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DANIELE BARTOLI
ANTONIO COSSIDENTE
GIUSEPPE MARINO
FRANCESCO PAVESE

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
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Line spreads of $\mathcal{W}(2n-1,q)$, q even

Daniele Bartoli[1] Antonio Cossidente[2] Giuseppe Marino[3] Francesco Pavese[4]

[1]Department of Mathematics and Computer Science, University of Perugia, Perugia, Italy.
daniele.bartoli@unipg.it
[2]Department of Engineering, University of Basilicata, Potenza, Italy.
antonio.cossidente@unibas.it
[3]Department of Mathematics and Applications "Renato Caccioppoli", University of Naples Federico II, Naples, Italy.
giuseppe.marino@unina.it
[4]Department of Mechanics, Mathematics and Management, Politecnico di Bari, Bari, Italy.
francesco.pavese@poliba.it

Abstract

Let q be an even prime power and $n \geq 2$. We construct line spreads of the symplectic polar space $\mathcal{W}(2n-1,q)$ by replacing pairwise disjoint reguli in a Desarguesian line spread. The construction is based on partitions of the points of $\mathrm{PG}(n-1,q^2)$ into points of a nondegenerate Hermitian variety and Baer sublines meeting that variety in exactly one point. In the plane case, the partition is invariant under a cyclic group of order q^2-q+1.

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1 Introduction

A line spread of a finite projective space is a set of pairwise disjoint lines partitioning its point set. A line spread of a symplectic polar space is required, in addition, to consist of totally isotropic lines. Spreads and their interaction with classical polar spaces are treated in [5, 6]; for the rank-two case, see also [8].

The general theory of ovoids and spreads in finite classical polar spaces provides a broader context for this problem [3]. Field reduction also gives a direct construction in certain dimensions: when n is even, a symplectic space of vector dimension n over $\mathbb{F}_{q^2}$ yields a line spread of $\mathcal{W}(2n-1,q)$ over $\mathbb{F}_q$. The case $n=3$ does not follow from this construction. Hering's line spreads of $\mathcal{W}(5,3)$ [2] are relevant small-field examples. De Winter and Metsch [1, Remark 8] recall these examples and observe that relatively few line spreads of $\mathcal{W}(5,q)$ were known in the literature. This provides motivation for a uniform construction.

Let q be even, let $n \geq 2$, and let $\mathcal{H}(n-1,q^2)$ be a non-degenerate Hermitian variety of $\mathrm{PG}(n-1,q^2)$. Put $\varepsilon = (-1)^n$. In this paper, we show that the points of $\mathrm{PG}(n-1,q^2)$ can be partitioned into

$$p_n = \frac{(q^{n-2}+\varepsilon)(q^n-\varepsilon)}{q^2-1}$$

points of $\mathcal{H}(n-1,q^2)$ and

$$b_n = \frac{q^{n-2}(q^n-\varepsilon)}{q+1}$$

Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point, where $\varepsilon = (-1)^n$. Under field reduction from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$, the points of $\mathrm{PG}(n-1,q^2)$ correspond to the lines of a Desarguesian

spread of $\mathrm{PG}(2n-1,q)$ [7], while the points of a Baer subline correspond to a regulus. As shown in Section 2, replacing the corresponding b_n reguli by their opposite reguli gives a line spread of $\mathcal{W}(2n-1,q)$. This spread contains p_n lines on the associated quadric $\mathcal{Q}^{\varepsilon}(2n-1,q)$ and $(q+1)b_n$ lines tangent to it.

There is also a connection with intriguing sets in dual polar graphs. In the dual polar graph of $\mathcal{W}(5,q)$, vertices are totally isotropic planes, with two vertices adjacent when the planes intersect in a line. For a line spread, take the planes containing at least one spread line. Each selected plane has q^2+q selected neighbours, while each unselected plane has q^2+q+1 selected neighbours. Consequently, this set defines a perfect two-colouring with nontrivial eigenvalue -1. In particular, the rank-three spreads constructed here yield such colourings for every even q; see [1, Example 7] for this graph construction.

Section 2 presents the geometric setting. Section 3 supplies the required partition and completes the construction. In particular, the plane case uses a cyclic model of the Hermitian curve, for which background can be found in [4].

2 Quadrics and regulus replacement

Let q be an even prime power, let $n \geq 2$, and let $\mathcal{W}(2n-1,q)$ be a symplectic polar space of $\mathrm{PG}(2n-1,q)$. Denote by $\mathcal{Q}^{\varepsilon}$ a nondegenerate quadric of $\mathrm{PG}(2n-1,q)$, with $\varepsilon=+1$ if the quadric is hyperbolic and $\varepsilon=-1$ if it is elliptic, and assume that its associated polar form defines $\mathcal{W}(2n-1,q)$. A line of $\mathrm{PG}(2n-1,q)$ is a line of $\mathcal{W}(2n-1,q)$ if and only if it is either tangent to or contained in $\mathcal{Q}^{\varepsilon}$, where a tangent line meets $\mathcal{Q}^{\varepsilon}$ in exactly one point.

Lemma 2.1. *Let $\mathcal{L}$ be a line spread of $\mathcal{W}(2n-1,q)$, and let $\mathcal{Q}^{\varepsilon}$ be a nondegenerate quadric of $\mathrm{PG}(2n-1,q)$ whose associated polar form defines $\mathcal{W}(2n-1,q)$. The numbers of lines of $\mathcal{L}$ contained in and tangent to $\mathcal{Q}^{\varepsilon}$ are*

$$\frac{(q^{n-2}+\varepsilon)(q^n-\varepsilon)}{q^2-1} \qquad \text{and} \qquad q^{n-2}(q^n-\varepsilon),$$

respectively.

Proof. Let x and y denote the numbers of lines of $\mathcal{L}$ contained in and tangent to $\mathcal{Q}^{\varepsilon}$, respectively. Since $\mathcal{L}$ partitions the points of $\mathrm{PG}(2n-1,q)$,

$$x+y=|\mathcal{L}|=\frac{q^{2n}-1}{q^2-1}.$$

On the other hand, counting the points of $\mathcal{Q}^{\varepsilon}$ on the lines of $\mathcal{L}$ gives

$$(q+1)x+y=|\mathcal{Q}^{\varepsilon}|=\frac{(q^{n-1}+\varepsilon)(q^n-\varepsilon)}{q-1}.$$

Subtracting these two identities yields

$$x=\frac{|\mathcal{Q}^{\varepsilon}|-|\mathcal{L}|}{q}=\frac{(q^{n-2}+\varepsilon)(q^n-\varepsilon)}{q^2-1}.$$

Consequently,

$$y=|\mathcal{L}|-x=q^{n-2}(q^n-\varepsilon),$$

as required. □

Embed $\mathrm{PG}(2n-1,q)$ as a Baer subgeometry Σ of $\mathrm{PG}(2n-1,q^2)$, and let σ be the Baer involution of $\mathrm{PG}(2n-1,q^2)$ fixing Σ pointwise. Let Π be an $(n-1)$-dimensional projective subspace of $\mathrm{PG}(2n-1,q^2)$ such that

$$\Pi \cap \Sigma = \varnothing \qquad \text{and} \qquad \Pi \cap \Pi^\sigma = \varnothing.$$

Then Π and Π^σ span $\mathrm{PG}(2n-1,q^2)$. For $P \in \Pi$, put $\ell_P = \langle P, P^\sigma \rangle_{q^2} \cap \Sigma$. The lines ℓ_P, $P \in \Pi$, form the Desarguesian spread $\mathcal{S}$. For each line ℓ of $\mathcal{W}(2n-1,q)$, let $\overline{\ell}$ denote its extension to $\mathrm{PG}(2n-1,q^2)$. Assume further that the point set

$$\bigcup_{\ell \text{ a line of } \mathcal{W}(2n-1,q)} (\overline{\ell} \cap \Pi)$$

is a nondegenerate Hermitian variety $\mathcal{H}(n-1,q^2)$ of Π. The lines of $\mathcal{S}$ corresponding to the points of $\mathcal{H}(n-1,q^2)$ partition a nondegenerate quadric $\mathcal{Q}^\varepsilon$ of Σ, with $\varepsilon = (-1)^n$; thus this quadric is elliptic, $\mathcal{Q}^-$, if n is odd and hyperbolic, $\mathcal{Q}^+$, if n is even.

Theorem 2.2. *Let $\mathcal{A} \subseteq \mathcal{H}(n-1,q^2)$ be a set of*

$$\frac{(q^{n-2}+(-1)^n)(q^n-(-1)^n)}{q^2-1}$$

points, and let $\mathcal{B}$ be a collection of

$$\frac{q^{n-2}(q^n-(-1)^n)}{q+1}$$

Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point, such that $\mathcal{A}$ and the members of $\mathcal{B}$ partition the points of Π. Then this partition gives rise to a line spread of $\mathcal{W}(2n-1,q)$.

Proof. Use the Desarguesian spread $\mathcal{S}$ defined above. For each Baer subline $b \in \mathcal{B}$, the set $\mathcal{R}_b = \{\ell_P : P \in b\}$ is a regulus in a solid of Σ. Let $\mathcal{R}'_b$ be its opposite regulus. The two reguli cover exactly the same points. Since the members of $\mathcal{B}$ are pairwise disjoint, the point sets covered by their reguli are pairwise disjoint. Hence

$$\mathcal{L} = \{\ell_P : P \in \mathcal{A}\} \cup \left(\bigcup_{b \in \mathcal{B}} \mathcal{R}'_b \right)$$

is a line spread of Σ.

For $P \in \mathcal{A}$, the line ℓ_P is contained in $\mathcal{Q}^\varepsilon$ and therefore belongs to $\mathcal{W}(2n-1,q)$. If $b \cap \mathcal{H}(n-1,q^2) = \{P_b\}$, the points covered by $\mathcal{R}_b$ meet $\mathcal{Q}^\varepsilon$ precisely in ℓ_{P_b}. Every line of $\mathcal{R}'_b$ meets ℓ_{P_b} in one point and thus meets $\mathcal{Q}^\varepsilon$ in exactly one point. Consequently, every line of $\mathcal{R}'_b$ also belongs to $\mathcal{W}(2n-1,q)$, proving the assertion. □

3 A partition of $\mathrm{PG}(n-1,q^2)$

3.1 A partition of the projective line

We begin with a partition of the projective line into two points of $\mathcal{H}(1,q^2)$ and $q-1$ Baer sublines that have exactly one point in common with $\mathcal{H}(1,q^2)$.

Theorem 3.1. *The points of $\mathrm{PG}(1,q^2)$ can be partitioned into two points of $\mathcal{H}(1,q^2)$ and $q-1$ Baer sublines, each meeting $\mathcal{H}(1,q^2)$ in exactly one point.*

Proof. In the elliptic-quadric model of the inversive plane (see [5]), the points of $\mathrm{PG}(1,q^2)$ correspond to the points of $\mathcal{Q}^-(3,q)$, and Baer sublines correspond to its plane conic sections. In particular, the $q+1$ points of $\mathcal{H}(1,q^2)$ correspond to the $q+1$ points of a conic C of the elliptic quadric $\mathcal{Q}^-(3,q)$. Let P_1, P_2 be two fixed points of C, and let π_i be the plane tangent to $\mathcal{Q}^-(3,q)$ at P_i, for $i=1,2$. Then the line $r=\pi_1\cap\pi_2$ is external to $\mathcal{Q}^-(3,q)$ and passes through the nucleus N of C. The remaining $q-1$ planes through r meet $\mathcal{Q}^-(3,q)$ in conics $C_1,\dots,C_{q-1}$. The plane of C does not contain r. Each plane of a conic C_i therefore meets the plane of C in a line through N, which meets C in exactly one point. Thus $|C_i\cap C|=1$ for $i=1,\dots,q-1$. The plane sections in this pencil partition the elliptic quadric into P_1, P_2 and $C_1,\dots,C_{q-1}$, giving the required partition. The two points may be prescribed arbitrarily on $\mathcal{H}(1,q^2)$. □

3.2 A cyclic partition of the projective plane

We next give a cyclic construction of an analogous partition of the projective plane.

Let q be an even prime power. We use the cyclic model

$$\mathcal{P}=\{P_x=(x,x^{q^2},x^{q^4}) : x\in\mathbb{F}_{q^6}^*\}\cong \mathrm{PG}(2,q^2),$$

where $P_x=P_y$ if and only if $y/x\in\mathbb{F}_{q^2}^*$. The auxiliary plane is

$$\pi=\{p_y=(y,y^{q^2},y^q) : y\in\mathbb{F}_{q^3}^*\}\cong \mathrm{PG}(2,q),$$

with $p_y=p_z$ if and only if $z/y\in\mathbb{F}_q^*$. The norm $N(x)=x^{q^3+1}$ induces the map

$$\begin{aligned}\nu:\mathcal{P}&\longrightarrow\pi,\\ P_x&\longmapsto p_{N(x)}=\left(x^{q^3+1},x^{q^5+q^2},x^{q^4+q}\right).\end{aligned}$$

We take

$$\mathcal{H}(2,q^2)=\{P_x : \mathrm{Tr}_{q^3|q}(N(x))=0\}.$$

This is a nondegenerate Hermitian curve of $\mathcal{P}$: its defining Hermitian form is given by $\mathrm{Tr}_{q^6|q^2}(xy^{q^3})$, where Tr denotes the trace function. For background on cyclic models of Hermitian curves, see [4]. Thus $\mathcal{H}(2,q^2)=\nu^{-1}(h)$, where

$$h=\{p_y : \mathrm{Tr}_{q^3|q}(y)=0\}$$

is a line of π.

For $c\in\mathbb{F}_{q^6}^*$ with $N(c)=1$, multiplication by c induces the projectivity

$$g_c : P_x\longmapsto P_{cx},\qquad (x,x^{q^2},x^{q^4})\longmapsto(cx,c^{q^2}x^{q^2},c^{q^4}x^{q^4}).$$

Since multiplication by c is an invertible $\mathbb{F}_{q^2}$-linear map on $\mathbb{F}_{q^6}$, the map g_c fixes $\mathcal{P}$ setwise. The norm-one elements form a cyclic group of order q^3+1, whose kernel on $\mathcal{P}$ consists of the $q+1$ elements of $\mathbb{F}_{q^2}^*$ having norm one. Thus the induced group is

$$G=\{c\in\mathbb{F}_{q^6}^* : N(c)=1\}/\{c\in\mathbb{F}_{q^2}^* : N(c)=1\},\qquad |G|=q^2-q+1.$$

Moreover, $N(cx)=N(c)N(x)=N(x)$, and hence

$$\mathrm{Tr}_{q^3|q}(N(cx))=\mathrm{Tr}_{q^3|q}(N(x)).$$

Consequently, G also fixes $\mathcal{H}(2,q^2)$ setwise and satisfies $\nu\circ g=\nu$ for every $g\in G$.

The group G acts semiregularly on $\mathcal{P}$. Indeed, if $g_c(P_x)=P_x$, then $c\in\mathbb{F}_{q^2}^*$, so g_c is the identity.

For every $p_y\in\pi$, the set $\nu^{-1}(p_y)$ is a G-orbit of size q^2-q+1. Indeed, if $\nu(P_x)=\nu(P_z)$, then $N(z)/N(x)\in\mathbb{F}_q^*$. As squaring permutes $\mathbb{F}_q^*$, one can choose $\lambda\in\mathbb{F}_q^*$ such that $N(\lambda z)=N(x)$. The element $c=\lambda z/x$ then has norm one and satisfies $g_c(P_x)=P_z$.

Theorem 3.2. *The plane $\mathcal{P}$ admits a partition into $q(q^2-q+1)$ Baer sublines having one point in common with $\mathcal{H}(2,q^2)$ and q^2-q+1 points of $\mathcal{H}(2,q^2)$, invariant under G.*

Proof. Choose $a, b \in \mathbb{F}_{q^3}$ such that

$$a \neq 0, \qquad \mathrm{Tr}_{q^3|q}(a) = 0, \qquad \mathrm{Tr}_{q^3|q}(b) = \mathrm{Tr}_{q^3|q}(ab) = 1,$$

and choose $\delta \in \mathbb{F}_q$ satisfying

$$\mathrm{Tr}_{q|2}(\delta) = 1.$$

Such a, b exist: a nonzero trace-zero element a does not belong to $\mathbb{F}_q$, and the $\mathbb{F}_q$-linear map

$$b \in \mathbb{F}_{q^3} \longmapsto \big(\mathrm{Tr}_{q^3|q}(b), \mathrm{Tr}_{q^3|q}(ab)\big) \in \mathbb{F}_q^2$$

is surjective, since the linear forms $b \mapsto \mathrm{Tr}_{q^3|q}(b)$ and $b \mapsto \mathrm{Tr}_{q^3|q}(ab)$ are linearly independent over $\mathbb{F}_q$.

The elements a^2, ab, a form an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^3}$. Indeed, a relation $\alpha a + \beta b + \gamma = 0$, with $\alpha, \beta, \gamma \in \mathbb{F}_q$, first gives $\beta + \gamma = 0$ upon taking traces, and then $\beta = 0$ upon multiplying by a and taking traces; hence all coefficients vanish. In the corresponding coordinates

$$p_{a^2X+abY+aZ} \longleftrightarrow (X, Y, Z)$$

of π, we have $h : Y = 0$. Set

$$p_a = (0, 0, 1), \qquad \ell : Z = 0,$$

and, for $s \in \mathbb{F}_q^*$, let C_s be the non-degenerate conic of π given by

$$X^2 + XY + \delta^2 Y^2 = s^2 Z^2.$$

These $q-1$ conics are pairwise disjoint, have common nucleus p_a, and each meets h in exactly one point. They are also disjoint from ℓ. Hence

$$\{p_a\} \ \cup \ \ell \ \cup \ \left(\bigcup_{s \in \mathbb{F}_q^*} C_s \right)$$

is a partition of π.

Let B_∞ be the Baer subline of $\mathcal{P}$ given by

$$B_\infty = \{P_{ra+t\sqrt{ab}} : (r, t) \in \mathrm{PG}(1, q)\}.$$

The defining vectors are independent over $\mathbb{F}_{q^2}$, since $\mathbb{F}_{q^3} \cap \mathbb{F}_{q^2} = \mathbb{F}_q$ and a^2, ab are independent over $\mathbb{F}_q$. As they belong to $\mathbb{F}_{q^3}$,

$$N(ra + t\sqrt{ab}) = a^2 r^2 + ab\, t^2, \qquad r, t \in \mathbb{F}_q,$$

so ν maps B_∞ bijectively onto ℓ.

For $s \in \mathbb{F}_q^*$, the polynomial

$$f(Z) = a(1 + sa)Z^2 + aZ + a(\delta + sb)$$

is irreducible over $\mathbb{F}_{q^3}$. Indeed,

$$\begin{aligned} \mathrm{Tr}_{q^3|2}\big((1 + sa)(\delta + sb)\big) &= \mathrm{Tr}_{q|2}\Big(\mathrm{Tr}_{q^3|q}\big((1 + sa)(\delta + sb)\big)\Big) \\ &= \mathrm{Tr}_{q|2}(\delta + s + s^2) \\ &= 1. \end{aligned}$$

Choose $z_s \in \mathbb{F}_{q^6} \setminus \mathbb{F}_{q^3}$ such that $f(z_s) = 0$. Then $z_s + z_s^{q^3} = 1/(1+sa)$ and $z_s^{q^3+1} = (\delta+sb)/(1+sa)$. Set $u_s \in \mathbb{F}_{q^6}^*$ with $N(u_s) = a(1+sa)$, and $v_s = u_s z_s$. Then

$$\begin{aligned} N(ru_s + tv_s) &= a(1+sa)r^2 + art + a(\delta+sb)t^2 \\ &= a^2 sr^2 + ab\, st^2 + a(r^2 + rt + \delta t^2), \qquad r,t \in \mathbb{F}_q. \end{aligned}$$

The vectors u_s, v_s are independent over $\mathbb{F}_{q^2}$, otherwise $z_s \in \mathbb{F}_{q^2}$, implying $z_s + z_s^{q^3} = z_s + z_s^q = 1/(1+sa) \in \mathbb{F}_q$, which is impossible since $a \notin \mathbb{F}_q$. Hence

$$B_s = \{P_{ru_s+tv_s} : (r,t) \in \mathrm{PG}(1,q)\}$$

is a Baer subline. In the chosen coordinates of π, we have

$$\nu(B_s) = \{(sr^2, st^2, r^2 + rt + \delta t^2) : (r,t) \in \mathrm{PG}(1,q)\} = C_s,$$

and this correspondence is bijective.

Whenever ν maps a subline B bijectively onto a set D, the action of G on each set $\nu^{-1}(p_y)$, with $p_y \in D$, yields

$$\nu^{-1}(D) = \bigcup_{g \in G} g(B).$$

Indeed, B meets $\nu^{-1}(p_y)$ in exactly one point for each $p_y \in D$, and its G-translates partition $\nu^{-1}(D)$. To see disjointness explicitly, if $g(P) = g'(P')$ with $P, P' \in B$, applying ν gives $P = P'$ by injectivity on B; semiregularity then gives $g = g'$.

Let

$$S = \nu^{-1}(p_a) = \{P_x : N(x)/a \in \mathbb{F}_q^\star\}.$$

Since $\mathrm{Tr}_{q^3|q}(a) = 0$, $S \subseteq \mathcal{H}(2,q^2)$, and $|S| = q^2 - q + 1$. The required partition is

$$S \;\cup\; \{g(B_\infty) : g \in G\} \;\cup\; \{g(B_s) : s \in \mathbb{F}_q^*,\ g \in G\}. \tag{3.1}$$

Disjointness and completeness follow from the partition of π given by $\{p_a\} \cup \ell \cup \left(\bigcup_{s \in \mathbb{F}_q^*} C_s\right)$ and the preceding observation.

Finally,

$$\mathrm{Tr}_{q^3|q}(N(ra + t\sqrt{ab})) = t^2, \qquad \mathrm{Tr}_{q^3|q}(N(ru_s + tv_s)) = s\,t^2.$$

Thus every initial subline meets $\mathcal{H}(2,q^2)$ exactly at the point corresponding to $(r,t) = (1,0)$. Since G preserves $\mathcal{H}(2,q^2)$, every subline in (3.1) is unisecant to $\mathcal{H}(2,q^2)$. There are $q(q^2-q+1)$ such sublines and $q^2 - q + 1$ points, as asserted. □

3.3 Higher dimensions

Here we extend the preceding constructions to higher dimensions.

In $\mathrm{PG}(2,q^2)$, let ℓ be a line and let P be a point of $\mathrm{PG}(2,q^2)$, with $P \notin \ell$. Let $\mathcal{H}(1,q^2)$ be a Hermitian variety of ℓ, and let ℓ_0 be a Baer subline of ℓ such that $\ell_0 \cap \mathcal{H}(1,q^2) = \{Q\}$. Denote by $\mathcal{C}_S$ the cone with vertex P and base the point set $S \subseteq \ell$.

Lemma 3.3. *The cone $\mathcal{C}_{\ell_0} \setminus \{P\}$ can be partitioned into q^2 Baer sublines. Every subline in the partition meets $\mathcal{C}_{\mathcal{H}(1,q^2)}$ in exactly one point.*

Proof. Choose $R \in \ell \setminus \ell_0$, and let $\mathcal{T}_R$ be the pencil of lines through R with PR omitted. It has q^2 members, none containing P. For $t \in \mathcal{T}_R$, put $\mathcal{B}_t = \{PX \cap t : X \in \ell_0\}$. The set $\{\mathcal{B}_t : t \in \mathcal{T}_R\}$ consists of q^2 pairwise disjoint Baer sublines partitioning $\mathcal{C}_{\ell_0} \setminus \{P\}$. Finally, $\ell_0 \cap \mathcal{H}(1,q^2) = \{Q\}$ implies $\mathcal{C}_{\ell_0} \cap \mathcal{C}_{\mathcal{H}(1,q^2)} = PQ$. Each $\mathcal{B}_t$ meets the generator PQ exactly once, and hence $\mathcal{B}_t \cap \mathcal{C}_{\mathcal{H}(1,q^2)} = \{PQ \cap t\}$. □

Theorem 3.4. *Let q be even and $n \geq 4$. Let $\mathcal{H}(n-1,q^2)$ be a non-degenerate Hermitian variety of $\Pi = \mathrm{PG}(n-1,q^2)$ and let $\mathcal{H}(n-3,q^2)$ be a non-degenerate Hermitian variety of $\mathrm{PG}(n-3,q^2)$. If $\mathrm{PG}(n-3,q^2)$ has a partition into*

$$\frac{(q^{n-4}+(-1)^n)(q^{n-2}-(-1)^n)}{q^2-1}$$

points of $\mathcal{H}(n-3,q^2)$ and

$$\frac{q^{n-4}(q^{n-2}-(-1)^n)}{q+1}$$

Baer sublines, each meeting $\mathcal{H}(n-3,q^2)$ in exactly one point, then Π has a partition into

$$\frac{(q^{n-2}+(-1)^n)(q^n-(-1)^n)}{q^2-1}$$

points of $\mathcal{H}(n-1,q^2)$ and

$$\frac{q^{n-2}(q^n-(-1)^n)}{q+1}$$

Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point.

Proof. Choose $P \in \mathcal{H}(n-1,q^2)$ and let T be the tangent hyperplane to $\mathcal{H}(n-1,q^2)$ at P. Choose a subspace $S \cong \mathrm{PG}(n-3,q^2)$ such that $T = \langle P,S\rangle$ and $\mathcal{H}(n-1,q^2)\cap T$ is the cone with vertex P and non-degenerate base $\mathcal{H}(n-1,q^2)\cap S$, which we identify with $\mathcal{H}(n-3,q^2)$. By hypothesis S has a partition into a subset $\mathcal{A}_0$ of $(q^{n-4}+(-1)^n)(q^{n-2}-(-1)^n)/(q^2-1)$ points of $\mathcal{H}(n-3,q^2)$ and a set $\mathcal{B}_0$ of $q^{n-4}(q^{n-2}-(-1)^n)/(q+1)$ Baer sublines, each meeting $\mathcal{H}(n-3,q^2)$ in exactly one point. We want to show the existence of a partition of Π consisting of a subset $\mathcal{A}$ of $(q^{n-2}+(-1)^n)(q^n-(-1)^n)/(q^2-1)$ points of $\mathcal{H}(n-1,q^2)$ and a set $\mathcal{B}$ of $q^{n-2}(q^n-(-1)^n)/(q+1)$ Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point.

For each $Q \in \mathcal{A}_0$, include all the q^2+1 points of the line PQ in $\mathcal{A}$. For each Baer subline $\ell_0 \in \mathcal{B}_0$, by Lemma 3.3, the cone $\mathcal{C}_{\ell_0}\setminus\{P\}$ can be partitioned into q^2 Baer sublines, each of them meeting $\mathcal{H}(n-1,q^2)$ in exactly one point. For each $\ell_0 \in \mathcal{B}_0$, include these q^2 Baer sublines in $\mathcal{B}$. Thus T is partitioned into $1+q^2|\mathcal{A}_0|$ points of $\mathcal{H}(n-1,q^2)$ and $q^2|\mathcal{B}_0|$ Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point.

For each line ℓ through P not contained in T, the line is secant to $\mathcal{H}(n-1,q^2)$, and $\ell\cap\mathcal{H}(n-1,q^2)$ is a Baer subline containing P. Choose $R_\ell \in (\ell\cap\mathcal{H}(n-1,q^2))\setminus\{P\}$. By Theorem 3.1, there is a partition of ℓ into the two points P, R_ℓ of $\mathcal{H}(n-1,q^2)$ and $q-1$ Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ in exactly one point. Include R_ℓ in $\mathcal{A}$ and these $q-1$ Baer sublines in $\mathcal{B}$. Distinct lines through P not contained in T meet only at P, and every point of $\Pi\setminus T$ belongs to exactly one such line. Thus these partitions are mutually disjoint and cover the complement of T.

There are q^{2n-2} points outside T, and each line through P not contained in T has q^2 such points. Hence there are q^{2n-4} such lines. Each contributes with one point of $\mathcal{A}$ and $q-1$ Baer sublines. Adding these contributions to the partition of T gives

$$|\mathcal{A}| = 1+q^2|\mathcal{A}_0|+q^{2n-4}, \qquad |\mathcal{B}| = q^2|\mathcal{B}_0|+(q-1)q^{2n-4}.$$

Substituting the values of $|\mathcal{A}_0|$ and $|\mathcal{B}_0|$ yields the stated formulas. □

The preceding extension theorem now yields the partition in every projective dimension.

Theorem 3.5. *Let q be even and $n \geq 2$, and let $\mathcal{H}(n-1,q^2)$ be a non-degenerate Hermitian variety in $\mathrm{PG}(n-1,q^2)$. Put $\varepsilon = (-1)^n$. Then there exists a partition of $\mathrm{PG}(n-1,q^2)$ into*

$$\frac{(q^{n-2}+\varepsilon)(q^n-\varepsilon)}{q^2-1}$$

points of $\mathcal{H}(n-1,q^2)$ *and*

$$\frac{q^{n-2}(q^n-\varepsilon)}{q+1}$$

Baer sublines, each meeting $\mathcal{H}(n-1,q^2)$ *in exactly one point.*

Proof. By induction on n, separately for even and odd n, the result follows. Theorem 3.1 provides the case $n=2$, with two points and $q-1$ Baer sublines; Theorem 3.2 provides the case $n=3$, with q^2-q+1 points and $q(q^2-q+1)$ Baer sublines. For $n\geq 4$, apply Theorem 3.4 to the partition of $\mathrm{PG}(n-3,q^2)$. This completes the induction. □

Combining Theorem 3.5 and Theorem 2.2, a line spread of $\mathcal{W}(2n-1,q)$ for every even prime power q is obtained.

Corollary 3.6. *For every even prime power* q *and every* $n\geq 2$*, the symplectic polar space* $\mathcal{W}(2n-1,q)$ *admits a line spread obtained by simultaneous opposite-regulus replacement in a Desarguesian line spread.*

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References

[1] S. De Winter and K. Metsch, *Perfect 2-colorings of the Grassmann graph of planes*, Electron. J. Combin. **27** (2020), Paper P1.21. doi:10.37236/8672.

[2] C. Hering, *Two new sporadic doubly transitive linear spaces*, in Finite Geometries (Winnipeg, 1984), Lecture Notes in Pure and Applied Mathematics, vol. 103, Dekker, New York, 1985, pp. 127–129.

[3] J. A. Thas, *Ovoids and spreads of finite classical polar spaces*, Geom. Dedicata **10** (1981), 135–143. doi:10.1007/BF01447417.

[4] S. Barwick and G. Ebert, *Unitals in Projective Planes*, Springer Monographs in Mathematics, Springer, New York, 2008. doi:10.1007/978-0-387-76366-8.

[5] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1985.

[6] J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries*, second edition, Springer Monographs in Mathematics, Springer, London, 2016. doi:10.1007/978-1-4471-6790-7.

[7] M. Lavrauw and G. Van de Voorde, *Field reduction and linear sets in finite geometry*, in Topics in Finite Fields, Contemporary Mathematics, vol. 632, American Mathematical Society, 2015.

[8] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, second edition, EMS Series of Lectures in Mathematics, European Mathematical Society, Zürich, 2009. doi:10.4171/066.