T]]$.

Moreover we choose a regular projective arithmetic surface $\pi : \mathcal{C} \rightarrow \mathrm{Spec} \mathbb{Z}$ which is a model of the curve C . In other words, $\mathcal{C}$ is an integral regular scheme of dimension 2, projective and flat over $\mathrm{Spec} \mathbb{Z}$, whose generic fiber $\mathcal{C}_{\mathbb{Q}}$ is identified to C .

The $\mathbb{Q}$ -point P of C uniquely extends to a section $\mathcal{P}$ of the morphism π . Its image lies in the smooth locus of this morphism, and we may consider the normal bundle $N_{\mathcal{P}\mathcal{C}} \simeq \mathcal{P}^*T_{\pi}$. It is a line bundle of rank one over $\mathrm{Spec} \mathbb{Z}$, and its generic fiber $(N_{\mathcal{P}\mathcal{C}})_{\mathbb{Q}}$ may be identified with the $\mathbb{Q}$ -line T_PC .

For every non-negative integer i , we shall denote by $\mathcal{P}_i$ the i -th infinitesimal neighborhood of $\mathcal{P}$ in $\mathcal{C}$. It is an affine subscheme of $\mathcal{C}$, finite and flat over $\mathbb{Z}$. The algebra $\mathcal{O}_{\mathcal{P}_i}$ of regular functions over $\mathcal{P}_i$ is a free $\mathbb{Z}$ -module of rank $i + 1$, and its generic fiber $\mathcal{O}_{\mathcal{P}_i, \mathbb{Q}}$ is the $\mathbb{Q}$ -algebra $\mathcal{O}_{\mathcal{P}_i} \simeq \mathbb{Q}[[T]]/T^{i+1}\mathbb{Q}[[T]]$ of functions on the i -th infinitesimal neighborhood P_i of P in C .

1.2. By a *denominator condition*, we mean a map:

$$\mathbf{d} : \mathbb{N} \longrightarrow \mathbb{Z}_{>0}$$

such that, for every $i \in \mathbb{N}$, $\mathbf{d}(i) \mid \mathbf{d}(i + 1)$. We shall say that a formal function $f \in \mathcal{O}_{\widehat{C}_P}$ satisfies the denominator condition $\mathbf{d}$ when, for every $i \in \mathbb{N}$, its truncation $f|_{P_i}$ in $\mathcal{O}_{P_i}$ satisfies:

$$\mathbf{d}(i) f|_{P_i} \in \mathcal{O}_{\mathcal{P}_i}.$$

1.3. Besides suitable denominator conditions, the formal functions in holonomy bounds are assumed to satisfy analytic conditions of overconvergence. These are formulated in terms of a marked compact Riemann surface with boundary (M, O) and of a complex analytic map:

$$(1) \quad \varphi : (M, O) \longrightarrow (C(\mathbb{C}), P_{\mathbb{C}}),$$

which we assume to be compatible with the markings — namely:

$$\varphi(O) = P_{\mathbb{C}}$$

— and to be étale at O .

Here M is a connected compact Riemann surface with non-empty boundary ∂M , defined as a domain with C^∞ boundary in some (germ of open) Riemann surface M^+ . By O we denote a point in the interior $\mathring{M} := M \setminus \partial M$ of M , and by φ a map which is $\mathbb{C}$ -analytic up to the boundary.¹

We shall say that a formal function f in $\mathcal{O}_{\widehat{C}_{\mathbb{C}}, P_{\mathbb{C}}} \simeq \mathcal{O}_{\widehat{C}_P} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C}$ is φ -overconvergent when f defines a germ of $\mathbb{C}$ -analytic function at the point $P_{\mathbb{C}}$ on the Riemann surface $C(\mathbb{C})$ and if its pull-back $\varphi^*(f)$ — which is a germ of $\mathbb{C}$ -analytic function

¹Or equivalently, a germ of analytic map from M^+ to $C(\mathbb{C})$ along M .

at the point O of M — analytically extends to some germ of meromorphic map on M^+ along M .

When f is a formal function in $\mathcal{O}_{\widehat{C}_P}$, we shall say that f is φ -overconvergent when its image in $\mathcal{O}_{\widehat{C}_{\mathbb{C}, P_{\mathbb{C}}}}$ is.

1.4. Recall that to a marked compact connected Riemann surface with non-empty boundary (M, O) as above is attached the *classical*, or Dirichlet, *Green function* $g_{M,O}$. It is defined as the unique function:

$$g_{M,O} : M \setminus \{O\} \longrightarrow \mathbb{R}$$

which satisfies the following conditions:

- (1) $g_{M,O}$ is harmonic on $\overset{\circ}{M} \setminus \{O\}$;
- (2) $g_{M,O}$ admits a logarithmic singularity at O ; namely, if (U, z) denotes a $\mathbb{C}$ -analytic chart on $\overset{\circ}{M}$ whose domain U contains O , the function $g_{M,O} - \log |z - z(O)|^{-1}$ is bounded on some punctured neighborhood of O in U ;
- (3) $g_{M,O}$ is continuous on $M \setminus \{O\}$ and vanishes on ∂M .

With the above notation, the difference $g_{M,O} - \log |z - z(O)|^{-1}$, initially defined on $U \setminus \{O\}$, actually extends to some harmonic function $h : U \rightarrow \mathbb{R}$.

According to its very definition, $g_{M,O}$ admits a continuous extension on the open Riemann surface $M^+ \setminus \{O\}$ defined by the equality:

$$g_{M,O}(x) = 0 \quad \text{for every } x \in M^+ \setminus M,$$

and then $g_{M,O}$ is a locally L^1 function on M^+ . As such, it satisfies the following equation of currents:

$$\frac{i}{\pi} \partial \bar{\partial} g_{M,O} + \delta_O = \mu_{M,O},$$

where δ_O is the Dirac measure at O , and where $\mu_{M,O}$ denotes the *harmonic measure* attached to (M, O) : it is a probability measure supported by the boundary ∂M , defined by a C^∞ everywhere positive one-form on ∂M .

To the Green function $g_{M,O}$ is attached a C^∞ Hermitian metric $\|\cdot\|_{M,O}$ on the line bundle $\mathcal{O}(O)$ on M , defined by the following equality on $M \setminus \{O\}$:

$$\|\mathbf{1}_{\mathcal{O}(O)}\|_{M,O} := \exp(-g_{M,O}).$$

Transported by the adjunction isomorphism $\mathcal{O}(O)_O \simeq T_O M$, the restriction to $\mathcal{O}(O)_O$ of the Hermitian metric $\|\cdot\|_{M,O}$ becomes the *capacitary norm* $\|\cdot\|_{M,O}^{\text{cap}}$ on $T_O M$. In terms of the above coordinate chart (U, z) and of the harmonic function h , it is also defined by the following equality:

$$\left\| \frac{\partial}{\partial z} \right\|_{M,O}^{\text{cap}} = \exp(-h(O)).$$

1.5. Using the data (M, O) and φ introduced in **1.3**, as well as the classical Green function $g_{M,O}$ and the capacitary norm $\|\cdot\|_{M,O}^{\text{cap}}$ on $T_O M$ defined in **1.4**, we may construct some natural Hermitian line bundles on the arithmetic curve $\mathcal{P}$ ($\simeq \text{Spec } \mathbb{Z}$) and on the arithmetic surface $\mathcal{C}$.

Firstly, since φ is étale at O , the isomorphism

$$D\varphi(O) : T_O M \xrightarrow{\sim} T_{P_C} C(\mathbb{C}) \simeq (T_P C)_{\mathbb{C}}.$$

allows us to transport the capacitary norm $\|\cdot\|_{M,O}^{\text{cap}}$ on $T_O M$ to a norm on $(T_P C)_{\mathbb{C}} \simeq (N_{\mathcal{P}} \mathcal{C})_{\mathbb{C}}$, which we shall denote by $\|\cdot\|_{M,O,\varphi}^{\text{cap}}$. Endowed with this norm, $N_{\mathcal{P}} \mathcal{C}$ becomes a Hermitian line bundle over $\mathcal{P} \simeq \text{Spec } \mathbb{Z}$:

$$\overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} := (N_{\mathcal{P}} \mathcal{C}, \|\cdot\|_{M,O,\varphi}^{\text{cap}}).$$

Secondly, the direct image $\varphi_* g_{M,O}$ of the classical Green function $g_{M,O}$ defines a Green function for the divisor $\mathcal{P}$ on $\mathcal{C}$, and we shall denote by $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ the corresponding Hermitian line bundle on $\mathcal{C}$. In other words, $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ is the pair $(\mathcal{O}_{\mathcal{C}}(\mathcal{P}), \|\cdot\|_{\varphi_* g_{M,O}})$, where $\|\cdot\|_{\varphi_* g_{M,O}}$ is the Hermitian metric on the line bundle $\mathcal{O}_{\mathcal{C}}(\mathcal{P})$ defined by the following equality, for every $x \in C(\mathbb{C})$:

$$\log \|\mathbf{1}_{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}(x)\|_{\varphi_* g_{M,O}}^{-1} = \varphi_* g_{M,O}(x) := \sum_{y \in \varphi^{-1}(x)} g_{M,O}(y).$$

where, in the last sum, the multiplicities of y in the divisor $\varphi^{-1}(x)$ have to be taken into account.

The Arakelov self-intersection of the Hermitian line bundle $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ over $\mathcal{C}$, the height $\text{ht}_{\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}}(P)$ of P with respect to this Hermitian line bundle, and the Arakelov degree $\widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi}$ of the Hermitian line bundle $\overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi}$ over $\mathcal{P}$ are related by the following equalities:

$$(2) \quad \overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi} \cdot \overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi} = \text{ht}_{\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}}(P) + \int_M g_{M,O} \varphi^* \varphi_* \mu_{M,O}$$

$$(3) \quad = \widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} + \text{Ex}(\varphi : (M, O) \rightarrow C(\mathbb{C})),$$

where $\text{Ex}(\varphi : (M, O) \rightarrow C(\mathbb{C}))$ denotes the *overflow* associated to the morphism of marked Riemann surfaces φ , introduced in [3, Chapter 5], and defined as follows:

$$\text{Ex}(\varphi : (M, O) \rightarrow C(\mathbb{C})) := \int_M g_{M,O} (\delta_{\varphi^*(P_C)-O} + \varphi^* \varphi_* \mu_{M,O}).$$

Equalities (2) and (3) are indeed straightforward consequences of the basic properties of the Arakelov intersection pairing, in the setting of [3, Chapter 4].

2. Eleutherian families

In this section, we denote by C a smooth projective integral curve over an arbitrary field k , endowed with an ample line bundle L and a point $P \in C(k)$.

Consider a tuple of formal functions $(f_{\alpha})_{1 \leq \alpha \leq m}$ in $\mathcal{O}_{\widehat{C}_P}^{\oplus m}$. For any $D \in \mathbb{N}$, we have an evaluation morphism:

$$\eta_D : \Gamma(C, L^{\otimes D})^{\oplus m} \longrightarrow L^{\otimes D}|_{\widehat{C}_P}, \quad (s_{\alpha})_{1 \leq \alpha \leq m} \longmapsto \sum_{\alpha=1}^m s_{\alpha}|_{\widehat{C}_P} \cdot f_{\alpha}.$$

The composition of η_D with the restriction morphism $L^{\otimes D}|_{\widehat{C}_P} \rightarrow L^{\otimes D}|_{P_{i-1}}$ for any $i > 0$ yields a collection of evaluation morphisms:

$$(4) \quad \eta_{D,i}: \Gamma(C, L^{\otimes D})^{\oplus m} \longrightarrow L^{\otimes D}|_{P_{i-1}}.$$

Some of the properties of the family $(f_\alpha)_{1 \leq \alpha \leq m}$ can be expressed in terms of these evaluation morphisms. For instance, the following conditions are equivalent:

- (1) $f_1, \dots, f_m \in \mathcal{O}_{\widehat{C}_P} \subset \text{Frac}(\mathcal{O}_{\widehat{C}_P})$ are linearly independent over $k(C)$;
- (2) for any $D \geq 0$, the morphism η_D is injective;
- (3) for any $D \geq 0$, and for any $i \gg_D 0$, the morphism $\eta_{D,i}$ is injective.

If one these equivalent conditions is satisfied, we set:

$$\iota_L(D) = \min\{i \in \mathbb{Z}_{>0} \mid \eta_{D,i} \text{ is injective}\}.$$

In particular, we have the inequality:

$$\iota_L(D) \geq \dim_k \Gamma(C, L^{\otimes D})^{\oplus m} \geq m \cdot (D \cdot \deg_C L + 1 - g).$$

We shall say that a family $(f_\alpha)_{1 \leq \alpha \leq m} \in \mathcal{O}_{\widehat{C}_P}^{\oplus m}$ is *eleutherian* if the following asymptotic holds:

$$\iota_L(D) \sim m \cdot D \cdot \deg_C L \quad \text{when } D \rightarrow +\infty.$$

Standard arguments indeed show that this condition does not depend on the choice of the ample line bundle L over C . It is also invariant under extension of the base field k .

When $C = \mathbb{P}_{\mathbb{C}}^1$ is the projective complex line and $P = [1 : 0]$, examples of eleutherian families already appear in the works of Hermite discussed in talk **B.4**. Consider the exponential functions $f_1 = e^{\gamma_1 T}, \dots, f_m = e^{\gamma_m T} \in \mathbb{C}[[T]]$, for pairwise distinct complex numbers $\gamma_1, \dots, \gamma_m$. In [6], Hermite proved that the family $(f_1, \dots, f_m)$ is eleutherian. Actually Hermite showed that, for this family, $\iota_{\mathcal{O}(1)}(D) = m \cdot D$ for every $D \in \mathbb{N}$.

When k is a field of characteristic 0 and C is $\mathbb{P}_k^1$, the Chudnovsky–Osgood theorem (see [5], [7]) asserts that a family $(f_1, \dots, f_m)$ of holonomic formal functions is eleutherian as soon as it is linearly independent over $k(C) \simeq k(T)$.

3. A basic bound on eleutherian families

Besides the data introduced in **1.1–3** above — which will be used to formulate the denominator and the overconvergence conditions required on the formal functions considered in holonomy bounds — the basic version of those involve some auxiliary data. By specific choices of these auxiliary data, our basic holonomy bounds will take more convenient forms, which specialized to $C = \mathbb{P}_{\mathbb{Q}}^1$ will cover various versions of the holonomy bounds in [4].

3.1. The first of these auxiliary data is a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ over the arithmetic surface $\mathcal{C}$, defined by a line bundle $\mathcal{L}$ over $\mathcal{C}$ — whose restriction $L := \mathcal{L}_{C_{\mathbb{Q}}}$ to the curve C is ample — and a Hermitian metric $\|\cdot\|$ (invariant under complex conjugation) on the line bundle $L_{\mathbb{C}}$ over the Riemann surface $C(\mathbb{C})$ locally of the form $\|\cdot\| = e^{-\psi}|\cdot|$ where ψ is a continuous subharmonic function. In particular the

“first Chern form” $c_1(\overline{L}_{\mathbb{C}})$ of $(L_{\mathbb{C}}, \|\cdot\|)$ is a well-defined positive measure on $C(\mathbb{C})$, locally given by $i\partial\bar{\partial}\psi/\pi$, of total mass $\deg_C L$.

For any integer D , we shall denote by $\overline{\mathcal{L}}^{\otimes D} := (\mathcal{L}^{\otimes D}, \|\cdot\|_D)$ the D -th tensor power of $\overline{\mathcal{L}}$, and by $\pi_*\overline{\mathcal{L}}^{\otimes D}$ the Euclidean lattice defined by the $\mathbb{Z}$ -module $\Gamma(\mathcal{C}, \mathcal{L}^{\otimes D})$ and the John norm $\|\cdot\|_D^J$ on $\Gamma(\mathcal{C}, \mathcal{L}^{\otimes D})_{\mathbb{C}} \simeq \Gamma(C(\mathbb{C}), L_{\mathbb{C}}^{\otimes D})$ attached to the L^∞ -norm $\|\cdot\|_{D, L^\infty}$ defined by $\|s\|_{D, L^\infty} := \max_{x \in C(\mathbb{C})} \|s(x)\|_D$. The slope

$$\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D}) := \frac{\widehat{\deg} \pi_*\overline{\mathcal{L}}^{\otimes D}}{\mathrm{rk} \pi_*\overline{\mathcal{L}}^{\otimes D}}$$

of this Euclidean lattice is easily seen to satisfy:

$$|\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D})| = O(D) \quad \text{when } D \rightarrow +\infty.$$

3.2. The second of these auxiliary data is associated to a tuple $(\mathbf{d}_\alpha)_{1 \leq \alpha \leq m}$ of denominator conditions. We shall say that a m -tuple $(\tilde{\mathbf{d}}_\alpha)_{1 \leq \alpha \leq m}$ of functions $\tilde{\mathbf{d}}_\alpha : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ and a non-decreasing function $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are *suited to* $(\mathbf{d}_\alpha)_{1 \leq \alpha \leq m}$ when, for every α in $\{1, \dots, m\}$, we have:

$$(5) \quad \tilde{\delta}_\alpha := \limsup_{D \rightarrow +\infty} D^{-1} \log \tilde{\mathbf{d}}_\alpha(D) < +\infty,$$

and when, for every $(i, D) \in \mathbb{N} \times \mathbb{Z}_{>0}$, the following inequality holds:

$$(6) \quad \lambda(i, D) := \log \mathrm{lcm}_{1 \leq \alpha \leq m} (\mathbf{d}_\alpha(i) \cdot \tilde{\mathbf{d}}_\alpha(D)) \leq D \chi(i/D).$$

3.3. We are now in position to state a basic version of the holonomy bounds.

Theorem 7. *Let us consider a marked smooth projective curve (C, P) over $\mathbb{Q}$, and a model $\pi : \mathcal{C} \rightarrow \mathrm{Spec} \mathbb{Z}$ of C and the section $\mathcal{P}$ of π as in 1.1. Let moreover $\varphi : (M, O) \rightarrow (C(\mathbb{C}), P_{\mathbb{C}})$ be as in 1.3.*

Consider a tuple $(f_\alpha)_{1 \leq \alpha \leq m}$ of formal functions in $\mathcal{O}_{\widehat{C}_P}$ such that, for every α in $\{1, \dots, m\}$, f_α satisfies some denominator condition $\mathbf{d}_\alpha$ and is φ -overconvergent. Moreover let $(\tilde{\mathbf{d}}_\alpha)_{1 \leq \alpha \leq m}$ and χ be functions suited to $(\mathbf{d}_\alpha)_{1 \leq \alpha \leq m}$.

If the family $(f_\alpha)_{1 \leq \alpha \leq m}$ is eleutherian, and if the following arithmetic pseudo-concavity condition is satisfied:

$$\widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_\varphi \geq 0,$$

then the following inequality holds:

$$(8) \quad m \deg_C L \left(\frac{1}{2} \widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_\varphi - \frac{1}{(m \deg_C L)^2} \int_0^{m \deg_C L} \chi(t) dt \right) \\ \leq \mathrm{ht}_{\overline{\mathcal{L}}}(P) + \int_M g_{M, O} \varphi^* c_1(\overline{L}_{\mathbb{C}}) - \frac{1}{m} \sum_{\alpha=1}^m \tilde{\delta}_\alpha - \limsup_{D \rightarrow +\infty} \frac{1}{D} \widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D}).$$

The inequality (8) follows from the slope estimates applied to the evaluation morphism $\eta_{D,i}$ introduced in (4) when $i = \iota_L(D)$ and by letting D go to infinity.

The derivation of bounds involving formal functions satisfying possibly different denominator conditions $(\mathbf{d}_\alpha)_{1 \leq \alpha \leq m}$ crucially relies on the choice of integral structures on the spaces of auxiliary polynomials $\Gamma(C, L^{\otimes D})^{\oplus m}$ introduced in [4, (7.3.6)] in terms of the $(\tilde{\mathbf{d}}_\alpha)_{1 \leq \alpha \leq m}$. Besides this construction, the proof of (8) relies on the basic formalism of slopes estimates, as presented in [1, 4.1-2], and the Schwarz Lemma in terms of Green functions and capacity metrics in [2, Section 10.5].

4. From basic bounds to holonomy bounds

4.1. The denominator conditions that appear in [4, Theorems 6.0.2 and 7.0.1], when $\mathbf{e} = 0$, are covered by the following choices of $\mathbf{d}_\alpha$, $\tilde{\mathbf{d}}_\alpha$ and χ in **3.2**, when $\deg_C L = 1$.

We consider a matrix $\mathbf{b} = (b_{\alpha\beta})_{1 \leq \alpha \leq m, 1 \leq \beta \leq r}$ of non-negative real numbers as in [4, Theorem 6.0.2], and we let, with the notation of [4, Section 7.3.3]:

$$\mathbf{d}_\alpha(i) := \prod_{\beta=1}^{h_\alpha} [1, \dots, b_\beta i] \quad \text{and} \quad \tilde{\mathbf{d}}_\alpha(D) := \prod_{\beta=h_\alpha+1}^r [1, \dots, y_\beta D].$$

Then conditions (5) and (6) are satisfied with:

$$\tilde{\delta}_\alpha = \sum_{\beta=h_\alpha+1}^r y_\beta \quad \text{and} \quad \chi(t) = \sum_{\beta=1}^r \max(y_\beta, b_\beta t).$$

When the y_β are chosen as in [4, Section 7.3.3, Proof of Theorem 7.0.1], namely when $y_\beta = u_\beta b_\beta$, we obtain:

$$\frac{1}{(m \deg_C L)^2} \int_0^{m \deg_C L} \chi(t) dt = \frac{1}{2m^2} \sum_{\beta=1}^r b_\beta (u_\beta^2 + m^2)$$

and

$$\frac{1}{m} \sum_{\alpha=1}^m \tilde{\delta}_\alpha = \frac{1}{m} \sum_{\beta=1}^r b_\beta u_\beta^2.$$

Finally if, as in [4, Theorem 6.0.2], we let:

$$\tau(\mathbf{b}) := \sum_{\beta=1}^r b_\beta - \frac{1}{m^2} \sum_{\beta=1}^r b_\beta u_\beta^2,$$

the estimate (8) becomes:

$$\begin{aligned} (9) \quad & m(\widehat{\deg \overline{N_{\mathcal{P}} \mathcal{C}}}_\varphi - \tau(\mathbf{b})) \\ & \leq 2 \text{ht}_{\overline{\mathcal{L}}}(P) + 2 \int_M g_{M,O} \varphi^* c_1(\overline{\mathcal{L}}_C) - 2 \limsup_{D \rightarrow +\infty} \frac{1}{D} \hat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}). \end{aligned}$$

4.2. When the auxiliary Hermitian line bundle $\overline{\mathcal{L}}$, besides fulfilling the conditions introduced in **3.1**, has an underlying line bundle $\mathcal{L}$ which is nef on the closed fibers $\mathcal{C}_{\mathbb{P}_p}$ of π , the arithmetic Hilbert–Samuel formula applies, as shown in [8], and accordingly we have:

$$(10) \quad \lim_{D \rightarrow +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{\overline{\mathcal{L}} \cdot \overline{\mathcal{L}}}{2 \deg_C L}.$$

4.3. When $\mathcal{C} = \mathbb{P}_{\mathbb{Z}}^1$ and $\overline{\mathcal{L}}$ is the Hermitian line bundle $\overline{\mathcal{O}}_{\mathbb{P}^1}(1) := (\mathcal{O}_{\mathbb{P}^1}(1), \|\cdot\|)$, defined by the standard Hermitian metric $\|\cdot\|$ on $\mathcal{O}_{\mathbb{P}^1}(1)$ whose first Chern form is the Fubini–Study 2-form:

$$\omega_{\text{FS}} := c_1(\overline{\mathcal{O}}_{\mathbb{P}^1}(1)),$$

then $\deg_C L = 1$, and, as a special case² of (10), we have:

$$(11) \quad \lim_{D \rightarrow +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{O}}_{\mathbb{P}^1}(D)) = 1/4.$$

When moreover $P = [1 : 0]$ and $(M, O) = (\overline{D}(0, 1), 0)$, then we have:

$$\widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} = \log |\varphi'(0)|,$$

the height $\text{ht}_{\overline{\mathcal{L}}}(P)$ vanishes, and the integral in the right-hand side of (8) is the Nevanlinna–Ahlfors–Shimizu characteristic function of φ :

$$\int_M g_{M,O} \varphi^* c_1(\overline{\mathcal{L}}_C) = \int_{\overline{D}(0,1)} \log |z|^{-1} \varphi^* \omega_{\text{FS}} =: T_{\varphi}(1).$$

Finally, when moreover the $\mathbf{d}_{\alpha}(i)$ and the $\widetilde{\mathbf{d}}_{\alpha}(D)$ are chosen as in **4.1**, the estimate (9) becomes:

$$m(\log |\varphi'(0)| - \tau(\mathbf{b})) \leq 2 T_{\varphi}(1) - 1/2.$$

This is an improved version of [4, (2.2.3)], with e replaced by 2, as mentioned at the end of [4, Section 2.2].

4.4. The conditions in **3.1** are also satisfied when $\overline{\mathcal{L}} = \overline{\mathcal{O}(\mathcal{P})}_{\varphi}$. Since $\mathcal{O}(\mathcal{P})$ is nef on the closed fiber of π , we may also apply the arithmetic Hilbert–Samuel formula (10), and accordingly:

$$\lim_{D \rightarrow +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{1}{2} \overline{\mathcal{O}_C(\mathcal{P})}_{\varphi} \cdot \overline{\mathcal{O}_C(\mathcal{P})}_{\varphi}.$$

Together with the relations (2) and (3), this shows that, with this choice of $\overline{\mathcal{L}}$, the estimate (9) becomes:

$$(12) \quad m(\widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} - \tau(\mathbf{b})) \leq \widehat{\deg} \overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} + \text{Ex}(\varphi : (M, O) \rightarrow C(\mathbb{C})).$$

²It is also easily derived by a direct computation.

In turn, when $\mathcal{C} = \mathbb{P}^1_{\mathbb{Z}}$, $P = [1 : 0]$, $(M, O) = (\overline{D}(0, 1), 0)$, and the image of φ lies in $\mathbb{C}$, the overflow of φ may be written:

$$\mathrm{Ex}(\varphi : (M, O) \rightarrow C(\mathbb{C})) = \iint_{\mathbb{T}^2} \log |\varphi(z) - \varphi(w)| \, d\mu(z) d\mu(w) - \log |\varphi'(0)|$$

(see [3, Proposition 5.4.3]), and therefore (12) becomes:

$$m(\log |\varphi'(0)| - \tau(\mathbf{b})) \leq \iint_{\mathbb{T}^2} \log |\varphi(z) - \varphi(w)| \, d\mu(z) d\mu(w),$$

which is precisely the estimate [4, (7.0.3)] with $\mathbf{e} = 0$.

The correspondence between our geometric framework and the original setting in [4] is summarized in the following table.

Geometric framework	Calegari–Dimitrov–Tang
Data	
C smooth projective curve/ $\mathbb{Q}$ $P \in C(\mathbb{Q})$	$\mathbb{P}^1_{\mathbb{Q}}$ $[1 : 0] \in \mathbb{P}^1(\mathbb{Q})$
$\mathcal{C}$ regular projective model of $C/ \mathbb{Z}$ $\mathcal{P} \in \mathcal{C}(\mathbb{Z})$ extending $P \in C(\mathbb{Q})$	$\mathbb{P}^1_{\mathbb{Z}}$ $[1 : 0] \in \mathbb{P}^1(\mathbb{Z})$
M connected compact Riemann surface with boundary $\partial M \neq \emptyset$ $O \in M \setminus \partial M$	$\overline{D}(0, 1) = \{z \in \mathbb{C} \mid z \leq 1\}$ $\mathbb{T} = \{z \in \mathbb{C} \mid z = 1\}$ $0 \in \overline{D}(0, 1) \setminus \mathbb{T}$
$\varphi \colon M \rightarrow C(\mathbb{C})$ holomorphic $\varphi(O) = P_{\mathbb{C}}$ $D\varphi(O) \colon T_O M \xrightarrow{\sim} (T_P C)_{\mathbb{C}}$ $\ \cdot\ _{M,O}^{\mathrm{cap}} \rightsquigarrow \ \cdot\ _{M,O,\varphi}^{\mathrm{cap}}$	$\varphi \colon \overline{D}(0, 1) \rightarrow \mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ holomorphic $\varphi(0) = [1 : 0]$ $\varphi'(0) \neq 0$
$\overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi} = (N_{\mathcal{P}} \mathcal{C}, \ \cdot\ _{M,O,\varphi}^{\mathrm{cap}})$ $\widehat{\deg}(\overline{N_{\mathcal{P}} \mathcal{C}}_{\varphi})$	$\log \varphi'(0) $
$(f_{\alpha})_{1 \leq \alpha \leq m} \in \mathcal{O}_{\widehat{C}_P}^{\oplus m}$	$(f_{\alpha})_{1 \leq \alpha \leq m} \in \mathbb{Q}[[T]]^{\oplus m}$
$\mathbf{d}_{\alpha} \colon \mathbb{N} \rightarrow \mathbb{Z}_{>0}, \quad \mathbf{d}_{\alpha}(i) \mid \mathbf{d}_{\alpha}(i+1)$ $\mathbf{d}_{\alpha}(i) \cdot f_{\alpha} _{P_i} \in \mathcal{O}_{P_i}$	$\mathbf{d}_{\alpha}(i) = \prod_{\beta=1}^{h_{\alpha}} [1, \dots, b_{\beta} i]$ $f_{\alpha}(T) = \sum_{i \geq 0} \frac{a_{\alpha,i}}{\mathbf{d}_{\alpha}(i)} T^i, \quad a_{\alpha,i} \in \mathbb{Z}$
$\varphi^*(f_{\alpha,\mathbb{C}})$ extends to a meromorphic function on M	$\varphi^*(f_{\alpha,\mathbb{C}})$ extends to a meromorphic function on $\overline{D}(0, 1)$
$(f_{\alpha})_{1 \leq \alpha \leq m} \in \mathcal{O}_{\widehat{C}_P}^{\oplus m}$ is an eleutherian family	$(f_{\alpha})_{1 \leq \alpha \leq m} \in \mathbb{Q}[[T]]$ is free / $\mathbb{Q}(T)$, and every f_{α} is holonomic

Geometric framework	Calegari–Dimitrov–Tang
Auxiliary data	
$\overline{\mathcal{L}}$ Hermitian line bundle on $\mathcal{C}$ $L = \mathcal{L} _{\mathcal{C}}$ $\limsup_{D \rightarrow +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D})$ $\mathrm{ht}_{\overline{\mathcal{L}}}(P) + \int_M g_{M,O} \varphi^* c_1(\overline{\mathcal{L}}_{\mathcal{C}})$	$\overline{\mathcal{O}(1)}_{\varphi} = (\mathcal{O}([1:0]), \varphi_* \log^+ z ^{-1})$ $\mathcal{O}(1)$ $\frac{1}{2}(\overline{\mathcal{O}(1)}_{\varphi} \cdot \overline{\mathcal{O}(1)}_{\varphi})$ $\int \int_{\mathbb{T}^2} \log \varphi(z) - \varphi(w) d\mu(z) d\mu(w)$
$\widetilde{\mathbf{d}}_{\alpha}: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ $\widetilde{\delta}_{\alpha} = \limsup_{D \rightarrow +\infty} \frac{\log \widetilde{\mathbf{d}}_{\alpha}(D)}{D} < +\infty$	$\widetilde{\mathbf{d}}_{\alpha}(D) = \prod_{\beta=h_{\alpha}+1}^r [1, \dots, y_{\beta} D]$ $\widetilde{\delta}_{\alpha} = \sum_{\beta=h_{\alpha}+1}^r y_{\beta}$
$\lambda(i, D) = \log \mathrm{lcm}_{\alpha}(\mathbf{d}_{\alpha}(i) \cdot \widetilde{\mathbf{d}}_{\alpha}(D))$ $\lambda(i, D) \leq D\chi(\frac{i}{D}),$ $\chi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ non-decreasing $\int_0^{m \deg_{\mathcal{C}} L} \chi(t) dt$	$\chi(t) = \sum_{\beta=1}^r \max(y_{\beta}, b_{\beta} t)$ $\frac{1}{2} \sum_{\beta=1}^r (\frac{y_{\beta}^2}{b_{\beta}} + b_{\beta} m^2)$

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D.1: The contour for the irrationality of $L(2, \chi_{-3})$

PABLO CHENAL

The goal of the talk is to present the final choice of contour $\varphi: (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 0)$ that appears in the holonomy bounds in [2] to prove the irrationality of the special L -function value $L(2, \chi_{-3})$.

For a well-chosen contour φ , we have a 14-dimensional $\mathbb{Q}(X)$ vector space

$$H(\varphi)$$

that is the span of 14 functions $(G_i)_{1 \leq i \leq 14}$ (assuming the linear dependence of 1, $\zeta(2)$, and $L(2, \chi_{-3})$ over $\mathbb{Q}$) on the modular curve $Y_0(2)$ that are linearly independent over $\mathbb{Q}(X)$ and that satisfy the following conditions:

- (1) $\varphi^* G_i \in \mathcal{O}(\overline{\mathbb{D}})$
- (2) G_i has fixed denominator type.

Under some conditions on φ , we have the holonomy bound

$$\dim_{\mathbb{Q}(X)} H(\varphi) \leq \frac{\iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z) \mu(w)}{\log |\varphi'(0)| - \tau}$$

where τ is a fixed real number that corresponds to the condition (2), T is the unit circle, and μ is the Haar measure on T .

We want to optimize the right-hand side so that it is smaller than 14 to get a contradiction.

1. SOME CONTEXT

1.1. Potential theory and Bost-Charles double integral. Let (V, P) be a Riemann surface with boundary pointed at P as defined by Bost and Charles in [1].

Definition 1.1. A classical Green function associated to (V, P) is a function $g_{V,P} : V \setminus P \rightarrow \mathbb{R}^+$ that satisfies the following conditions:

- (1) $g_{V,P}$ is continuous on $V \setminus P$, equal to zero on ∂V
- (2) $g_{V,P}$ is harmonic on $\overset{o}{V} \setminus P$
- (3) near P , $g_{V,P} - \log |z - P|^{-1}$ is harmonic

Proposition 1.2. Let (V, P) be a compact, connected, Riemann surface with non-empty boundary pointed at P . Then there exists a unique classical Green function on (V, P) .

Remark 1. Conditions 2 and 3 imply that the classical Green function is locally L^2 on V and we define the harmonic measure associated to (V, P) as

$$\mu_{V,P} := \frac{i}{\pi} \partial \bar{\partial} g_{V,P} + \delta_P .$$

It is a probability measure supported on the smooth curve ∂V .

Example 1. In the case of interest of [2], we have $(V, P) = (\overline{\mathbb{D}}, 0)$, the classical Green function is $g_{V,P} = \log^+ |z|^{-1}$ and the harmonic measure $\mu_{V,P}$ is the Haar measure μ on the circle T .

We can now define the *Overflow* as introduced by Bost and Charles in [1].

Definition 1.3. Let N be a Riemann surface, and let $\alpha : (V, P) \rightarrow N$ be a non constant analytic morphism, étale at 0. We define the Overflow of α as

$$\text{Ex}(\alpha : (V, P) \rightarrow N) := \int_V g_{V,P} (\alpha^* \alpha_* \mu_{V,P} + \delta_{\alpha^{-1}(\alpha(P)) - P}) \in \mathbb{R}^+ .$$

In the case of interest in [2], the following theorem proven in a more general framework (theorem 5.1, [1]) explains the link between Bost-Charles double integral and the overflow.

Theorem 1. *Let $\varphi : (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic morphism, étale at 0. Then we have the formula*

$$\text{Ex}(\varphi : (\overline{\mathbb{D}}, 0) \rightarrow (\mathbb{C}, 0)) = \iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z) \mu(w) - \log |\varphi'(0)| \, .$$

Remark 2. In the different holonomy bounds, the other expression with the rearrangement integral

$$J := \int_0^1 2t \cdot (\log |\varphi(e^{2i\pi t})|)^* dt$$

appears and has been compared with Bost-Charles double integral

$$I := \iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z) \mu(w)$$

by Nazarov. For any continuous map $\varphi : T \rightarrow \mathbb{C}$, we have

$$I \leq J \, .$$

For example, in the proof of the irrationality of products of logarithms, the final estimate gives $I \simeq 9,963$ and $J \simeq 9,972$.

1.2. Explicit conditions on the optimization problem. The 14 functions $(G_i)_{1 \leq i \leq 14}$ are defined on the modular curve $Y_0(2)$ and have a singularity at $-1/72$. To apply the holonomy bound, we need that

$$\varphi^* G_i \in \mathcal{O}(\overline{\mathbb{D}}) \, .$$

The modular function $h := \lambda + \frac{\lambda}{\lambda-1}$ is the function that ensures that we have this extension property. We will define $\varphi := h \circ \psi$ as the composition of a conformal isomorphism $\psi : \mathbb{D} \rightarrow \Omega$ such that $\overline{\Omega} \subset \mathbb{D}$ with the modular function h .

Let h be the Hauptmodul for $Y_0(2)$ defined as

$$h : \mathbb{D} \rightarrow \mathbb{C} \setminus \{0, 1\}, \, q \mapsto -256 \prod_{n=1}^{+\infty} (1 + q^n)^{24} \, .$$

If

$$\psi : (\overline{\mathbb{D}}, 0) \rightarrow (\overline{\Omega}, 0)$$

is a conformal isomorphism that avoids all the preimages of $-1/72$ of h (which is the only singularity of the G_i 's on $\mathbb{D}$) except for the one real preimage

$$\{0.000054\dots =: x_0\} \, ,$$

then the point x_0 is an "overconvergent" point for the $(G_i)_{1 \leq i \leq 14}$ in the sense that

$$(h \circ \psi)^* G_i \in \mathcal{O}(\overline{\mathbb{D}}) \, .$$

Therefore, it suffices to choose a contour $\partial\psi : \partial\Omega \rightarrow \mathbb{D}$ (the conformal isomorphism $\psi : \mathbb{D} \rightarrow \Omega$ will exist by the Riemann mapping theorem) that satisfies:

- (1) $h^{-1}(-1/72) \cap \psi(\overline{\mathbb{D}}) \subset \{x_0\}$ (to get convergence on $\overline{\mathbb{D}}$)
- (2) $\log |256\psi'(0)| - \tau > 0$ (to apply the holonomy bound).

2. THE FINAL CHOICE OF CONTOUR

- (1) To optimize the bound, we want to have a large conformal radius $|\psi'(0)|$. But the problem is that near any cusp $\alpha = e^{2i\pi \frac{a}{b}}$ with b odd of $Y_0(2)$, the function h grows as

$$\log |h(z)| \sim \frac{1}{2\pi^2 b^2 (1 - |z|)}$$

and therefore, the numerator

$$\iint_{T^2} \log |h(\psi(z)) - h(\psi(w))| \mu(z) \mu(w)$$

is large near the cusps. The contour needs to be a large circle inside $\mathbb{D}$ but not too large to avoid the horoballs near the cusps.

- (2) The preimages of $-1/72$ under h different from $\{x_0\}$ are distributed on horoballs near the boundary T . If we exclude the horoball at -1 , these horoballs are contained in the ring $\{z \in \mathbb{D}, 77/100 < |z| < 1\}$.

If we choose the contour as the circle of radius $R := 77/100$, the only preimages of $-1/72$ that are in the disk of radius R are the ones in the horoball near -1 , and we have only four of them. For the problem of cusps, the only cusp α for which the "large h values set" horoball intersects the disk of radius R is the cusp 1 .

Therefore, we will define the domain Ω as the disk of radius R inside $\mathbb{D}$ and remove the following parts of the disk that won't change the conformal radius so much:

- (1) remove 4 slits from the boundary of the disk to the 4 preimages of $-1/72$ in the horoball near -1
- (2) a lune near R to remove the intersection of the horoball near 1 with the disk.

Remark 3. The Riemann mapping theorem guarantees that such a domain exists, but we have to give explicit formulas to make computations. Therefore, the final choice will be an approximation of this domain, obtained by composing the operations Slits and Lunes that we define now.

Definition 2.1. (Lune) Fix $c > 1$. Let $L(c) := \mathbb{D} \setminus \mathbb{D} \cap D\left(-\frac{c^2+1}{c^2-1}, \frac{2c}{c^2-1}\right)$ be the lune of parameter c . The application

$$\text{Lune}(z, c) := \frac{z(1+c^2) - 1 - c^2 + \sqrt{(1+c^2)^2(1+z^2)^2 - 16c^2z}}{2(c^2-1)}$$

is a conformal isomorphism from the disk $\mathbb{D}$ to the lune $L(c)$ of conformal radius $\frac{c^2-1}{c^2+1}$.

Definition 2.2. (Slit) For $0 < r < 1$, we have a conformal isomorphism between the disk $\mathbb{D}$ and the Slit $\mathbb{D} \setminus]-1; -r]$ given by

$$\begin{aligned} \text{Slit}(z, r) := & \frac{(r+1)^2 - 2(r-1)^2 z + (r+1)^2 z^2}{8rz} \\ & + \frac{(1+r)(-1+z)\sqrt{(1+r)^2 - 2(1-6r+r^2)z + (1+r)^2 z^2}}{8rz} . \end{aligned}$$

The conformal radius of the Slit is $\frac{4r}{(1+r)^2}$.

Define the map from $\mathbb{D}$ to Ω as

$$G(z) := -R \cdot \text{Lune}(z_1 \cdot \text{Slit}(z_2 \cdot \text{Slit}(z_3 \cdot \text{Slit}(z_4 \cdot \text{Slit}(z, r_1), r_2), r_3), r_4), c)$$

where $R = 77/100$, the parameters z_i are in T to rotate the circle to remove the slits, the radius r_i are chosen so that $-r_i$ is near the i^{th} preimage, and the lune parameter c is chosen to remove the horoball near 1.

As it is not possible to determine with perfect precision the 4 preimages of $-1/72$, we define

$$\psi(z) := G\left(\frac{995}{1000} \cdot z\right)$$

as the final choice of contour so that this dilatation transforms the slits into "open" slits.

Proposition 2.3. *The constructed map $\varphi := h \circ \psi$ satisfies the conditions (1) and (2) of the introduction and we have the estimate with this choice of φ :*

$$\frac{\iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z) \mu(w)}{\log |256\psi'(0)| - \tau} = 13,9938... < 14 .$$

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