Snapshots of modern mathematics
from Oberwolfach

№ 5/2016

Symmetry and characters of finite groups

Eugenio Giannelli • Jay Taylor

Over the last two centuries mathematicians have developed an elegant abstract framework to study the natural idea of symmetry. The aim of this snapshot is to gently guide the interested reader through these ideas. In particular, we introduce finite groups and their representations and try to indicate their central role in understanding symmetry.

1 Symmetry

Everyday our brains use symmetry when processing and understanding the natural world around us; often without us even noticing. For example, studies have shown that we are more likely to find a person attractive if their facial features are symmetric [11]. In science, symmetry plays an active role for many reasons. One reason in particular is that the existence of symmetry can be used to simplify problems.

Consider the problem of counting all the red dots in Figure 1. If we observe that the diagram has a reflection symmetry in the dashed line then our work is cut in half! Indeed, with this symmetry we know that the total number of red dots in Figure 1 is precisely twice the number of red dots lying to the left, or right, of the dashed line.

There are, in fact, two other notable symmetries contained in Figure 1; can you find them[1]?

[1] Hint: It suffices to count 29 of the red dots to determine the total number.

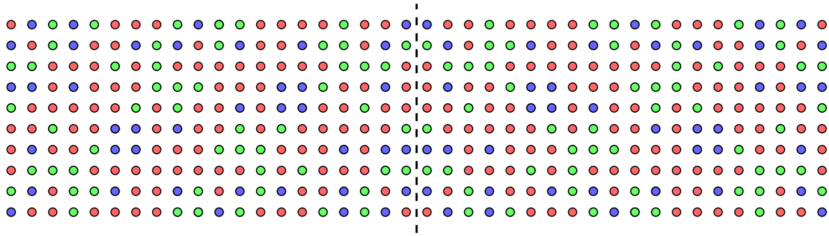


Figure 1: Coloured dots with a reflection symmetry.

This idea may seem simple but it can be extremely powerful. For instance, in chemistry, similar applications of symmetry are used to count the total number of chemical compounds that can be made from a given collection of elements [9]. Recent research in biology has shown that symmetry can be used to predict and model the RNA structure of emerging viruses [10] and in particle physics, symmetry forms the key framework for the standard model [1]. For the fledgling scientist, one thing is abundantly clear: never underestimate symmetry!

2 Group theory

At the beginning of the 19th century the French mathematician Évariste Galois[2] introduced a set of ideas which paved the way for the systematic study of symmetry in a rigorous mathematical framework. The branch of mathematics born from the work of Galois, and developed immediately afterwards by Cauchy, is now known as *group theory*.

One of the most important achievements of this new mathematical language was the introduction of the concept of an *indivisible component* of a symmetric object. The ancient Greeks were aware that every natural number can be expressed as the product of some smaller indivisible constituents, the prime numbers. Group theory showed that, analogous to the natural numbers, every symmetric object (*group*) can be decomposed into smaller *indivisible* symmetric objects (*simple groups*).

Let us consider an example. Recall that a regular polygon is a shape, all of whose sides have the same length and all of whose interior angles are equal. Now consider the regular polygon with 15 sides[3] in Figure 2. It is easy to check that every rotation symmetry of this polygon can be obtained by combining rotation symmetries of the blue triangle and red pentagon which it contains. Exactly as 3

[2] Who sadly only lived until the age of 20 (1811–1832).

[3] This is called a pendedecagon.

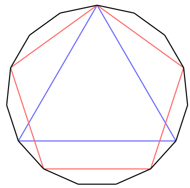


Figure 2: A regular pendedecagon, pentagon and triangle.

and 5 are prime numbers, the rotations of the triangle and pentagon correspond to distinct simple groups of symmetries. As soon as it became clear that every group was obtained by gluing together distinct simple groups, mathematicians started dreaming about the possibility of writing down a complete *periodic table* of symmetry, recording **all** simple groups.

The enormous effort and innovative ideas of many researchers in the last century finally led to the completion of the desired Classification of Finite Simple Groups in the early 1980s. The proof is spread over roughly 10,000 pages and is considered by many to be one of the finest achievements of 20th century mathematics. It marks a major milestone towards our goal of trying to understand the initially simple concept of symmetry. However, we are still very far from completing this ambitious goal!

This snapshot will be a guided tour of the main ideas that led mathematicians to such important discoveries. As we will see, these ideas provide us with as many questions as they do answers. The first thing we need to do, before discussing symmetry, is to understand what a group is. To arrive at this we will consider a more familiar idea.

In high school one of the first things we learn about is multiplication; it forms a basic tool of our everyday lives. However, when developing rigorous new ideas mathematicians question everything. For example, they ask questions such as: Is multiplication special to numbers? Can other things be "multiplied"? What should "multiplication" mean for other objects?

The formal answer to these questions is the notion of a *group*, which is a pair $(G, \star)$ consisting of a set G and a "multiplication rule" $\star$ on G such that:

- we have an *identity element*, usually denoted by e, such that $e \star a = a \star e = a$ for all $a \in G$,
- every $g \in G$ has an *inverse*, usually denoted by g^{-1}, satisfying the condition $g \star g^{-1} = g^{-1} \star g = e$,
- the multiplication is *associative*, so that $a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in G$.

This last condition is a natural condition that will be satisfied by all our examples; we will not mention it further in this snapshot.

We explain this by way of example. Let $\mathbb{Q}^\times$ be the set of all rational numbers a/b, with a and b *non-zero* integers, then the pair $(\mathbb{Q}^\times, \times)$ is a group where $\times$ is the usual multiplication rule given by

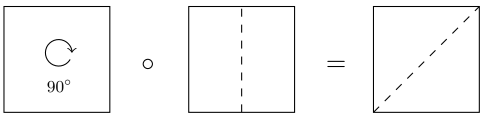
$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

The identity element in this case is $e = 1$, as a shorthand for $1/1$, and the inverse of a/b is $(a/b)^{-1} = b/a$. Note that the multiplication rule does not always have to be multiplication as we think of it. For example, let $\mathbb{Z}$ be the set of all integers, including 0, then the pair $(\mathbb{Z}, +)$ is a group with the multiplication rule given by addition. In this case the identity element is $e = 0$ and the inverse of n is $n^{-1} = -n$.

The groups we have listed are examples of infinite groups, in the sense that they contain infinitely many elements. However, we will be interested in *finite groups.* These are groups $(G, \star)$ such that G contains finitely many elements. In this situation it will be helpful to denote by $|G|$ the number of elements contained in G.

3 Groups of symmetries

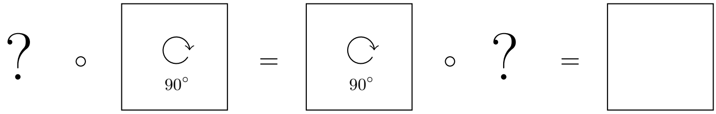
To describe how symmetries fit in to the language of groups let us consider the set D_8 of symmetries of the square, as described in Figure 3. We want to show that this has a group structure, so we need to define a multiplication rule on these objects. However, we have a natural multiplication rule given by *composition.* For example, reading from *right to left* we have



Or, in other words, reflecting in the vertical line then rotating clockwise through 90° is the same as reflecting in the diagonal. As drawing pictures is not very efficient, we label these symmetries σ and τ and denote their composition by $\sigma \circ \tau$. In Figure 3 we have described all the symmetries of the square by composing σ and τ in various ways, where $\sigma^2 = \sigma \circ \sigma$, $\sigma^3 = \sigma \circ \sigma \circ \sigma$, ... and so on. You should check that these are correct and that we did not overlook any.

To show that the pair $(\mathsf{D}_8, \circ)$ is a group we need to know that there is an identity element and that every element has an inverse. The identity element here is denoted by e and is simply the symmetry that does nothing.

Now let us consider inverses. For instance, is there an element satisfying the following equation



The answer is yes, namely the clockwise rotation through 270°. Hence, the inverse of σ is $\sigma^{-1} = \sigma^3$. It is easy to check that every element in Figure 3 has an inverse.

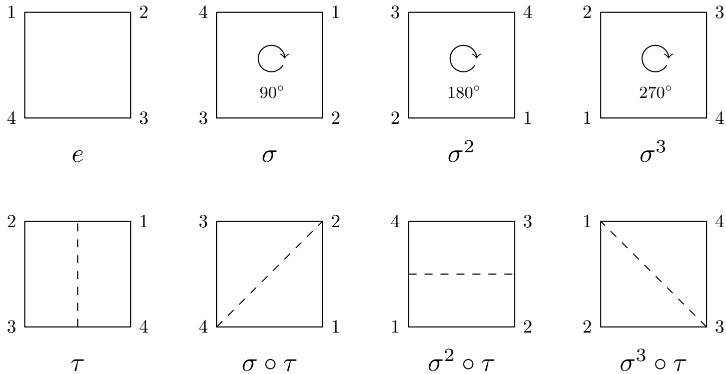


Figure 3: The symmetries of a square.

Let us note here that there is a stark difference between $(\mathsf{D}_8, \circ)$ and the arithmetic examples such as $(\mathbb{Q}^\times, \times)$ we gave in the previous section[4]. Namely, in $\mathbb{Q}^\times$ we have all elements *commute*, in the sense that $a \times b = b \times a$ for all a, $b \in \mathbb{Q}^\times$. However, this is certainly not the case for D_8! For example, in D_8 we have $\tau \circ \sigma = \sigma^3 \circ \tau$ and $\sigma^3 \circ \tau \neq \sigma \circ \tau$, which is easily checked using Figure 3.

The group D_8 fits into an infinite family of finite groups known as the *dihedral groups*. For any integer $n \geqslant 3$ we define D_{2n} to be the symmetries of the regular polygon with n sides; we call D_{2n} the *dihedral group of order* $2n$. The set D_{2n} is again a group with the multiplication rule given by composition. If we define τ to be any reflection symmetry of the n-gon and σ to be the clockwise rotation through $(360/n)^\circ$ then we have

$$\mathsf{D}_{2n} = \{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma \circ \tau, \sigma^2 \circ \tau, \ldots, \sigma^{n-1} \circ \tau\}.$$

The subscript $2n$ denotes that the set D_{2n} contains $2n$ elements.

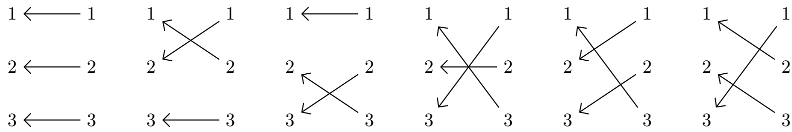
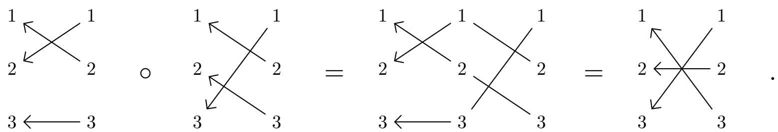


Figure 4: The 6 permutations of $\{1, 2, 3\}$.

4 The symmetric group and permutation characters

The principal example of a finite group is the *symmetric group* $\mathcal{S}_n$. This is the set consisting of all *permutations* $f : \{1, \dots, n\} \to \{1, \dots, n\}$, which are functions such that $f(1), \dots, f(n)$ is simply the list $1, \dots, n$ but in a different order. We may visualise a permutation by writing $1, \dots, n$ as two vertical lists and drawing an arrow $f(i) \leftarrow i$. All the permutations of $\{1, 2, 3\}$ are described in this way in Figure 4.

We can define a multiplication rule $\circ$ on $\mathcal{S}_n$ by composition of functions. In other words, given any two permutations $f, g \in \mathcal{S}_n$ we denote by $g \circ f$ the permutation given by $(g \circ f)(i) = g(f(i))$. In this case, the identity element is the permutation mapping i to i. If we think of permutations as diagrams, as in Figure 4, then this multiplication rule is simply given by concatenating and collapsing diagrams. For example, in $\mathcal{S}_3$ we have

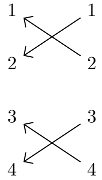


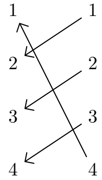
This explains why we draw the arrows from right to left because composition of functions is read from right to left.

In general, an abstract finite group is a very complicated and difficult thing to understand. To get a snapshot of what a finite group looks like we ask in what ways it can act on different objects. For example, we already know that the dihedral group D_8 acts on the whole square via rotations and reflections. However, is this the only thing it can act on?

Well, we could restrict our attention just to the vertices of the square. If we number the vertices of the square, as in Figure 3, then to each element of the dihedral group D_8 we obtain a well-defined permutation of the set $\{1, 2, 3, 4\}$. Hence, we have a well-defined map $\rho : \mathsf{D}_8 \to \mathcal{S}_4$ assigning to each element of D_8 the permutation just described. For example, using the diagrammatic notation of Figure 4 we have

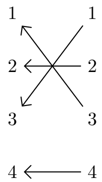
[4] Aside from one being finite and one being infinite.

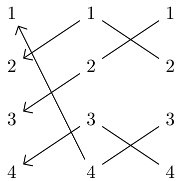
$\rho(\tau) \quad =$ 

and $\rho(\sigma) \quad =$ 

.

What does it mean to say that D_8 *acts* on the vertices of the square as permutations? Intuitively we do not simply mean that each element of D_8 determines a permutation but that it does so in a way which is compatible with the multiplication rule. For instance, using the diagrams in Figure 3 we see that

$\rho(\sigma \circ \tau) \quad =$ 

$=$ 

$= \quad \rho(\sigma)\circ\rho(\tau)$.

In fact, it is easy to see using the diagrams in Figure 3 that we have $\rho(a \circ b) = \rho(a) \circ \rho(b)$ for all $a, b \in \mathsf{D}_8$.

Now assume $(G, \star)$ is any finite group then, inspired by this, we say a map $\rho : G \to \mathcal{S}_n$ is a *permutation representation* of G if

$$\rho(g \star h) = \rho(g) \circ \rho(h) \text{ for all } g, h \in G. \tag{1}$$

In other words, giving a permutation representation is the same as giving an action of G on the set $\{1, \ldots, n\}$ via permutations.

Why do we care about permutation representations? Firstly, they can be used to give us information about our finite group and secondly, from a computational perspective, they are much simpler! In general, given any finite group $(G, \star)$ it is difficult to get a computer to perform calculations with the multiplication rule $\star$. However, people have worked hard to develop very efficient algorithms for computing with permutations in symmetric groups. Many of these are implemented in the freely available computer algebra system GAP [4], which is an indispensable tool in the study of finite groups.

An amazing idea, which was introduced by Frobenius in 1896 [3], is that we can encapsulate most of the information contained in $\rho : G \to S_n$ through an associated function $\chi_\rho : G \to \mathbb{Z}$ called the *permutation character* of ρ. For each $g \in G$ we define $\chi_\rho(g)$ to be the number of integers $i \in \{1, \ldots, n\}$ fixed by $\rho(g)$, that is, the number of integers i for which $\rho(g)(i) = i$ holds. For example, if $\rho : \mathsf{D}_8 \to \mathcal{S}_4$ is the permutation representation given above then the values of the corresponding permutation character are given in Table 1.

	e	σ	σ^2	σ^3	τ	$\sigma \circ \tau$	$\sigma^2 \circ \tau$	$\sigma^3 \circ \tau$
χ_ρ	4	0	0	0	0	2	0	2

Table 1: The values of the permutation character of $\rho : \mathsf{D}_8 \to \mathcal{S}_4$.

We give an example of how this simple numerical function can be used to give information about ρ: The action of G on $\{1, \ldots, n\}$ defined by ρ is called *transitive* if for any $i, j \in \{1, \ldots, n\}$ there exists an element $g \in G$ such that $\rho(g)(i) = j$. It is easily seen that D_8 acts transitively on the vertices of the square. Indeed, try picking any two vertices of the square and use the symmetries described in Figure 3 to get from one to the other; you can always do this! It turns out that the action of a finite group G given by $\rho : G \to \mathcal{S}_n$ is transitive if and only if

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = 1 \tag{2}$$

(this is a special case of what is usually called Burnside's Lemma).

Let us check this in the case of the permutation representation $\rho : \mathsf{D}_8 \to \mathcal{S}_4$ considered above. We already know that the corresponding action is transitive so the number on the left hand side of Equation 2 should be 1. Using the values of χ_ρ given in Table 1 we see that

$$\frac{1}{|\mathsf{D}_8|} \sum_{g \in \mathsf{D}_8} \chi_\rho(g) = \frac{1}{8}(4 + 0 + 0 + 0 + 0 + 2 + 0 + 2) = 1.$$

Using the elements of D_8 try and construct groups which do not act transitively on the vertices of the square. (Hint: These groups should contain 2 elements.) For each such group compute the left hand side of Equation 2 and check it is not 1. Using what you have computed can you guess what the left hand side of Equation 2 counts in general?

5 The character table of a finite group

Looking at the values in Table 1 we see there is a lot of repetition. This is because many of the symmetries of the square are similar. For instance, consider the reflections τ and $\sigma^2 \circ \tau$ in the vertical and horizontal lines. If we allow ourselves to tilt our heads by 90° then these symmetries are essentially the same. A more precise way to say this is that $\tau = \sigma^{-1} \circ (\sigma^2 \circ \tau) \circ \sigma$. In general, if $(G, \star)$ is a finite group then we say $h \in G$ is *conjugate* to $g \in G$ if

$$g = a^{-1} \star h \star a \text{ for some } a \in G.$$

This generalises our intuitive notion of being similar. Using this notion we can break up our group G into smaller disjoint subsets called *conjugacy classes*. These are sets (g) obtained by starting with an element $g \in G$ and computing $a^{-1} \star g \star a$ for all $a \in G$. For example, in D_8 we have the conjugacy classes

$$\{e\}, \quad \{\sigma, \sigma^3\}, \quad \{\sigma^2\}, \quad \{\tau, \sigma^2 \circ \tau\}, \quad \{\sigma \circ \tau, \sigma^3 \circ \tau\}.$$

You should check that these are correct! Looking at Table 1 we see that the permutation character χ_ρ is constant on conjugacy classes, in the sense that $\chi_\rho(g) = \chi_\rho(h)$ whenever $g, h \in G$ are conjugate. In fact, any permutation character of a finite group has this property. Can you see why?

The study of permutation characters of a finite group fits into the wider picture of *class functions*. These are *complex*-valued functions $\chi : G \to \mathbb{C}$[5] which are constant on conjugacy classes. A class function $\chi : G \to \mathbb{C}$ is called an *irreducible character* if the following property holds

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1}) = 1.$$

Every class function can be broken up as a unique linear combination of irreducible characters and, just like the permutation characters, every irreducible character arises from the action of G on some "hidden" geometric object. The irreducible characters form the atomic constituents for studying how a group can act on a given object via symmetries.

An important basic fact in this theory is that the number of irreducible characters, which we denote by k, is the same as the number of conjugacy classes. Therefore we can record the values of all irreducible characters in a $k \times k$ square matrix, known as the *character table*, with rows labelled by the irreducible characters and columns labelled by the conjugacy classes of G.

The character table of a finite group G encodes an impressive amount of information about its algebraic structure. For example, the sum of the squares of the entries in the column labelled by (e) is the same as the number of the elements $|G|$ in the group. You can check this fact for D_8 by using the character table given in Table 2. Notice also that the permutation character χ_ρ, described in Table 1, can be obtained as a linear combination of irreducible characters. Again just by inspecting the character table we deduce that

$$\chi_\rho = \chi_1 + \chi_2 + \chi_5.$$

The irreducible character χ_5 is obtained from the natural action of the dihedral group D_8 on the square.

---

[5] By extension of the (one-dimensional) line of real numbers to two dimensions we arrive at the *complex plane* consisting of all complex numbers $\mathbb{C}$. If you like, have a look at Bruce Reznick's Snapshot 4/2014 *What does ">" really mean?* for a brief introduction to $\mathbb{C}$.

	(e)	(σ)	(σ^2)	(τ)	$(\sigma \circ \tau)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	-1	1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2	0	-2	0	0

Table 2: The character table of D_8.

The study of character tables and more generally *character theory* has had a huge impact on the study of finite groups. In particular, the use of character theoretic arguments was crucial in achieving the Classification of Finite Simple Groups (CFSG), which we discussed in Section 2. For instance, the character tables of all 26 'sporadic' simple groups have been computed and are contained in the Atlas of finite groups [2]. However it is an open and difficult problem to determine the character tables of all finite simple groups.

Over the last 50 years some simple to state, yet beautiful and deep conjectures, have been proposed about the irreducible characters of finite groups. In recent years amazing new progress has been made using the CFSG. A highlight at the last Oberwolfach workshop on representations of finite groups in April 2015 has been the announcement, by Malle and Späth, of a proof of the McKay conjecture for $p = 2$ [8], formulated in 1972. This is based on the landmark paper by Isaacs–Malle–Navarro [5] which reduces this problem to questions about finite simple groups.

The strategy of using the CFSG to solve the above mentioned conjectures has shone a light on the problem of determining the character tables of finite simple groups and more generally determining explicit information about the irreducible characters of these groups. This problem turns out to have many fascinating relationships with other branches of mathematics such as number theory, algebraic geometry and algebraic topology [7].

Over the next couple of years, hundreds of mathematicians from around the world will meet at the Bernoulli centre [6] in Lausanne, Switzerland, and the Mathematical Sciences Research Institute [12] in Berkeley, California, USA, for two six month long research programs to intensely study these problems. It is hoped that these meetings will culminate in substantial breakthroughs in the study of finite groups. At such a moment in our history, it is interesting to think what questions mathematicians will be asking about finite groups over the next 50 or 100 years as part of our long term endeavour to understand symmetry.

Acknowledgements

Eugenio Giannelli thanks the London Mathematical Society for financial support through Postdoctoral Mobility Grant PMG14-15 02. Jay Taylor thanks INdAM and the European Research Council for financial support through an INdAM-COFUND Marie Curie Fellowship. We also thank Meinolf Geck and Gabriel Navarro for their comments on an earlier version of this snapshot.

References

[1] J. Baez and J. Huerta, *The algebra of grand unified theories*, Bulletin of the American Mathematical Society **47** (2010), 483–552.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[3] F. G. Frobenius, *Über Gruppencharaktere*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1896), 985–1021.

[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.7*, 2015, http://www.gap-system.org.

[5] I. M. Isaacs, G. Malle and G. Navarro, *A reduction theorem for the McKay conjecture*, Inventiones Mathematicae **170** (2007), no. 1, 33–101.

[6] R. Kessar, G. Malle and D. Testerman, *Local representation theory and simple groups*, 1st July to 31st Dec. 2016, Centre Interfacultaire Bernoulli.

[7] G. Lusztig, *Algebraic and geometric methods in representation theory*, `arXiv:1409.8003 [math.RT]`, http://arxiv.org/abs/1409.8003v1.

[8] G. Malle and B. Späth, *Characters of odd degree*, `arXiv:1506.07690 [math.RT]`, http://arxiv.org/abs/1506.07690.

[9] G. Pólya and R. C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Springer, New York, 1987.

[10] P. G. Stockley and R. Twarock, *Emerging Topics in Physical Virology*, Imperial College Press, 2010.

[11] R. Thornhill and S. W. Gangestad, *Facial attractiveness*, Trends in cognitive sciences **3** (1999), no. 12, 452–460.

[12] P. Tiep et al., *Group representation theory and applications*, 16th January to 25th May 2018, Mathematical Sciences Research Institute, http://www.msri.org/programs/293, [Online; accessed 12-December-2015].

Eugenio Giannelli *is a postdoc at the TU Kaiserslautern supported by the London Mathematical Society.*

Jay Taylor *is a Marie Curie fellow of the Istituto Nazionale di Alta Matematica at the Università degli Studi di Padova.*

Mathematical subjects
Algebra and Number Theory

DOI
10.14760/SNAP-2016-005-EN

Snapshots of modern mathematics from Oberwolfach are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the general public worldwide. It started as part of the project "Oberwolfach meets IMAGINARY" in 2013 with a grant by the Klaus Tschira Foundation. The project has also been supported by the Oberwolfach Foundation and the MFO. All snapshots can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.
