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Oberwolfach Preprints

Number 2026-02

NAIARA V. DE PAULO
SEONGCHAN KIM
PEDRO A. S. SALOMÃO
ALEXSANDRO SCHNEIDER

Transverse Foliations for
Two-Degree-of-Freedom
Mechanical Systems

March 2026

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ISSN 1864-7596

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany
<https://www.mfo.de>

DOI 10.14760/OWP-2026-02



TRANSVERSE FOLIATIONS FOR TWO-DEGREE-OF-FREEDOM MECHANICAL SYSTEMS

NAIARA V. DE PAULO, SEONGCHAN KIM, PEDRO A. S. SALOMÃO,
AND ALEXSANDRO SCHNEIDER

ABSTRACT. We investigate the dynamics of a two-degree-of-freedom mechanical system for energies slightly above a critical value. The critical set of the potential function is assumed to contain a finite number of saddle points. As the energy increases across the critical value, a disk-like component of the Hill region gets connected to other components precisely at the saddles. Under certain convexity assumptions on the critical set, we show the existence of a weakly convex foliation in the region of the energy surface where the interesting dynamics takes place. The binding of the foliation is formed by the index-2 Lyapunov orbits in the neck region about the rest points and a particular index-3 orbit. Among other dynamical implications, the transverse foliation forces the existence of periodic orbits, homoclinics, and heteroclinics to the Lyapunov orbits. We apply the results to the Hénon-Heiles potential for energies slightly above $1/6$. We also discuss the existence of transverse foliations for decoupled mechanical systems, including the frozen Hill's lunar problem with centrifugal force, the Stark problem, the Euler problem of two centers, and the potential of a chemical reaction.

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1. INTRODUCTION AND MAIN RESULTS

We study the dynamics of a mechanical system

$$(1.1) \quad \ddot{x} = -\nabla V(x), \quad x \in \mathbb{R}^2,$$

where the potential V is smooth. Let $x(t)$ be a solution of (1.1), and let $y(t) := \dot{x}(t)$. Then $(x(t), y(t)) \in \mathbb{R}^4$ solves Hamilton's equations for the Hamiltonian $H(x, y) = \frac{|y|^2}{2} + V(x)$, that is

$$\dot{x} = \frac{\partial H}{\partial y}(x, y), \quad \dot{y} = -\frac{\partial H}{\partial x}(x, y).$$

We fix the energy $E \in \mathbb{R}$ and study the dynamics of the three-dimensional energy surface $H^{-1}(E)$. The projection of $H^{-1}(E)$ to the x -plane is the Hill region $\Omega_E = \{V(x) \leq E\} \subset \mathbb{R}^2$, and its boundary $\partial\Omega_E$ is the zero-velocity curve.

Let $v \in V^{-1}(0)$ be a critical point of V . Then $p = (v, 0) \in H^{-1}(0)$ is an equilibrium point of the Hamiltonian flow of H . If v is a saddle point, then p is a saddle-center equilibrium, i.e., the linearization $J_0\mathcal{H}(p)$ admits a pair of real eigenvalues $\pm\alpha$ and a pair of purely imaginary eigenvalues $\pm i\omega$. Here, J_0 is the complex matrix

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and \mathcal{H} is the Hessian of H . For every $E > 0$ sufficiently small, $H^{-1}(E)$ has a unique index-2 hyperbolic orbit $P_{2,E}$ in the neck region about p , called Lyapunov orbit. For the definition of the Conley-Zehnder index of a periodic orbit, see section 2.2. The projection of $P_{2,E}$ to Ω_E is a simple arc in the neck region around v connecting distinct points of the zero-velocity curve $\partial\Omega_E$, i.e., $P_{2,E}$ is a brake orbit.

We are particularly interested in the case where V has precisely $l \geq 1$ saddle points $v_1, \dots, v_l \in V^{-1}(0)$, which are vertices of a simple closed curve $\partial K_0 \subset V^{-1}(0)$ bounding a disk-like compact region $K_0 \subset \{V \leq 0\}$. Except for the saddles, the points of ∂K_0 are regular, and $V < 0$ in $K_0 \setminus \partial K_0$. In that case, K_0 is the projection of a singular sphere-like subset $S_0 \subset H^{-1}(0)$, with precisely l singularities of saddle-center type at $p_i = (v_i, 0)$, $i = 1, \dots, l$. As the energy E changes from negative to positive, a sphere-like subset $S_E \subset H^{-1}(E)$, $E < 0$, projecting to the interior of K_0 , gets connected to other components at p_i 's for $E = 0$. For $E > 0$ small, the neck region around each p_i contains a Lyapunov orbit $P_{2,E}^i$. This orbit bounds a pair of disks, forming a two-sphere $\mathcal{S}_i \subset H^{-1}(E)$ that locally separates $H^{-1}(E)$. The flow is transverse to both disks in opposite directions. See section 2.5 for a more detailed description. The energy surface $H^{-1}(E)$, $E > 0$, thus contains a compact subset, bounded by $\cup_i \mathcal{S}_i$ and also denoted by S_E , which is diffeomorphic to a three-sphere with l disjoint open balls removed. The left images in Figures 4.1, 4.2, and 4.3 represent the projections of S_E to their Hill regions in concrete examples.

In general, the dynamics of S_E is rather complicated, combining trajectories that escape S_E through some \mathcal{S}_i forward and/or backward in time, with invariant

subsets presenting rich dynamics. We aim to study the complex dynamics of S_E via transverse foliations.

Definition 1.1. *Let $S_E \subset H^{-1}(E)$ be the compact subset bounded by $\cup_i S_i$ as above. A weakly convex foliation adapted to S_E is a singular foliation \mathcal{F}_E of S_E satisfying the following properties:*

- (i) *The singular set \mathcal{P}_E of \mathcal{F}_E consists of the Lyapunov orbits $P_{2,E}^1, \dots, P_{2,E}^l$ and an unknotted periodic orbit $P_{3,E}$ of index 3. These $l+1$ periodic orbits form the binding of \mathcal{F}_E and are called binding orbits.*
- (ii) *The complement $S_E \setminus \cup_{P \in \mathcal{P}_E} P$ is foliated by properly embedded planes and cylinders, called regular leaves of \mathcal{F}_E . They consist of: a) l pairs of (rigid) planes $U_{1,E}^j, U_{2,E}^j \subset \mathcal{S}_j, j = 1, \dots, l$, each pair asymptotic to $P_{2,E}^j$; b) l (rigid) cylinders $V_E^j, j = 1, \dots, l$, each one asymptotic to $P_{3,E}$ and to $P_{2,E}^j$; c) l families of planes $D_{\tau,E}^j, \tau \in (0, 1), j = 1, \dots, l$, all asymptotic to $P_{3,E}$. The family $D_{\tau,E}^j$ breaks onto the closure of $U_{1,E}^j \cup V_E^j$ as $\tau \rightarrow 0^+$, and onto the closure of $U_{2,E}^{j+1} \cup V_E^{j+1}$ as $\tau \rightarrow 1^-$. Here, we use the convention $l+1 = 1$.*
- (iii) *All regular leaves are transverse to the flow.*

See the cases $l = 1$ and $l = 2$ in Figure 1.1.

A natural question is under which conditions S_E admits a weakly convex foliation. A motivation for this question comes from the works of Albers, Fish, Frauenfelder, Hofer, and van Koert on the restricted three-body problem [2, section 2]. A weakly convex foliation with the Lyapunov orbit near the first Lagrange point as a binding orbit is expected to exist for energies slightly above the first Lagrange value. Note that a weakly convex foliation implies the existence of homoclinics and/or heteroclinics to the Lyapunov orbits, see [29] and also [10, 12].

Finding a weakly convex foliation is, in general, quite challenging. Here, we point out some issues. First, it is not known for an arbitrary Hamiltonian H whether the energy surface has contact type. Such a condition enables the use of J -holomorphic curves in symplectizations, as developed by Hofer, Wysocki, and Zehnder in [23, 24, 25, 27, 28]. Secondly, the compact subset $S_E \subset H^{-1}(E)$ may admit low index orbits other than the Lyapunov orbits, and these orbits may obstruct the existence of a weakly convex foliation.

In general, results concerning the existence of transverse foliations for Reeb flows require the periodic orbits to be non-degenerate. This is the case, for instance, in the pioneering work by Hofer, Wysocki, and Zehnder [29] where such foliations are obtained on the tight 3-sphere. Although the non-degeneracy condition is C^∞ -generic, it is typically very difficult to check it in concrete examples. Here we do not assume that the flow is non-degenerate.

In [11, 12], the authors found weakly convex foliations assuming that the critical subset $S_0 \subset H^{-1}(0)$ is strictly convex and has a unique singularity ($l = 1$). Such foliations are projections of the so-called finite energy foliations in the symplectization of the energy surface. Fish and Siefring [17] studied the connected sum of finite energy foliations and constructed weakly convex foliations. See also [30, 32, 34]. Wendl [47] constructed finite energy foliations for overtwisted contact three-manifolds. Finite energy foliations were first studied on star-shaped hyper-surfaces of \mathbb{R}^4 by Hofer, Wysocki, and Zehnder [26, 29]. Colin, Dehornoy, and

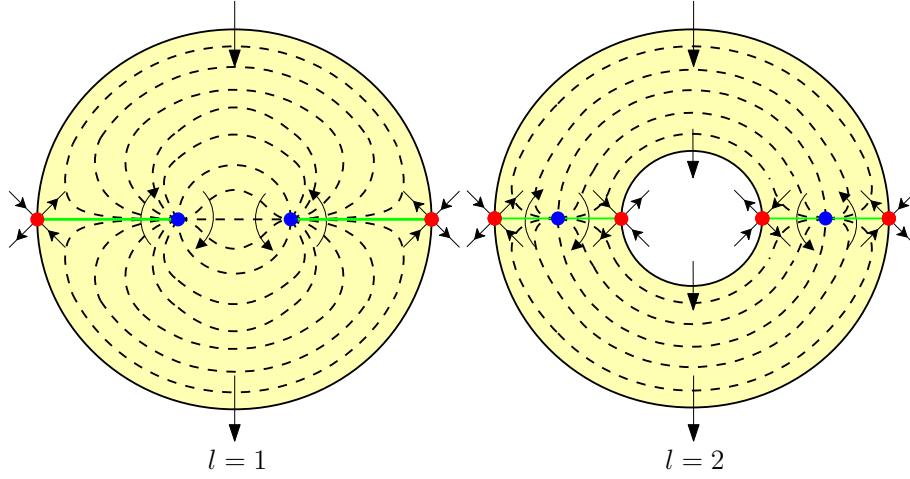


FIGURE 1.1. A cross section of a weakly convex foliation with $l = 1$ (left) and $l = 2$ (right). The red dots represent the Lyapunov orbits, and the blue dots represent index-3 orbits. The black and green curves represent rigid planes and cylinders, respectively. The dashed curves represent the one-parameter families of planes asymptotic to the index-3 orbits.

Rechtman introduced in [9] the broken book decompositions which generalize the transverse foliations obtained by projections of finite energy foliations.

Many concrete examples in the literature – including most of those arising in Celestial Mechanics – do not satisfy the geometric condition of strict convexity assumed in [11] and, moreover, several of them present more than one saddle-center equilibrium point in the critical level. Heading toward overcoming this difficulty, the present paper extends results in [11] for $l \geq 2$ under weaker assumptions on the critical set $S_0 \subset H^{-1}(0)$. To explain it, observe that S_0 is invariant by the flow, and its rest points $p_i = (v_i, 0), i = 1, \dots, l$, are saddle-center equilibrium points. Here we assume that:

- H1. S_0 is dynamically convex, i.e., every (non-constant) periodic orbit in S_0 has index at least 3.
- H2. S_0 admits a positive frame, i.e., the transverse linearized flow on $S_0 \setminus \{p_1, \dots, p_l\}$ strictly rotates counterclockwise on a suitable symplectic frame.

Conditions H1 and H2 hold if the singular set S_0 is strictly convex, as proved in [11, 20, 26], see Remark 2.2 below. These conditions may hold for more general systems, such as the Hénon-Heiles potential, even though S_0 is not strictly convex.

The main result of this paper states that if S_0 satisfies conditions H1 and H2, then a weakly convex foliation adapted to $S_E \subset H^{-1}(E)$ exists for every $E > 0$ sufficiently small.

Theorem 1.2. *Let V be a real-analytic potential on \mathbb{R}^2 , and let $H = |y|^2/2 + V(x)$. Assume that V admits precisely $l \geq 1$ saddle points $v_1, \dots, v_l \in \partial K_0 \subset \{V = 0\}$ as above, so that the singular sphere-like subset $S_0 \subset H^{-1}(0)$ projecting to $K_0 \subset \{V \leq 0\}$ satisfies conditions H1 and H2. Then, for every $E > 0$ sufficiently small, $H^{-1}(E)$ contains a compact subset S_E admitting a weakly convex foliation \mathcal{F}_E as*

in Definition 1.1. The binding orbits are the Lyapunov orbits $P_{2,E}^1, \dots, P_{2,E}^l \subset \partial S_E$ and an index-3 periodic orbit $P_{3,E} \subset S_E \setminus \partial S_E$. Moreover, the projection $K_E \subset \mathbb{R}^2$ of S_E to the x -plane converges in the Hausdorff topology to K_0 as $E \rightarrow 0^+$. If the actions of the Lyapunov orbits in S_E coincide, then S_E carries infinitely many periodic orbits, and every Lyapunov orbit has infinitely many homoclinics or heteroclinics to another Lyapunov orbit.

Conditions H1 and H2 imply the following weaker condition:

H3. For every $E > 0$ sufficiently small, the energy surface $H^{-1}(E)$ carries no index-2 orbit near S_0 other than the Lyapunov orbits.

Remark 1.3. The conclusions of Theorem 1.2 also hold under the weaker conditions H1 and H3. Nevertheless, with the aim of stating a result with broader applicability to concrete examples, we formulate the main theorem under assumptions H1 and H2.

Remark 1.4. The real-analyticity condition on the potential function V in Theorem 1.2 ensures the existence of suitable local coordinates near the saddle-center critical points of H , which are important for obtaining ideal index estimates in the neck regions of S_E (as explored in Section 3.2). In addition, the global real-analyticity of the Hamiltonian H is relevant in [10, 12] for deriving dynamical consequences from the weakly convex foliation \mathcal{F}_E when the actions of the Lyapunov orbits coincide.

The proof of Theorem 1.2 explores the fact that the energy surfaces of a mechanical system have contact type, and thus, the Hamiltonian flow is equivalent to a Reeb flow. Recall that a contact form on a three-manifold M is a 1-form λ so that $\lambda \wedge d\lambda$ never vanishes. The Reeb vector field R of λ is the unique vector field on M determined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. The tangent plane distribution $\xi = \ker \lambda$ is the contact structure, and the pair (M, ξ) is a contact manifold. In canonical coordinates $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, the standard tight contact structure on $S^3 = \{x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1\}$ is $\xi_0 = \ker \lambda_0 \subset TS^3$, where λ_0 is the contact form given by the restriction of the Liouville form $\frac{1}{2} \sum x_i dy_i - y_i dx_i$ to S^3 . Given $f: S^3 \rightarrow (0, +\infty)$, the Reeb flow of $\lambda := f\lambda_0$ is equivalent to the Hamiltonian flow on the star-shaped hypersurface $\sqrt{f}S^3 \subset \mathbb{R}^4$. We say that λ is weakly convex if its periodic orbits have an index of at least 2. A crucial ingredient in the proof of Theorem 1.2 is the following key result from [10].

Theorem 1.5 (de Paulo, Hryniewicz, Kim, and Salomão [10]). *Let $\lambda = f\lambda_0$ be a weakly convex contact form on the tight three-sphere (S^3, ξ_0) . Assume that λ admits precisely $l \geq 1$ index-2 periodic orbits $P_{2,1}, \dots, P_{2,l}$, and they are all unknotted, mutually unlinked, hyperbolic, and have self-linking number -1 . Assume that the actions of $P_{2,i}$ are smaller than that of any other periodic orbit and that every index-3 orbit is unlinked with any $P_{2,i}$. Then the Reeb flow of λ admits a weakly convex foliation \mathcal{F} in the following sense: each $P_{2,i}$ is a binding orbit and bounds a pair of rigid planes $U_{1,i}, U_{2,i}$ whose closures form a regular embedded two-sphere $S_i \subset S^3$. The remaining binding orbits are index-3 orbits $P_{3,j}, j = 1, \dots, l+1$, contained in distinct components $U_j, j = 1, \dots, l+1$, of $S^3 \setminus \cup_i S_i$. The closure of U_j is called a chamber, and the foliation \mathcal{F} restricted to a chamber is as in Definition 1.1.*

Remark 1.6. *The transverse foliation given in Theorem 1.5 is the projection to S^3 of a finite energy foliation $\tilde{\mathcal{F}}$ in the symplectization $(\mathbb{R} \times S^3, d(e^a \lambda))$, where a is the \mathbb{R} -coordinate. An \mathbb{R} -invariant almost complex structure J on $\mathbb{R} \times S^3$ is adapted to λ if $J \cdot \partial_a = R$ and $J \cdot \xi_0 = \xi_0$ is $d\lambda$ -compatible. The set of such J 's, denoted by $\mathcal{J}(\lambda)$, is contractible in the C^∞ -topology. The leaves of $\tilde{\mathcal{F}}$ are the image of embedded finite energy J -holomorphic curves $\tilde{u} = (a, u): \mathbb{C}P^1 \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, where $\Gamma \subset \mathbb{C}P^1$ is finite, see section 2.4 for definitions. An important fact in the proof of Theorem 1.2 is that for special choices of J , there exist precisely two finite energy J -holomorphic planes asymptotic to each $P_{2,i}$ through opposite directions. Moreover, there exists a generic subset $\mathcal{J}_{\text{reg}}(\lambda) \subset \mathcal{J}(\lambda)$ so that if $J \in \mathcal{J}_{\text{reg}}(\lambda)$, then $\mathbb{R} \times S^3$ admits a finite energy foliation by J -holomorphic curves whose projection to S^3 is a weakly convex foliation as in Theorem 1.5, see [10, Theorem 5.3].*

Let us sketch out in a few brief sentences the main ideas of the proof of Theorem 1.2. The potential V can be modified away from an arbitrarily small neighborhood of K_0 in such a way that the new critical set $\hat{H}^{-1}(0)$ projects to the union of K_0 and l mutually disjoint disk-like compact domains $K_1, \dots, K_l \subset \mathbb{R}^2$. Each K_i is connected to K_0 through the saddle point v_i which is the unique singularity of K_i . For $E > 0$ sufficiently small, $\hat{H}^{-1}(E)$ contains a regular sphere-like hypersurface W_E whose projection contains $K_0 \cup \bigcup_{i=1}^l K_i$ and whose flow is equivalent to the Reeb flow of a contact form λ_E on the tight three-sphere. Conditions H1 and H2 imply that we can choose \hat{H} so that for every $E > 0$ sufficiently small, λ_E is weakly convex and the l Lyapunov orbits $P_{2,E}^1, \dots, P_{2,E}^l$ are the only index-2 orbits in W_E . A thorough study of the linearized flow near the saddle-center equilibria, using the real-analiticity of V , shows that the longer a periodic orbit stays near a saddle-center, the larger its index is. In particular, for $E > 0$ sufficiently small, any periodic orbit in W_E linking with some Lyapunov orbit must stay an arbitrarily long time near the saddle-center and thus have an arbitrarily large index. In particular, index-3 orbits in W_E are unlinked with all the Lyapunov orbits if $E > 0$ is sufficiently small. Finally, the action of each Lyapunov orbit goes to zero as $E \rightarrow 0$, and they are the shortest periodic orbits in W_E . Thus, Theorem 1.5 gives a weakly convex foliation \mathcal{F} for λ_E , which is the projection of a finite energy foliation in the symplectization of W_E . For some special choices of almost complex structure J , the foliation \mathcal{F} admits a chamber S_E bounded by all the pairs of rigid planes asymptotic to the Lyapunov orbits, which is contained in the original energy surface $H^{-1}(E)$ and projects near K_0 . The desired weakly convex foliation given in Theorem 1.2 is the restriction of \mathcal{F} to S_E .

We apply Theorem 1.2 to many mechanical systems. The Hénon-Heiles potential and the frozen Hill's lunar problem with centrifugal force meet the assumptions on the critical level and admit weakly convex foliations for energies slightly above the critical value. We also consider decoupled mechanical systems and describe the weakly convex foliations obtained from certain gradient flow lines. They include the Stark problem, the Euler problem of two centers, and a chemical reaction model.

The paper is organized as follows. In section 2, we review the basic properties of Reeb dynamics on contact-type hypersurfaces in \mathbb{R}^4 and pseudo-holomorphic curves. In section 3, we establish the main steps in the proof of Theorem 1.2 as follows: in section 3.1, we modify the potential away from K_0 to obtain, for $E > 0$ sufficiently small, a sphere-like energy surface W_E . The weak convexity of W_E and the unlinking condition between index-2 and index-3 periodic orbits are proved in

section 3.2. In section 3.3, we establish the contact property of W_E and construct a particular contact form λ_E satisfying certain suitable normal form, used in section 3.4 to define an almost complex structure on $\mathbb{R} \times W_E$, which admits a pair of finite energy planes asymptotic to each Lyapunov orbit and whose projections to W_E are contained in the unchanged region of the potential. We then gather all these results to prove Theorem 1.2 in section 3.5. The applications are left to section 4.

2. BASICS

Let us introduce some basic facts about the geometry of energy surfaces in \mathbb{R}^4 .

2.1. The quaternion frame. Let (x_1, x_2, y_1, y_2) be coordinates in \mathbb{R}^4 and let $\omega_0 := dy_1 \wedge dx_1 + dy_2 \wedge dx_2$ be the standard symplectic form on \mathbb{R}^4 . Let $j_0 := I, j_1, j_2, j_3: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the orthogonal maps defined by

$$(2.1) \quad \begin{aligned} j_1 \cdot (a_1, a_2, b_1, b_2) &= (b_2, -b_1, a_2, -a_1) \\ j_2 \cdot (a_1, a_2, b_1, b_2) &= (a_2, -a_1, -b_2, b_1) \\ j_3 \cdot (a_1, a_2, b_1, b_2) &= (b_1, b_2, -a_1, -a_2) \end{aligned}$$

Let $H: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a Hamiltonian, and let $w \in S_E := H^{-1}(E)$ be a regular point of H . Let $X_0(w) = \frac{\nabla H(w)}{|\nabla H(w)|}$ be the unit vector normal to S_E at w . The vectors

$$(2.2) \quad X_i(w) := j_i \cdot X_0(w), \quad i = 1, 2, 3,$$

span the tangent space $T_w S_E$ and $X_3(w)$ is parallel to the Hamiltonian vector field X_H , determined by $i_{X_H} \omega_0 = -dH$. Also, $\{X_1, X_2\}$ is a symplectic basis of a plane transverse to X_H .

Let $w(t) \in S_E$ be a nonconstant solution of the Hamiltonian flow, that is, $w(t)$ satisfies $\dot{w}(t) = j_3 \cdot \nabla H(w(t)), \forall t$, and let $u(t) = \alpha_1(t)X_1(w(t)) + \alpha_2(t)X_2(w(t)) + \alpha_3(t)X_3(w(t))$ be a linearized solution along $w(t)$, that is, $u(t) \in T_{w(t)} S_E$ satisfies $\dot{u}(t) = j_3 \cdot \mathcal{H}(w(t))u(t), \forall t$. Here, \mathcal{H} denotes the Hessian of H . Denoting by $\alpha = (\alpha_1, \alpha_2)^T$ the transverse linearized solution in the frame $\{X_1, X_2\}$, we have

$$(2.3) \quad \dot{\alpha} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S(t)\alpha, \quad S(t) := \begin{pmatrix} \kappa_{11} + \kappa_{33} & \kappa_{12} \\ \kappa_{12} & \kappa_{22} + \kappa_{33} \end{pmatrix} \Big|_{w(t)},$$

where $\kappa_{ij} := \langle \mathcal{H} \cdot X_i, X_j \rangle, i, j = 1, 2, 3$. See, for example, [13, Proposition D.1].

Let $\theta(t)$ be a continuous argument of $\alpha_1(t) + i\alpha_2(t) \in \mathbb{R}^+ e^{i\theta(t)}$. Then

$$(2.4) \quad \begin{aligned} \dot{\theta} &= \frac{\alpha_1 \dot{\alpha}_2 - \alpha_2 \dot{\alpha}_1}{|\alpha|^2} = \frac{\alpha^T S(t) \alpha}{|\alpha|^2} = (\cos \theta \ \sin \theta) S(t) (\cos \theta \ \sin \theta)^T \\ &= (\kappa_{11} + \kappa_{33}) \cos^2 \theta + 2\kappa_{12} \cos \theta \sin \theta + (\kappa_{22} + \kappa_{33}) \sin^2 \theta. \end{aligned}$$

Lemma 2.1. [20, Theorem 5] *If the Hessian \mathcal{H} of H restricted to $T_{w(t)} S_E$ is positive-definite along the non-constant trajectory $w(t) \in S_E$, then a continuous argument $\theta(t)$ of any non-trivial transverse linearized solution $0 \neq \alpha(t) \equiv \alpha_1(t) + i\alpha_2(t) \in \mathbb{R}^+ e^{i\theta(t)}$ of (2.3) satisfies $\dot{\theta}(t) > 0, \forall t$. In particular, the frame $\{X_1, X_2\}$ is positive along $w(t)$.*

Proof. Since \mathcal{H} restricted to $T_{w(t)} S_E$ is positive-definite along $w(t)$, we have $\kappa_{ii} > 0, \forall i = 1, 2, 3$. Hence $\text{tr}(S(t)) = \kappa_{11} + \kappa_{22} + 2\kappa_{33} > 0, \forall t$. Since

$$\langle \mathcal{H} \cdot (X_1 - \lambda X_2), X_1 - \lambda X_2 \rangle \Big|_{w(t)} = \kappa_{22} \lambda^2 - 2\kappa_{12} \lambda + \kappa_{11} > 0 \quad \forall \lambda \in \mathbb{R},$$

we obtain $\kappa_{11}\kappa_{22} - \kappa_{12}^2 > 0$. This implies $\det(S(t)) = \kappa_{11}\kappa_{22} - \kappa_{12}^2 + \kappa_{33}(\kappa_{11} + \kappa_{22} + \kappa_{33}) > 0, \forall t$, proving that $S(t)$ is positive-definite and thus $\dot{\theta}(t) > 0, \forall t$, see (2.4). \square

Remark 2.2. *As a consequence of Lemma 2.1, if the critical subset $S_0 \subset H^{-1}(0)$ is strictly convex, then it admits a positive frame in the sense of condition H2 in the introduction. We say that a sphere-like subset $S_0 \subset H^{-1}(0)$ admitting a finite number of saddle-centers is strictly convex if it bounds a convex subset in \mathbb{R}^4 and the Hessian \mathcal{H} restricted to the tangent space at any regular point of S_0 is positive definite. In this particular case, it also follows from [11, Proposition 4.7] and [26, Theorem 3.4] that S_0 satisfies condition H1 concerning its dynamical convexity. It turns out that if S_0 is strictly convex, then it satisfies the assumptions of our main result (Theorem 1.2).*

For a mechanical Hamiltonian $H = \frac{1}{2}|y|^2 + V(x)$, let $x \in \mathbb{R}^2$ be the projection of a regular point $w \in S_E = H^{-1}(E)$. Then

$$(2.5) \quad \begin{aligned} X_0 &= g \cdot (V_{x_1}, V_{x_2}, y_1, y_2), & X_1 &= g \cdot (y_2, -y_1, V_{x_2}, -V_{x_1}), \\ X_2 &= g \cdot (V_{x_2}, -V_{x_1}, -y_2, y_1), & X_3 &= g \cdot (y_1, y_2, -V_{x_1}, -V_{x_2}), \end{aligned}$$

where $V_{x_i} = \partial_{x_i} V$ and $g = (V_{x_1}^2 + V_{x_2}^2 + y_1^2 + y_2^2)^{-1/2}$. Then

$$(2.6) \quad \begin{aligned} \kappa_{11} &= g^2 \cdot (V_{x_1 x_1} y_2^2 - 2V_{x_1 x_2} y_1 y_2 + V_{x_2 x_2} y_1^2 + V_{x_1}^2 + V_{x_2}^2), \\ \kappa_{12} &= g^2 \cdot (V_{x_1 x_1} y_2 V_{x_2} - V_{x_1 x_2} V_{x_1} y_1 V_{x_2} - V_{x_1 x_2} y_2 V_{x_1} \\ &\quad + V_{x_2 x_2} y_1 V_{x_1} - V_{x_2} y_2 - V_{x_1} y_1), \\ \kappa_{22} &= g^2 \cdot (V_{x_1 x_1} V_{x_2}^2 - 2V_{x_1 x_2} V_{x_1} V_{x_2} + V_{x_2 x_2} V_{x_1}^2 + y_1^2 + y_2^2), \\ \kappa_{33} &= g^2 \cdot (V_{x_1 x_1} y_1^2 + 2V_{x_1 x_2} y_1 y_2 + V_{x_2 x_2} y_2^2 + V_{x_1}^2 + V_{x_2}^2), \end{aligned}$$

where $V_{x_i x_j} = \partial_{x_i x_j} V$. If $y_1 = y_2 = 0$, then

$$(2.7) \quad \kappa_{11} = \kappa_{33} = 1, \quad \kappa_{12} = 0 \quad \text{and} \quad \kappa_{22} = \frac{V_{x_1 x_1} V_{x_2}^2 - 2V_{x_1 x_2} V_{x_1} V_{x_2} + V_{x_2 x_2} V_{x_1}^2}{V_{x_1}^2 + V_{x_2}^2}.$$

Hence $\text{tr}(S) = 3 + \kappa_{22}$ and $\det(S) = 2 + 2\kappa_{22}$. In particular, the following implications are valid

$$(2.8) \quad \kappa_{22} > -1 \quad \Rightarrow \quad S \text{ is positive-definite} \quad \Rightarrow \quad \dot{\theta} > 0,$$

provided $y_1 = y_2 = 0$. The implications above will be useful in the Hénon-Heiles potential to show that the singular sphere-like subset of the critical energy surface admits a positive frame even though it is not strictly convex. See Proposition 4.1.

2.2. The Conley-Zehnder index. We can use the quaternion frame to define the index of a periodic orbit $P = (w, T) \subset H^{-1}(E) \subset \mathbb{R}^4$. Let $\theta(t)$ be a continuous argument of a non-vanishing solution $\alpha(t) \equiv \alpha_1(t) + i\alpha_2(t)$ of (2.3). Let

$$(2.9) \quad I := \left\{ \frac{\theta(T) - \theta(0)}{2\pi} \mid \alpha(0) \neq 0 \right\}.$$

Then, I is a compact interval, and its length is less than $1/2$. Given $\varepsilon > 0$ sufficiently small, let $I_\varepsilon := I - \varepsilon$. Then the index of P

$$\mu(P) := \begin{cases} 2k + 1 & \text{if } I_\varepsilon \subset (k, k + 1), \\ 2k & \text{if } k \in \text{int}(I_\varepsilon), \end{cases}$$

is independent of $\epsilon > 0$ sufficiently small. If S_E has contact-type with contact form λ_E , then $\mu(P)$ coincides with the (generalized) Conley-Zehnder index [26], see also Long's book [35]. Indeed, let $\xi \subset TS_E$ be the contact structure. Every $v \in \xi$ can be written in the quaternion frame as $v = a_1X_1 + a_2X_2 + a_3X_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$. Recall that X_3 is parallel to X_H . We define the isomorphism

$$\pi_\xi: \xi \rightarrow \text{span}\{X_1, X_2\}, \quad v = \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 \mapsto \alpha_1X_1 + \alpha_2X_2.$$

Set $\tilde{X}_j := \pi_\xi^{-1}(X_j)$, $j = 1, 2$, so that $\xi = \text{span}\{\tilde{X}_1, \tilde{X}_2\}$. Since $X_3 \subset \ker \omega_0|_{TS_E}$, we compute $\omega_0(\tilde{X}_1, \tilde{X}_2) = \omega_0(X_1, X_2) = 1$. Hence, the frame $\{\tilde{X}_1, \tilde{X}_2\}$ induces a symplectic trivialization $\Psi: \xi \rightarrow S_E \times \mathbb{C}$

$$\Psi: \alpha_1\tilde{X}_1 + \alpha_2\tilde{X}_2 \mapsto \alpha_1 + i\alpha_2,$$

and the linearized flow on ξ is determined by (2.3).

Denoting by $P^k = (w, kT)$ the k -th iterate of $P = (w, T)$, the rotation number of P is the well-defined limit

$$\rho(P) = \lim_{k \rightarrow +\infty} \frac{\mu(P^k)}{2k}.$$

2.3. Self-linking number. Let $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow M$ be a null-homologous transverse knot on a contact three-manifold $(M, \xi = \ker \lambda)$, i.e., $T\gamma \cap \xi = 0$. Choose a Seifert surface $i: \Sigma \hookrightarrow M$ for γ , i.e., Σ is an embedded oriented surface bounded by γ , and take a non-vanishing section X of $i^*\xi$. Assume that M is oriented by $\lambda \wedge d\lambda > 0$, γ is oriented by $\lambda > 0$, and Σ is oriented by $\gamma = \partial\Sigma$. Let \exp be an exponential map on M . If X is sufficiently small, then $\gamma(\mathbb{R}/\mathbb{Z}) \cap \gamma_X(\mathbb{R}/\mathbb{Z}) = \emptyset$, where $\gamma_X(t) := \exp_{\gamma(t)} X(t), \forall t \in \mathbb{R}/\mathbb{Z}$. The self-linking number of γ , denoted by $\text{sl}(\gamma)$, is defined as the algebraic intersection number $\gamma_X \cdot i$, where γ_X inherits the orientation of γ . Notice that $\text{sl}(\gamma)$ is independent of X and \exp . If the Euler class $e(\xi)$ vanishes on $H_2(M)$, then $\text{sl}(\gamma)$ is independent of Σ .

2.4. Pseudo-holomorphic curves in symplectizations. Let $(M, \xi = \ker \lambda)$ be a contact three-manifold, and let $(\mathbb{R} \times M, d(e^a\lambda))$ be its symplectization, where a is the \mathbb{R} -coordinate. Denote by $\pi_\xi: TM \rightarrow \xi$ the projection along the Reeb vector field R . As above, a periodic orbit of λ is a pair $P = (x, T)$, where x is a periodic trajectory of the Reeb flow of λ and $T > 0$ is a period of x . We identify periodic orbits with the same image and period. Let $\mathcal{J}(\lambda)$ be the space of \mathbb{R} -invariant $d\lambda$ -compatible almost complex structures J on $\mathbb{R} \times M$ satisfying $J \cdot \partial_a = R$ and $J(\xi) = \xi$. Let (S, j) be a closed Riemann surface and $\Gamma \subset S$ a finite set. A map $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$ is called a finite energy J -holomorphic curve if

$$(2.10) \quad \bar{\partial}_J(\tilde{u}) := \frac{1}{2}(d\tilde{u} + J(\tilde{u}) \circ d\tilde{u} \circ j) = 0,$$

and its Hofer's energy is finite

$$0 < E(\tilde{u}) := \sup_{\phi \in \Lambda} \int_{S \setminus \Gamma} \tilde{u}^* d(\phi\lambda) < \infty.$$

Here, $\Lambda := \{\phi \in C^\infty(\mathbb{R}, [0, 1]) \mid \phi' \geq 0\}$. If $S = \mathbb{C}P^1$ and $\#\Gamma = 1$, then \tilde{u} is called a finite energy plane. Near each point $z \in S$, one may consider holomorphic

coordinates $s + it$ and rewrite (2.10) as

$$(2.11) \quad \begin{cases} \pi_\xi u_s(s, t) + J(u(s, t))\pi_\xi u_t(s, t) = 0, \\ \lambda(u_t(s, t)) = a_s(s, t), \\ \lambda(u_s(s, t)) = -a_t(s, t). \end{cases}$$

The elements in Γ are called punctures. A puncture $z_0 \in \Gamma$ is called removable if \tilde{u} can be smoothly extended over z_0 . Otherwise, we say that z_0 is non-removable. In the following, we assume that every puncture is non-removable. Given $z_0 \in \Gamma$, we choose a neighborhood $U_0 \subset S \setminus \Gamma$ of z_0 and a bi-holomorphism $\phi_{z_0}: (\mathbb{D}, 0) \rightarrow (U_0, z_0)$ such that $\phi_{z_0}^* j = i$, where $\mathbb{D} \subset \mathbb{C}$ is the closed unit disk centered at 0 and i is the canonical complex structure on \mathbb{C} . In cylindrical coordinates $[0, +\infty) \times \mathbb{R}/\mathbb{Z} \ni (s, t) \mapsto \phi_{z_0}(e^{-2\pi(s+it)})$ near z_0 , we write $\tilde{u}(s, t) = (a(s, t), u(s, t)) = \tilde{u} \circ \phi_{z_0}(e^{-2\pi(s+it)})$.

The following result is due to Hofer establishing a deep connection between finite energy J -holomorphic curves and periodic Reeb orbits.

Theorem 2.3 (Hofer [23]). *Let $(s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}$ be cylindrical coordinates near a puncture $z_0 \in \Gamma$ of a finite energy curve $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$. Write $\tilde{u}(s, t) = (a(s, t), u(s, t))$ in cylindrical coordinates as above. Given a sequence $s_n \rightarrow +\infty$, there exists a subsequence, still denoted by s_n , and a periodic orbit $P = (w, T)$ such that $u(s_n, \cdot) \rightarrow w(\varepsilon T \cdot)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, M)$ as $n \rightarrow +\infty$. The sign $\varepsilon = \varepsilon(z_0) \in \{\pm 1\}$ depends only on z_0 .*

A non-removable puncture $z_0 \in \Gamma$ is said to be positive or negative if $\varepsilon(z_0) = +1$ or $\varepsilon(z_0) = -1$, respectively. Then $a(z) \rightarrow \pm\infty$ as $z \rightarrow z_0$ according to the sign $\varepsilon(z_0) = \pm 1$. A periodic orbit $P = (w, T)$ as in Theorem 2.3 is an asymptotic limit of \tilde{u} at z_0 . The set of asymptotic limits of \tilde{u} at z_0 is compact and connected; see [19, Lemma 13.3.1] and [44].

The following theorem of Hofer, Wysocki, and Zehnder shows that if an asymptotic limit of \tilde{u} at z_0 is nondegenerate, then it is the unique asymptotic limit of \tilde{u} at z_0 .

Theorem 2.4 (Hofer-Wysocki-Zehnder [25]). *Let $z_0 \in \Gamma$ be a puncture of a finite energy curve $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$ and let $P = (w, T)$ be an asymptotic limit of \tilde{u} at z_0 . If P is non-degenerate, then P is the unique asymptotic limit at z_0 .*

2.5. Re-scaled coordinates near a saddle-center. Throughout the paper we shall re-scale coordinates near the saddle-center $p = (v, 0) \in H^{-1}(0)$, where v represents any saddle point v_1, \dots, v_l of V . We may assume that $v = 0$ and $V = ax_1^2/2 + bx_2^2/2 + R(x)$, where $a < 0$, $b > 0$ and $R(x) = O(|x|^3)$. Taking new coordinates $(x, y) = \sqrt{\varepsilon}(\hat{x}, \hat{y})$, with $\varepsilon > 0$ small, we consider the Hamiltonian

$$\hat{H}_\varepsilon(\hat{x}, \hat{y}) = \frac{1}{\varepsilon}H(x, y) = \frac{\hat{y}_1^2 + \hat{y}_2^2}{2} + \frac{a\hat{x}_1^2}{2} + \frac{b\hat{x}_2^2}{2} + \frac{1}{\varepsilon}\hat{R}_\varepsilon(\hat{x}),$$

where $\hat{R}_\varepsilon(\hat{x}) = R(\sqrt{\varepsilon}\hat{x})$. The rescaled potential $V(\sqrt{\varepsilon}\hat{x})/\varepsilon$ is denoted by $\hat{V}_\varepsilon(\hat{x})$. Notice that $H^{-1}(\varepsilon E)$ corresponds to $\hat{H}_\varepsilon^{-1}(E), \forall E \in \mathbb{R}$. We fix $E > 0$. Since $\hat{R}_\varepsilon(\hat{x})/\varepsilon$ converges in $C_{\text{loc}}^\infty(\mathbb{R}^2)$ to 0 as $\varepsilon \rightarrow 0^+$, \hat{H}_ε converges in $C_{\text{loc}}^\infty(\mathbb{R}^4)$ to

$$\hat{H}_0(\hat{x}, \hat{y}) := \frac{\hat{y}_1^2 + \hat{y}_2^2}{2} + \frac{a\hat{x}_1^2}{2} + \frac{b\hat{x}_2^2}{2}$$

as $\epsilon \rightarrow 0^+$. The potential of \hat{H}_0 is denoted by $\hat{V}_0(\hat{x}) = a\hat{x}_1^2/2 + b\hat{x}_2^2/2$. The equilibrium point $(\hat{x}, \hat{y}) = (0, 0)$ of \hat{H}_0 projects into the $\hat{x}_1\hat{y}_1$ -plane as a saddle and into the $\hat{x}_2\hat{y}_2$ -plane as a center. The orbit $\hat{P}_{2,0,E} = \{\hat{x}_1 = \hat{y}_1 = 0, b\hat{x}_2^2 + \hat{y}_2^2 = 2E\} \subset \hat{H}_0^{-1}(E)$ is an index-2 hyperbolic orbit. It is the limit as $\epsilon \rightarrow 0^+$ of index-2 hyperbolic orbits $\hat{P}_{2,\epsilon,E} \subset \hat{H}_\epsilon^{-1}(E)$ corresponding to the Lyapunov orbits $P_{2,\epsilon E} \subset H^{-1}(\epsilon E)$ near 0. Notice that both the disks

$$\hat{U}_{1,0,E} := \hat{H}_0^{-1}(E) \cap \{\hat{x}_1 = 0, \hat{y}_1 > 0\} \quad \text{and} \quad \hat{U}_{2,0,E} := \hat{H}_0^{-1}(E) \cap \{\hat{x}_1 = 0, \hat{y}_1 < 0\},$$

have $\hat{P}_{2,0,E}$ as boundary and are transverse to the flow on $\hat{H}_0^{-1}(E)$. See Figure 2.1. The behavior of $\hat{U}_{1,0,E}$ and $\hat{U}_{2,0,E}$ near $\hat{P}_{2,0,E}$ and the C_{loc}^∞ -convergence of \hat{H}_ϵ to \hat{H}_0 imply that $\hat{P}_{2,\epsilon,E}$ bounds embedded disks $\hat{U}_{1,\epsilon,E}, \hat{U}_{2,\epsilon,E} \subset \hat{H}_\epsilon^{-1}(E)$ transverse to the flow and C^∞ -close to $\hat{U}_{1,0,E}$ and $\hat{U}_{2,0,E}$, respectively, for every $\epsilon > 0$ sufficiently small. In particular, if $Q_{\epsilon E} \subset H^{-1}(\epsilon E)$ is a periodic orbit linked with $P_{2,\epsilon E}$, then $\hat{Q}_{\epsilon,E} := Q_{\epsilon E}/\sqrt{\epsilon} \subset \hat{H}_\epsilon^{-1}(E)$ is linked with $\hat{P}_{2,\epsilon,E}$ and therefore intersects $\hat{U}_{1,\epsilon,E}$ and $\hat{U}_{2,\epsilon,E}$.

We shall see later that such a linked orbit $Q_{\epsilon E}$ has an arbitrarily high index if $\epsilon, E > 0$ are sufficiently small. Also, we shall choose the disks $\hat{U}_{1,\epsilon,E}, \hat{U}_{2,\epsilon,E}$ as projections of finite energy planes in the symplectization for suitable almost complex structures.

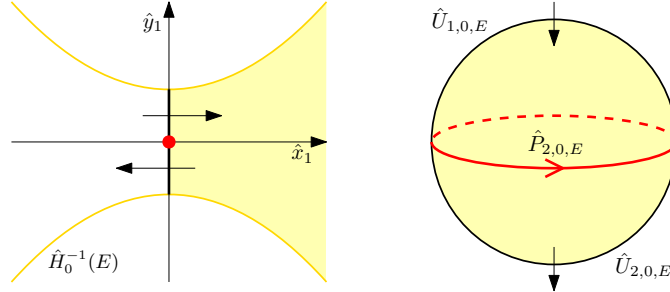


FIGURE 2.1. Projections of the disks $\hat{U}_{1,0,E}$ and $\hat{U}_{2,0,E}$ to the $\hat{x}_1\hat{y}_1$ -plane represented in bold black lines. The flow goes across the disks in different directions (left). The hyperbolic orbit $\hat{P}_{2,0,E}$ is the equator of a two-sphere that locally separates the energy level and mediates the entry/exit of the flow into/from a chamber (right).

3. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is organized in several steps:

- We start by changing the Hamiltonian H away from S_0 to cap the critical set near the saddle-centers. To do that, we change the potential V away from K_0 so that the Hill region $\{V \leq 0\}$ becomes the union of K_0 and l disjoint compact disks K_1, \dots, K_l , touching K_0 at the saddles v_1, \dots, v_l . The new Hamiltonian has the same saddle-centers $p_1, \dots, p_l \in H^{-1}(0)$ and, for $E > 0$ sufficiently small, $H^{-1}(E)$ is a sphere-like hypersurface $W_E \subset \mathbb{R}^4$ whose projection contains $K_0 \cup \bigcup_{i=1}^l K_i$. This is the content of section 3.1.

- In section 3.2, we show that W_E is weakly convex, and the Lyapunov orbits are the only index-2 orbits for every $E > 0$ sufficiently small. Moreover, we study the linearized flow near a saddle-center and show that index-3 orbits in W_E cannot be linked with any Lyapunov orbit if $E > 0$ is sufficiently small.
- A special contact form λ_E on W_E is constructed in section 3.3. This contact form is crucial in the location of rigid planes asymptotic to the Lyapunov orbits.
- In section 3.4, we choose a special almost complex structure J_E in the symplectization of W_E , which is adapted to λ_E and admits a pair of rigid planes asymptotic to each Lyapunov orbit. The projections of the rigid planes to W_E lie in the unchanged domain of the Hamiltonian H for every $E > 0$ sufficiently small.
- The weakly convex foliation in the domain of $H^{-1}(E)$ determined by the rigid planes then follows from Theorem 1.5 and the previous steps. See Remark 1.6.

3.1. Capping the energy surface. Recall that $K_0 \subset \{V \leq 0\}$ is the compact subset given by the projection to the x -plane of the singular sphere-like subset $S_0 \subset H^{-1}(0)$ containing l saddle-centers of H which project to l saddle points of V in the boundary of K_0 . Since the energy surface $H^{-1}(E)$, $E > 0$ small, may not be compact at first, it will be convenient to change the potential V away from K_0 so that $H^{-1}(E)$ becomes a sphere-like regular component projecting near K_0 , for every $E > 0$ small.

We may assume that $0 \in \partial K_0$ is one of the saddles of V and $V(x) = ax_1^2/2 + bx_2^2/2 + R(x)$ near 0, where $a < 0$, $b > 0$ and $R(x) = O(|x|^3)$. We can also assume that K_0 is locally contained in $\{x_1 \leq 0\}$. We locally change V on $\{x_1 > 0\}$. For each $\epsilon > 0$ small, consider the re-scaled Hamiltonian \hat{H}_ϵ and re-scaled potential \hat{V}_ϵ as in section 2.5. Choose a smooth function $f: (-\infty, 2) \rightarrow [0, +\infty)$ so that

$$(3.1) \quad f \equiv 0 \text{ on } (-\infty, 1], \quad f''' > 0 \text{ on } (1, 2), \quad \text{and} \quad \lim_{t \rightarrow 2^-} f(t) = +\infty,$$

and let

$$(3.2) \quad \tilde{V}_\epsilon(\hat{x}) := \hat{V}_\epsilon(\hat{x}) + f(\hat{x}_1), \quad \forall \hat{x} = (\hat{x}_1, \hat{x}_2) \text{ with } \hat{x}_1 < 2.$$

Note that \tilde{V}_ϵ coincides with \hat{V}_ϵ on $\{\hat{x}_1 \leq 1\}$ and thus $0 \in \mathbb{R}^2$ is a saddle point of \tilde{V}_ϵ .

Lemma 3.1. *For every $\epsilon > 0$ sufficiently small, $\{\tilde{V}_\epsilon \leq 0\}$ contains a disk-like compact set $\tilde{K}_\epsilon \subset \{0 \leq \hat{x}_1 < 2\}$ with a unique singularity at 0. Except for the origin, every point in $\partial \tilde{K}_\epsilon$ is a regular point of \tilde{V}_ϵ , and $\tilde{V}_\epsilon < 0$ in $\tilde{K}_\epsilon \setminus \partial \tilde{K}_\epsilon$.*

Proof. Notice that \tilde{V}_ϵ converges in $C_{\text{loc}}^\infty(\{\hat{x}_1 < 2\})$ to $\tilde{V}_0(\hat{x}_1, \hat{x}_2) := \hat{V}_0(\hat{x}_1, \hat{x}_2) + f(\hat{x}_1) = a\hat{x}_1^2/2 + b\hat{x}_2^2/2 + f(\hat{x}_1)$ as $\epsilon \rightarrow 0^+$. By (3.1), $g(\hat{x}_1) := a\hat{x}_1^2/2 + f(\hat{x}_1)$ has a unique critical point \hat{q}_1 in the interval $(0, 2)$, which is a nondegenerate minimum. Hence \tilde{V}_0 has a unique nondegenerate minimum $\tilde{q} = (\hat{q}_1, 0)$ in $\{0 < \hat{x}_1 < 2\}$ with $\tilde{V}_0(\tilde{q}) = g(\hat{q}_1) < 0$. For every $E \in (\tilde{V}_0(\tilde{q}), 0)$, $\tilde{V}_0^{-1}(E)$ contains a regular circle-like level on $\{0 < \hat{x}_1 < 2\}$. Such a family of circles develops a singularity at $0 \in \mathbb{R}^2$ as $E \rightarrow 0^-$, and thus $\tilde{V}_0^{-1}(0)$ contains a singular circle-like subset $\partial \tilde{K}_0$ with a unique singularity at 0, bounding a disk-like compact subset $\tilde{K}_0 \subset \{\tilde{V}_0 \leq 0\} \cap \{0 \leq \hat{x}_1 < 2\}$.

Since both singularities at 0 and \tilde{q} are nondegenerate critical points of \tilde{V}_0 , similar conclusions hold for \tilde{V}_ϵ , for every $\epsilon > 0$ sufficiently small, that is $\tilde{V}_\epsilon^{-1}(0)$ contains a

singular circle-like subset $\partial\tilde{K}_\epsilon$ with a unique singularity at 0, bounding a disk-like compact subset $\tilde{K}_\epsilon \subset \{\tilde{V}_\epsilon \leq 0\}$ and $\tilde{V}_\epsilon|_{\tilde{K}_\epsilon \setminus \partial\tilde{K}_\epsilon} < 0$. This finishes the proof. \square

In the original coordinates $x = (x_1, x_2)$, the compact set \tilde{K}_ϵ corresponds to $K_\epsilon := \sqrt{\epsilon}\tilde{K}_\epsilon$ and can be made arbitrarily close to the saddle point. Performing this construction near each saddle v_1, \dots, v_l of V we end up with l disk-like compact sets $K_{i,\epsilon} := \sqrt{\epsilon}\tilde{K}_{i,\epsilon}$, $i = 1, \dots, l$, each one admitting a unique singularity at the corresponding v_i . The following proposition follows directly from Lemma 3.1.

Proposition 3.2. *Let $\mathcal{U}_0 \subset \mathbb{R}^2$ be an open neighborhood of the disk-like compact set $K_0 \subset \mathbb{R}^2$. For every $\epsilon > 0$ sufficiently small, there exists a potential $V_\epsilon: \mathbb{R}^2 \rightarrow \mathbb{R}$ coinciding with V near K_0 so that*

- (i) *The Hill region $\{V_\epsilon \leq 0\}$ contains a subset formed by K_0 and the union of disjoint disk-like compact sets $K_{1,\epsilon}, \dots, K_{l,\epsilon} \subset \mathcal{U}_0$, touching K_0 precisely at the saddles v_1, \dots, v_l . The saddles v_1, \dots, v_l are the unique critical points of V in $V^{-1}(0)$.*
- (ii) *For every $E > 0$ sufficiently small, the Hamiltonian $H_\epsilon(x, y) := |y|^2/2 + V_\epsilon(x)$ admits a regular sphere-like component $W_{\epsilon,E} = H_\epsilon^{-1}(E)$ projecting to a disk-like compact set $K_{\epsilon,E} \supset K_0 \cup \bigcup_{i=1}^l K_{i,\epsilon}$. See Figure 3.1.*
- (iii) *In re-scaled coordinates $(\hat{x}, \hat{y}) = (x/\sqrt{\epsilon}, y/\sqrt{\epsilon})$ near the saddle-centers, the Hamiltonian $H_\epsilon(\sqrt{\epsilon}\hat{x}, \sqrt{\epsilon}\hat{y})/\epsilon$ converges in $C_{loc}^\infty(\{\hat{x}_1 < 2\})$ to*

$$\tilde{H}_0(\hat{x}, \hat{y}) := \frac{|\hat{y}|^2}{2} + \frac{a\hat{x}_1^2}{2} + \frac{b\hat{x}_2^2}{2} + f(\hat{x}_1),$$

as $\epsilon \rightarrow 0^+$, where $a < 0, b > 0$, and f satisfies (3.1).

Proof. The new potential V_ϵ near each saddle v_1, \dots, v_l is obtained in local coordinates as in Lemma 3.1 by defining $V_\epsilon(x) := \epsilon\tilde{V}_\epsilon(x/\sqrt{\epsilon})$. Locally, each compact subset of $\{V_\epsilon \leq 0\}$ is given by $K_{i,\epsilon} = \sqrt{\epsilon}\tilde{K}_{i,\epsilon}$. Hence $K_{i,\epsilon}$ lies in \mathcal{U}_0 if $\epsilon > 0$ is taken sufficiently small. We may assume that $V_\epsilon > 0$ on $\mathbb{R}^2 \setminus (K_0 \cup \bigcup_{i=1}^l K_{i,\epsilon})$. For every $\epsilon > 0$ fixed sufficiently small, V_ϵ coincides with V near K_0 and the projection $K_{\epsilon,E}$ of the sphere-like component $W_{\epsilon,E} := H_\epsilon^{-1}(E)$ is contained in \mathcal{U}_0 for every $E > 0$ sufficiently small. \square

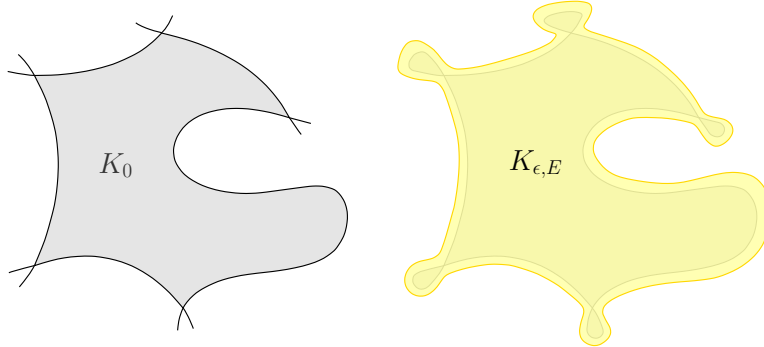


FIGURE 3.1. The compact subset $K_0 \subset \{V \leq 0\}$ before changing the potential (left). The Hill region $K_{\epsilon,E} = \{V_\epsilon \leq E\}$ after capping the critical set (right), with $\epsilon, E > 0$ small.

3.2. Weak convexity. Consider the Hamiltonian $H_\epsilon = |y|^2/2 + V_\epsilon(x)$ as in Proposition 3.2, where $\epsilon > 0$ is sufficiently small. Let $W_{\epsilon,E} = H_\epsilon^{-1}(E)$, $E > 0$ small, be the regular sphere-like hypersurface whose projection to the x -plane contains $K_0 \cup \bigcup_{i=1}^l K_{i,\epsilon}$. Our goal in this section is to prove the following proposition.

Proposition 3.3. *Assume that the Hamiltonian H is real-analytic and the singular sphere-like subset S_0 satisfies conditions H1 and H2 as stated in the introduction. For every $\epsilon > 0$ sufficiently small, the following holds: if $E > 0$ is sufficiently small, then*

- (i) *The Hamiltonian flow of H_ϵ on $W_{\epsilon,E}$ is weakly convex.*
- (ii) *The Lyapunov orbits in $W_{\epsilon,E}$ around the saddle-centers p_1, \dots, p_l are the only index-2 periodic orbits in $W_{\epsilon,E}$.*
- (iii) *If $P \subset W_{\epsilon,E}$ is an index-3 orbit, then P is not linked with any Lyapunov orbit.*
- (iv) *The Lyapunov orbits have the smallest actions among the actions of all periodic orbits in $W_{\epsilon,E}$.*

Before proving Proposition 3.3, we study the linearized flow of a general Hamiltonian $H : \mathbb{R}^4 \rightarrow \mathbb{R}$, not necessarily mechanical, near a saddle-center. We follow [11] and assume that H is real-analytic near the saddle-center.

Lemma 3.4. *Let $0 \in \mathbb{R}^4$ be a saddle-center of a real-analytic Hamiltonian $H = H(x, y)$. Let $U_0 \subset \mathbb{R}^4$ be a small open neighborhood of 0, and let $\{X_1, X_2\}$ be the transverse frame induced by the quaternions, see (2.2), defined on $\text{closure}(U_0) \setminus \{0\}$. Given $M > 0$, there exist compact neighborhoods $U_M \subset U_1$ of 0 contained in the interior of U_0 with the following significance. Let $0 < \lambda \leq 1$, and let $\gamma(t)$, $t \in [-T, T]$, be a trajectory of H satisfying $\gamma(t) \in (\lambda U_1) \setminus \partial(\lambda U_1)$, $\forall t \in (-T, T)$, $\gamma(\pm T) \in \partial(\lambda U_1)$, and $\gamma([-T, T]) \cap (\lambda U_M) \neq \emptyset$. Let $\theta(t)$ be a continuous argument of a non-trivial linearized solution along γ in the frame $\{X_1, X_2\}$, see (2.3). Then $\theta(T) - \theta(-T) > M$.*

Proof. Since H is real-analytic near 0, there exist real-analytic symplectic coordinates $(q, p) = \varphi(x, y)$ near $0 \in \mathbb{R}^4$ so that H (or $-H$) takes the form

$$(3.3) \quad K = -\alpha I_1 + \omega I_2 + R(I_1, I_2),$$

where $\alpha, \omega > 0$, $I_1 = q_1 p_1$, $I_2 = (q_2^2 + p_2^2)/2$ and $R(I_1, I_2) = O(I_1^2 + I_2^2)$. These coordinates are due to J. Moser [38] and H. Rüssmann [41]. The symplectic form in the new coordinates (q, p) is $\sum_i dp_i \wedge dq_i$ and the equations of motion are

$$\begin{cases} \dot{q}_1 = -\bar{\alpha} q_1 \\ \dot{p}_1 = \bar{\alpha} p_1 \end{cases} \quad \begin{cases} \dot{q}_2 = \bar{\omega} p_2 \\ \dot{p}_2 = -\bar{\omega} q_2 \end{cases}$$

where $\bar{\alpha} := \alpha - \partial_{I_1} R$ and $\bar{\omega} := \omega + \partial_{I_2} R$. Since I_1 and I_2 are preserved by the flow, $\bar{\alpha}$ and $\bar{\omega}$ do not depend on time, and the solutions are

$$(3.4) \quad \begin{aligned} q_1(t) &= q_1(0)e^{-\bar{\alpha}t}, & p_1(t) &= p_1(0)e^{\bar{\alpha}t}, \\ q_2(t) + ip_2(t) &= e^{-i\bar{\omega}t}(q_2(0) + ip_2(0)). \end{aligned}$$

Let $\delta > 0$ be small so that in coordinates (q, p) we have $B_\delta(0) \subset \varphi(U_0)$. Let $0 < \lambda \leq 1$. Consider a solution $\gamma(t) = (q_1(t), q_2(t), p_1(t), p_2(t)) \in B_{\lambda\delta}(0)$ satisfying

$$(3.5) \quad \begin{aligned} q_1(0) = p_1(0) = \lambda b > 0 \quad \text{or} \quad q_1(0) = -p_1(0) = \lambda b > 0, \\ \text{and } q_2(0) + ip_2(0) = 0 + i\lambda r, \end{aligned}$$

for some $b > 0$ and $r \geq 0$, so that $r^2 + 2b^2 < \delta^2$. The case $b < 0$ is analogous. Such trajectories escape $B_{\lambda\delta}(0)$ both forward and backward in time, so they are called escaping trajectories.

Consider the symplectic frame $\{Y_1, Y_2\}$ induced by the quaternions in coordinates (q, p) . Notice that it may not coincide with the initial frame $\{X_1, X_2\}$. Let $\eta(t)$ be a continuous argument of a non-trivial linearized solution along γ in the frame $\{Y_1, Y_2\}$. Observe that

$$(3.6) \quad q_1(t)^2 + p_1(t)^2 = 2\lambda^2 b^2 \cosh(2\bar{\alpha}t) < \lambda^2 \delta^2$$

for every t satisfying $\gamma(t) \in B_{\lambda\delta}(0)$. Let $T > 0$ be such that $\gamma(\pm T) \in \partial B_{\lambda\delta}(0)$.

Claim. If $\delta > 0$ is sufficiently small, then $\eta(T) - \eta(-T) > \omega T - 2\pi$.

We follow the computation in [11] to prove the claim. Since

$$\begin{aligned} \kappa_{11} &= \frac{1}{|\nabla K|^2} (\bar{\alpha}^2 \bar{\omega} (q_1^2 + p_1^2) - 2\bar{\alpha} \bar{\omega}^2 q_2 p_2 + \bar{r} (q_1 q_2 + p_1 p_2)^2), \\ \kappa_{12} &= \frac{1}{|\nabla K|^2} (\bar{\alpha} \bar{\omega}^2 (p_2^2 - q_2^2) + \bar{r} (q_2 p_1 - q_1 p_2) (q_1 q_2 + p_1 p_2)), \\ \kappa_{22} &= \frac{1}{|\nabla K|^2} (\bar{\alpha}^2 \bar{\omega} (q_1^2 + p_1^2) + 2\bar{\alpha} \bar{\omega}^2 q_2 p_2 + \bar{r} (q_2 p_1 - q_1 p_2)^2), \\ \kappa_{33} &= \frac{1}{|\nabla K|^2} (2\bar{\alpha}^3 q_1 p_1 + \bar{\omega}^3 (q_2^2 + p_2^2)), \end{aligned}$$

where the functions $\bar{r} := r_{11}\bar{\omega}^2 + 2r_{12}\bar{\alpha}\bar{\omega} + r_{22}\bar{\alpha}^2$, $r_{ij} := \partial_{I_i I_j}^2 R$, $i, j = 1, 2$, are constant along the trajectories, we use (2.4), (3.4), and (3.5) to find

$$\begin{aligned} \dot{\eta} = \bar{\omega} + \frac{1}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t) + \bar{\omega}^2 r^2} \{ \epsilon 2\bar{\alpha}^3 b^2 + (\bar{\alpha} \bar{\omega}^2 r^2 - \epsilon \lambda^2 b^2 r^2 \bar{r}) \sin(2\eta(t) - 2\bar{\omega}t) + \\ \lambda^2 b^2 r^2 \bar{r} [\cosh(2\bar{\alpha}t) + \sinh(2\bar{\alpha}t) \cos(2\eta(t) - 2\bar{\omega}t)] \}. \end{aligned}$$

Notice that \bar{r} is uniformly bounded on $B_\delta(0)$ by some constant $c_1 > 0$ that does not depend on δ . The sign $\epsilon \in \{+1, -1\}$ in the expression for $\dot{\eta}$ depends on the initial conditions in (3.5). Since $\bar{\alpha} > 0$ for $\delta > 0$ small, we may assume that $\epsilon = -1$. The case $\epsilon = +1$ is simpler due to the positivity of the term $\epsilon 2\bar{\alpha}^3 b^2$ and was treated in [11]. First we consider the case $\bar{\omega}^2 r^2 \leq 4\bar{\alpha}^2 b^2$. Using (3.6), we obtain for every $\delta > 0$ sufficiently small

$$\begin{aligned} \dot{\eta} - \bar{\omega} &> \frac{-\bar{\alpha}(2\bar{\alpha}^2 b^2 + \bar{\omega}^2 r^2) - 2c_1 r^2 \delta^2}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t) + \bar{\omega}^2 r^2} \\ &\geq \frac{-\bar{\alpha}(2\bar{\alpha}^2 b^2 + 4\bar{\alpha}^2 b^2) - 2c_1 \delta^2 4\bar{\alpha}^2 b^2 / \bar{\omega}^2}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t) + \bar{\omega}^2 r^2} \\ &> \frac{-4\bar{\alpha}}{\cosh(2\bar{\alpha}t)}. \end{aligned}$$

Since

$$0 < \int_{-T}^T \frac{\bar{\alpha}}{\cosh(2\bar{\alpha}t)} dt < \int_{-\infty}^{+\infty} \frac{\bar{\alpha}}{\cosh(2\bar{\alpha}t)} dt = \frac{\pi}{2},$$

we obtain $\eta(T) - \eta(-T) > \bar{\omega} 2T - 2\pi > \omega T - 2\pi$, proving the claim in the case $\bar{\omega}^2 r^2 \leq 4\bar{\alpha}^2 b^2$.

Now assume that $\bar{\omega}^2 r^2 > 4\bar{\alpha}^2 b^2$. If $\delta > 0$ is sufficiently small and $\eta(t_*) = \bar{\omega}t_* + \frac{\pi}{4} + k_0\pi$ for some $k_0 \in \mathbb{Z}$, then

$$\begin{aligned} \dot{\eta}(t_*) &= \bar{\omega} + \frac{\bar{\alpha}(\bar{\omega}^2 r^2 - 2\bar{\alpha}^2 b^2) + \lambda^2 b^2 r^2 \bar{r} + \lambda^2 b^2 r^2 \bar{r} \cosh(2\bar{\alpha}t_*)}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t_*) + \bar{\omega}^2 r^2} \\ &= \bar{\omega} + \frac{\bar{\alpha}(\bar{\omega}^2 r^2 - 2\bar{\alpha}^2 b^2) + r^2 O(\delta^2)}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t_*) + \bar{\omega}^2 r^2} \\ &> \bar{\omega} + \frac{\bar{\alpha}\bar{\omega}^2 r^2/2 + r^2 O(\delta^2)}{2\bar{\alpha}^2 b^2 \cosh(2\bar{\alpha}t_*) + \bar{\omega}^2 r^2} \\ &> \bar{\omega}. \end{aligned}$$

This forces $\eta(T) - \eta(-T) > \bar{\omega}2T - \pi > \omega T - \pi$, and the claim is proved.

We see from (3.4) that given $M_0 > 0$, there exists $0 < \delta_{M_0} \ll \delta$ so that if $\gamma([-T, T]) \cap B_{\lambda\delta_{M_0}}(0) \neq \emptyset$, then $T > (M_0 + 2\pi)/\omega$ and thus in view of the claim above we obtain $\eta(T) - \eta(-T) > M_0$.

Now consider the frame $\{X_1, X_2\}$ as in the statement, defined in local coordinates (x, y) . Up to a projection along the Hamiltonian vector field, we may assume that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ span the same plane field transverse to the flow. Hence we may write $Y_1 = aX_1 + bX_2$, where $a, b: \text{closure}(B_\delta(0)) \setminus \{0\} \rightarrow \mathbb{R}$ are smooth. Since $\text{closure}(B_\delta(0)) \setminus \{0\}$ is simply connected, $Y_1 \equiv a + ib$ admits a well-defined continuous argument, that is $a + ib \in \mathbb{R}^+ e^{i\hat{\zeta}}$ for some smooth function $\hat{\zeta}: \text{closure}(B_\delta(0)) \setminus \{0\} \rightarrow \mathbb{R}$. Let $C_\lambda := \sup_{z_1, z_2 \in \partial B_{\lambda\delta}(0)} |\hat{\zeta}(z_1) - \hat{\zeta}(z_2)| < +\infty$. Since the frames $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ have a C_{loc}^∞ -limit under the re-scaling $K(\lambda q, \lambda p)/\lambda^2 \rightarrow -\alpha I_1 + \omega I_2$ as $\lambda \rightarrow 0^+$, we may assume that $\sup_{0 < \lambda \leq 1} C_\lambda < C < +\infty$. Let $\gamma(t), t \in [-T, T]$, be an escaping trajectory as above so that $\gamma(0) \in B_{\lambda\delta_{M_0}}(0)$ and $\gamma(\pm T) \in \partial B_{\lambda\delta}(0)$. Consider a non-trivial solution to the linearized flow along $\gamma(t)$ in the frame $\{X_1, X_2\}$, let $\theta(t), \eta(t)$ be continuous arguments of this solution in the frames $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$, respectively. Then $\theta(T) - \theta(-T) > \eta(T) - \eta(-T) - 2C - 2\pi$. Hence, choosing $M_0 > 0$ as above so that $M_0 > 2C + \pi + M$, we obtain that $\theta(T) - \theta(-T) > M$, as desired.

Finally, taking $\delta_M > 0$ even smaller if necessary, we obtain the desired sets in local coordinates by defining $U_1 := \text{closure}(B_\delta(0))$ and $U_M := \text{closure}(B_{\delta_M}(0))$. \square

Now, we use Lemma 3.4 and some properties of mechanical systems to show that every periodic orbit sufficiently close to a saddle-center has an arbitrarily high index.

Proposition 3.5. *Let $0 \in \mathbb{R}^4$ be a saddle-center of a real-analytic mechanical Hamiltonian $H(x, y) = |y|^2/2 + V(x)$ and let $M > 0$ be given. Then there exists a small compact neighborhood $V_M \subset \mathbb{R}^4$ of 0 such that if $\gamma \subset H^{-1}(E), \gamma \neq P_{2,E}$, is a periodic orbit intersecting V_M , then $\mu(\gamma) > M$. Here, $P_{2,E} \subset H^{-1}(E)$ is the Lyapunov orbit near 0 .*

Proof. Consider the transverse frame $\{X_1, X_2\}$ induced by the quaternions and defined on the regular points of H . Consider also the transverse frame on $TH^{-1}(E) \setminus \{y_1 = y_2 = 0\}$ induced by the vertical and horizontal lifts of $(y_1, y_2)^\perp = (-y_2, y_1)$ under the projection $(x, y) \mapsto x$. Indeed, there exist non-vanishing independent

vector fields \hat{X}_1, \hat{X}_2 tangent to $H^{-1}(E) \setminus \{y_1 = y_2 = 0\}$ given by

$$\begin{aligned}\hat{X}_1 &= (0, 0, -y_2, y_1) \\ &= -(y_1 V_{x_1} + y_2 V_{x_2})gX_1 + 2(E - V(x))gX_2 + (y_2 V_{x_1} - y_1 V_{x_2})X_3, \\ \hat{X}_2 &= (2(E - V(x)))^{-1}(-y_2, y_1, -V_{x_2}, V_{x_1}) \\ &= -(2(E - V(x))g)^{-1}X_1,\end{aligned}$$

where $V_{x_i} = \partial_{x_i} V$, $i = 1, 2$, $g = (V_{x_1}^2 + V_{x_2}^2 + y_1^2 + y_2^2)^{-1/2}$, so that $\text{span}\{\hat{X}_1, \hat{X}_2\}$ is transverse to the Hamiltonian vector field $X_H = (y_1, y_2, -V_{x_1}, -V_{x_2})$. A linearized solution $\eta = \alpha_1 \hat{X}_1 + \alpha_2 \hat{X}_2 + \alpha_3 X_H$, with $\alpha_1^2 + \alpha_2^2 \neq 0$, along a trajectory γ satisfies the following condition: if $\alpha_2 = 0$ and $\alpha_1 > 0$, then $\dot{\alpha}_2 > 0$. Indeed, from $\dot{\eta} = DX_H(\gamma)\eta$ and $DX_H \cdot X_H = \frac{d}{dt}X_H \circ \gamma$, we obtain

$$\alpha_1 \langle DX_H \cdot \hat{X}_1, \hat{X}_2 \rangle = \alpha_1 \langle \hat{X}'_1, \hat{X}_2 \rangle + \dot{\alpha}_2 \langle \hat{X}_2, \hat{X}_2 \rangle,$$

where $\hat{X}'_1 = \frac{d}{dt}\hat{X}_1 \circ \gamma$. Since

$$\langle DX_H \cdot \hat{X}_1, \hat{X}_2 \rangle = \langle (-y_2, y_1, 0, 0), \hat{X}_2 \rangle > 0,$$

and

$$\langle \hat{X}'_1, \hat{X}_2 \rangle = \langle (0, 0, V_{x_2}, -V_{x_1}), \hat{X}_2 \rangle \leq 0,$$

we conclude that $\dot{\alpha}_2 > 0$ if $\alpha_2 = 0$ and $\alpha_1 > 0$. In particular, the linearized flow in the frame $\{\hat{X}_1, \hat{X}_2\}$ does not rotate backward more than π . This means that if $\eta(t)$ is a continuous argument of $\alpha_1(t) + i\alpha_2(t) \neq 0$ and $\eta(t) = \eta_*$, then $\eta(s) > \eta_* - \pi$ for every $s > t$. Since \hat{X}_2 is parallel to X_1 , the frame induced by $\{\hat{X}_1, \hat{X}_2\}$ and $\{X_1, X_2\}$ do not wind with respect to each other. Hence, the argument $\theta(t)$ of a linearized solution in the frame $\{X_1, X_2\}$ satisfies a similar property. As a final remark, if a simple periodic orbit touches the boundary of the Hill region, then the contribution to the variation of the argument $\theta(t)$ is bounded since that happens at most twice along the minimal period.

Given $M' > 0$, we know from Lemma 3.4 that there exist small neighborhoods $U_{M'} \subset U_1 \subset \mathbb{R}^4$ of 0 so that if $\gamma \neq P_{2,E} \subset H^{-1}(E)$, $E > 0$ small, is a simple periodic trajectory intersecting $U_{M'}$, and not contained in U_1 , then the variation in the argument $\eta(t)$ of a non-trivial transverse linearized solution inside U_1 in the frame $\{X_1, X_2\}$ is greater than M' . From the reasoning above, the total variation of $\eta(t)$ along the whole period of γ is greater than $M' - C$, for some fixed $C > 0$ that does not depend on γ or M' . If $M' > 0$ is taken sufficiently large, then $\mu(\gamma) > M$, as desired. \square

Corollary 3.6. *Let $0 \in \mathbb{R}^4$ be a saddle-center of a real-analytic mechanical Hamiltonian $H(x, y) = |y|^2/2 + V(x)$. Given $M > 0$, there exists $E_M > 0$ so that if $0 < E < E_M$ and $\gamma \subset H^{-1}(E)$ is a simple periodic orbit, linked with the Lyapunov orbit $P_{2,E} \subset H^{-1}(E)$ near 0, then $\mu(\gamma) > M$. Iterates of γ also have index greater than M .*

Proof. Take V_M as in Proposition 3.5. Consider the local coordinates (q, p) as in the proof of Lemma 3.4, so that the Hamiltonian has the form $K = -\alpha I_1 + \omega I_2 + R(I_1, I_2)$, where $I_1 = q_1 p_1$ and $I_2 = (q_2^2 + p_2^2)/2$. Recall that in these coordinates, the Lyapunov orbit is given by $P_{2,E} = \{q_1 = p_1 = 0\} \cap K^{-1}(E)$, where $E > 0$ is

small. Notice that $P_{2,E}$ is the boundary of the embedded disks

$$U_{1,E} := \{p_1 = -q_1 \geq 0, I_1(E) \leq q_1 p_1 \leq 0, I_2 \leq I_2(E)\} \cap K^{-1}(E),$$

$$U_{2,E} := \{p_1 = -q_1 \leq 0, I_1(E) \leq q_1 p_1 \leq 0, I_2 \leq I_2(E)\} \cap K^{-1}(E),$$

where $I_1(E) = -\frac{E}{\alpha} + O(E^2) < 0$ solves $E = -\alpha I_1 + R(I_1, 0)$, and $I_2(E) > 0$ solves $E = \omega I_2 + R(0, I_2)$, for every $E > 0$ small. Observe that the interior of such disks is transverse to the flow, and $I_1(E), I_2(E) \rightarrow 0$ as $E \rightarrow 0$. Hence we find $E_M > 0$ sufficiently small such that $U_{1,E}, U_{2,E} \subset V_M$ for every $0 < E < E_M$. In particular, any periodic orbit $\gamma \subset H^{-1}(E)$, with $0 < E < E_M$, that is linked with $P_{2,E}$, must intersect $U_{1,E}$ and $U_{2,E}$ and thus intersects V_M . This implies $\mu(\gamma) > M$. \square

Proof of Proposition 3.3. Recall from the proof of Lemma 3.1 that in re-scaled coordinates $\hat{x} = x/\sqrt{\epsilon}$ near the saddle v_i , the re-scaled potential $\tilde{V}_\epsilon(\hat{x})$ converges in C_{loc}^∞ to $\tilde{V}_0(\hat{x}) := a\hat{x}_1^2/2 + b\hat{x}_2^2/2 + f(\hat{x}_1)$ as $\epsilon \rightarrow 0^+$, where $a < 0$, $b > 0$, and f satisfies (3.1). Denote $\tilde{H}_\epsilon(\hat{x}, \hat{y}) := |\hat{y}|^2/2 + \tilde{V}_\epsilon(\hat{x})$. Recall that $\{\tilde{V}_\epsilon \leq 0\}$ contains a disk-like region $\tilde{K}_{i,\epsilon} \subset \{0 \leq \hat{x}_1 < 2\}$ with a singularity at $0 \in \mathbb{R}^2$, corresponding to the capping of K_0 near v_i .

For $M > 0$ large, consider the sets $U_M \subset U_1$ as in Lemma 3.4. In re-scaled coordinates (\hat{x}, \hat{y}) , they become $\tilde{U}_M := \epsilon^{-1}U_M \subset \tilde{U}_1 := \epsilon^{-1}U_1$. There exists $\delta_1 > 0$ such that for every $\epsilon > 0$ sufficiently small, we find $0 < \lambda \leq 1$ satisfying $B_{\delta_1}(0) \subset \lambda\tilde{U}_M$ and $\lambda\tilde{U}_1 \subset \{\hat{x}_1 < 1\}$. This implies that for $M > 0$ large, any periodic orbit $P \subset \tilde{H}_\epsilon^{-1}(E)$, $P \neq P_{2,E}$, intersecting $B_{\delta_1}(0)$, has index greater than 3.

We shall find $\epsilon > 0$ small enough so that $\tilde{H}_\epsilon^{-1}(E)$ is dynamically convex on $0 < \hat{x}_1 < 2$ for every $E > 0$ sufficiently small. Indeed, assume by contradiction the existence, for every $\epsilon > 0$ small and $E > 0$ arbitrarily close to 0, of an orbit $\tilde{Q}_{2,\epsilon,E}$ contained in the capped region $\tilde{H}_\epsilon^{-1}(E) \cap \{0 < \hat{x}_1 < 2\}$ so that its index is ≤ 2 . From the considerations above, we find $\epsilon_1 > 0$ small so that $\tilde{Q}_{2,\epsilon,E} \subset \{\epsilon_1 < \hat{x}_1 < 2\}$ for every $\epsilon > 0$ sufficiently small and $E > 0$ arbitrarily small. The limiting dynamics as $\epsilon \rightarrow 0^+$ is determined by the decoupled Hamiltonian $\tilde{H}_0 = \tilde{H}_1(\hat{x}_1, \hat{y}_1) + \tilde{H}_2(\hat{x}_2, \hat{y}_2)$, where $\tilde{H}_1 = \hat{y}_1^2/2 + a\hat{x}_1^2/2 + f(\hat{x}_1)$ and $\tilde{H}_2 = \hat{y}_2^2/2 + b\hat{x}_2^2/2$. Notice that the orbit $\tilde{Q}_{2,\epsilon,E} \subset \tilde{H}_\epsilon^{-1}(E)$, $E > 0$ small, cannot get arbitrarily close to 0 in the (\hat{x}_2, \hat{y}_2) -plane as $E \rightarrow 0^+$, otherwise it would intersect $\hat{x}_1 = \epsilon_1$, a contradiction. Hence, if $\epsilon > 0$ is fixed sufficiently small, the action of $\tilde{Q}_{2,\epsilon,E}$ is uniformly bounded. Otherwise, its index would be > 2 due to the contribution to the index of the linearized flow on the (\hat{x}_2, \hat{y}_2) -plane. Taking the limit $E \rightarrow 0^+$, and then $\epsilon \rightarrow 0^+$, we obtain a periodic orbit $\tilde{Q} \subset \tilde{H}_0^{-1}(0) \cap \{\epsilon_1 \leq \hat{x}_1 < 2\}$ with index ≤ 2 . This is a contradiction with Proposition 4.3 below; in fact, it follows from (3.1) and Proposition 4.3 that, except for the index-2 orbit $\hat{P}_{2,0,E} = \{\hat{x}_1 = \hat{y}_1 = 0, b\hat{x}_2^2 + \hat{y}_2^2 = 2E\}$, all periodic orbits in $\tilde{H}_0^{-1}(E)$ have index ≥ 3 . We conclude that $\tilde{H}_\epsilon^{-1}(E)$ is dynamically convex in the capped region $0 < \hat{x}_1 < 2$ for every $\epsilon > 0$ fixed sufficiently small and $E > 0$ sufficiently small.

For the original capped Hamiltonian H_ϵ , with $\epsilon > 0$ sufficiently small as above, we see that since S_0 is dynamically convex and admits a positive frame, $W_{\epsilon,E}$ is weakly convex and the Lyapunov orbits near p_i are the only index-2 orbit for $E > 0$ sufficiently small. Indeed, if another orbit $Q_{2,\epsilon,E} \subset W_{\epsilon,E}$ exists, then the argument above shows that it cannot be contained in the capped region and stays away from all p_i as $E \rightarrow 0^+$. Due to the positive frame for S_0 , its period is uniformly bounded in $E > 0$, and thus converges up to a subsequence to a periodic orbit

$Q_{2,\epsilon,0} \subset S_0 \setminus \{p_1, \dots, p_l\}$. Its index is ≤ 2 , contradicting the dynamical convexity of S_0 .

Finally, let us check that the Lyapunov orbits have the smallest actions among all periodic orbits. We shall see in Section 3.3 below that $H_\epsilon^{-1}(E)$ admits a contact form λ_E which converges in C_{loc}^∞ to λ_0 as $E \rightarrow 0^+$, where λ_0 is a contact form on $H_\epsilon^{-1}(0) \setminus \{p_1, \dots, p_l\}$. The action of a periodic orbit Q of λ_E is defined by $\int_Q \lambda_E$. Fix $\epsilon > 0$ sufficiently small as before and consider re-scaled coordinates $(\tilde{x}, \tilde{y}) = (x/\sqrt{E}, y/\sqrt{E})$ near some p_i . Let $\tilde{H}_E(\tilde{x}, \tilde{y}) := H_\epsilon(x, y)/E$. Then $H_\epsilon^{-1}(E)$ corresponds to $\tilde{H}_E^{-1}(1)$. As $E \rightarrow 0$, the Lyapunov orbit $P_{2,E} \subset H_\epsilon^{-1}(E)$ in coordinates (\tilde{x}, \tilde{y}) converges in C^∞ to $\{\tilde{x}_1 = \tilde{y}_1 = 0, b\tilde{x}_2^2 + \tilde{y}_2^2 = 2\}$. Hence, for $E > 0$ sufficiently small, $P_{2,E}$ has action $2\pi E/\sqrt{b} + O(E^2)$, that is $\mathcal{A}(P_{2,E}) = O(E) \rightarrow 0$ as $E \rightarrow 0$. Suppose, by contradiction, that, for every $E > 0$, there exists a periodic orbit $Q_E \subset H_\epsilon^{-1}(E)$ so that Q_E is not a Lyapunov orbit and $\mathcal{A}(Q_E) \rightarrow 0$ as $E \rightarrow 0$. Since the Lyapunov orbits are hyperbolic and Q_E is not a Lyapunov orbit, we see that Q_E cannot converge to any p_i . Hence, up to a subsequence, Q_E converges to non-constant trajectories of λ_0 , possibly homoclinic/heteroclinic orbits connecting the saddle-centers, contradicting $\mathcal{A}(Q_E) \rightarrow 0$ as $E \rightarrow 0^+$. \square

3.3. The contact property. In this section, $H = H_\epsilon = \frac{1}{2}|y|^2 + V_\epsilon(x)$ is as in Proposition 3.3, where $\epsilon > 0$ is fixed sufficiently small. We show that for $E > 0$ sufficiently small, the regular sphere-like hypersurface $W_E = H^{-1}(E)$ near S_0 has contact type, and the induced contact structure $\xi = \ker \lambda_E$ is tight; see, for instance, [1, Lemma 4.1]. The critical set $H^{-1}(0)$ is denoted by W_0 . It contains S_0 whose projection to the x -plane is the compact disk-like set K_0 .

In the next section, we shall define an almost complex structure J_E on $\mathbb{R} \times W_E$, which admits a pair of holomorphic planes asymptotic to the Lyapunov orbits, see section 2.4. To control the location of these holomorphic planes, it will be necessary to choose a particular contact form λ_E on W_E as in the following statement.

Proposition 3.7. *The following assertions hold.*

- (i) *For every $E > 0$ sufficiently small, there exists a Liouville vector field X_E defined on a neighborhood of the sphere-like hypersurface $W_E = H^{-1}(E) \subset \mathbb{R}^4$, which is transverse to W_E and the induced contact form $\lambda_E := -\iota_{X_E}\omega_0$ on W_E is tight. Moreover, for suitable symplectic coordinates near each saddle-center p_i so that the Hamiltonian has the form $H = \frac{1}{2}(y_1^2 + y_2^2 + ax_1^2 + bx_2^2) + R(x)$, where $a < 0, b > 0, R = O(|x|^3)$, we can assume that $X_E = \frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2} + y_1\partial_{y_1} + y_2\partial_{y_2})$ and thus $\lambda_E = \frac{1}{2}(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)|_{W_E}$ for every $|x_1| < c\sqrt{E}$, where $c > 0$ is independent of $E > 0$.*
- (ii) *There exists a Liouville vector field X_0 defined on a neighborhood of $\dot{W}_0 := H^{-1}(0) \setminus \{p_1, \dots, p_l\}$, which is transverse to \dot{W}_0 . Given any neighborhood $\mathcal{U} \subset \mathbb{R}^4$ of $\{p_1, \dots, p_l\}$, we have $X_0 = X_E$ outside \mathcal{U} for every $E > 0$ sufficiently small. In particular, $\lambda_E \rightarrow \lambda_0$ in $C_{loc}^\infty(\dot{W}_0)$ as $E \rightarrow 0^+$, where $\lambda_0 := -\iota_{X_0}\omega$ is the induced contact form on \dot{W}_0 .*

Proof. We need to interpolate a few Liouville vector fields to obtain the desired X_E . We start with the Liouville vector field

$$\mathcal{Z}_0 := y_1\partial_{y_1} + y_2\partial_{y_2},$$

which is transverse to \dot{W}_0 except at the points of \dot{W}_0 projecting to $\partial\Omega_0$. The set $\dot{W}_0 \cap \{y = 0\} \subset H^{-1}(0)$ is formed by finitely many embedded curves $\gamma_1, \dots, \gamma_n, n = 2l$, projecting to the corresponding embedded curves $\hat{\gamma}_1, \dots, \hat{\gamma}_n \subset \partial\Omega_0 \setminus \{v_1, \dots, v_l\}$.

Fix $\gamma = \gamma_j$ for some j , and consider the quaternion vector fields X_0, X_1, X_2, X_3 defined on regular points of H , see (2.5). Since $T\gamma = \mathbb{R}X_2|_\gamma$, the plane field $\text{span}\{X_1, X_3\}|_\gamma \subset T\dot{W}_0$ is transverse to γ . Take a parametrization $\gamma = \gamma(t), t \in I = [0, 1]$, so that $\gamma(0) = p_i = 0$ for some i , and consider coordinates $(\tilde{x}, \tilde{y}, \tilde{z}, t)$ on a tubular neighborhood $\mathcal{U} \equiv \tilde{B} := B_r(0) \times I$, $r > 0$ small, of γ , given by $(\tilde{x}, \tilde{y}, \tilde{z}, t) \mapsto \exp_{\gamma(t)}(\tilde{x}X_1 + \tilde{y}X_3 + \tilde{z}X_0) \in \mathbb{R}^4$. Here, $B_r(0) \subset \mathbb{R}^3$ is the open ball of radius $r > 0$ centered at 0 and \exp is the exponential map induced by the Euclidean metric. Let $\beta = \beta(\tilde{x}, \tilde{y}, \tilde{z}, t) \in [0, 1]$ be a smooth function so that $\beta = 0$ near $\partial\tilde{B}$ and $\beta = 1$ near $L := 0 \times (r, 1 - r)$. Let $f(\tilde{x}, \tilde{y}, \tilde{z}, t) := -\tilde{y}\beta(\tilde{x}, \tilde{y}, \tilde{z}, t)$ be defined on \tilde{B} . Then X_f vanishes near $\partial\tilde{B}$ and $X_f = X_0$ is positively transverse to \dot{W}_0 near L . Hence

$$\mathcal{Z}_\varepsilon := X_{\varepsilon f} + \mathcal{Z}_0$$

is transverse to \dot{W}_0 near L for every $\varepsilon > 0$ sufficiently small. Repeating the same construction near each γ_j , we obtain a Liouville vector field still denoted by \mathcal{Z}_ε , which is positively transverse to \dot{W}_0 away from arbitrarily small neighborhoods of $p_i, i = 1, \dots, l$. We may assume the existence of functions $f_i = f$ as above, defined on mutually disjoint open neighborhoods $\mathcal{U}_i = \mathcal{U}$ of γ_i and they vanish near $\partial\mathcal{U}_i$. They give rise to a smooth function f supported in the interior of $\cup_i \mathcal{U}_i$.

Let $p_c := p_i$ for some i . We may assume that $p_c = 0 \in \mathbb{R}^4$ and near p_c the Hamiltonian writes as $H(x, y) = y_1^2/2 + y_2^2/2 + ax_1^2/2 + bx_2^2/2 + R(x)$, where $a < 0, b > 0$ and $R = O(|x|^3)$. We consider the subset of \dot{W}_0 near 0 that is contained in $\{x_1 > 0\}$. Then there exists $C > 0$ such that

$$(3.7) \quad x_1 \geq C|x_2| \geq 0$$

for every $(x_1, x_2, y_1, y_2) \in \dot{W}_0 \cap \{x_1 > 0\} \cap B_{\delta_1}(0)$, where δ_1 is fixed sufficiently small. Let

$$\mathcal{Z} := \frac{1}{2}(x_1 - \delta_1)\partial_{x_1} + \frac{1}{2}x_2\partial_{x_2} + \frac{1}{2}y_1\partial_{y_1} + \frac{1}{2}y_2\partial_{y_2}$$

be the radial Liouville vector field centered at $(\delta_1, 0, 0, 0)$. Then \mathcal{Z} is positively transverse to \dot{W}_0 on $\{(x_1 - \delta_1)^2 + x_2^2 < (3\delta_1)^2, x_1 > 0\}$, provided $\delta_1 > 0$ is small enough. Indeed, we find

$$dH \cdot \mathcal{Z}|_{\dot{W}_0} = -\frac{1}{2}\delta_1 ax_1 - R + \frac{1}{2}(x_1 - \delta_1)\partial_{x_1}R + \frac{1}{2}x_2\partial_{x_2}R,$$

which is positive in that region if $\delta_1 > 0$ is sufficiently small. We have used (3.7). Furthermore, this also shows that for every $E > 0$ sufficiently small, we have

$$(3.8) \quad dH \cdot \mathcal{Z} = E - \frac{1}{2}\delta_1 ax_1 - R + \frac{1}{2}(x_1 - \delta_1)\partial_{x_1}R + \frac{1}{2}x_2\partial_{x_2}R > 0$$

on $W_E \cap \{(x_1 - \delta_1)^2 + x_2^2 < (3\delta_1)^2, x_1 \geq 0\}$, if $\delta_1 > 0$ is fixed sufficiently small, see (3.7). On the other hand, from the construction above, we may assume that \mathcal{Z}_ε is positively transverse to \dot{W}_0 on $\{(2\delta_1)^2 \leq (x_1 - \delta_1)^2 + x_2^2 \leq (4\delta_1)^2, x_1 > 0\}$. This can be achieved by taking $r > 0$ sufficiently small.

Now we construct an interpolation between \mathcal{Z} and \mathcal{Z}_ε on $\{(2\delta_1)^2 \leq (x_1 - \delta_1)^2 + x_2^2 \leq (3\delta_1)^2, x_1 > 0\}$. Let $\ell(x_1, x_2, y_1, y_2) := -(x_1 - \delta_1)y_1/2 - x_2y_2/2 + \varepsilon f$, where f is as above. Notice that the Hamiltonian vector field of ℓ satisfies $X_\ell = \mathcal{Z}_0 + X_{\varepsilon f} - \mathcal{Z} =$

$\mathcal{Z}_\varepsilon - \mathcal{Z}$. Let $\tilde{h} := h((x_1 - \delta_1)^2 + x_2^2)$ be defined near $0 \in \mathbb{R}^4$, where $h: \mathbb{R} \rightarrow [0, 1]$ is smooth and satisfies $h(t) = 0$ if $t \leq (2\delta_1)^2$, $h(t) = 1$ if $t \geq (3\delta_1)^2$, and $h'(t) > 0$ if $(2\delta_1)^2 < t < (3\delta_1)^2$. Let

$$\tilde{\mathcal{Z}}_\varepsilon := \mathcal{Z} + X_{\ell\tilde{h}}.$$

Then $\tilde{\mathcal{Z}}_\varepsilon = \mathcal{Z}$ on $\{(x_1 - \delta_1)^2 + x_2^2 \leq (2\delta_1)^2, x_1 > 0\}$ and $\tilde{\mathcal{Z}}_\varepsilon = \mathcal{Z}_\varepsilon$ on $\{(x_1 - \delta_1)^2 + x_2^2 \geq (3\delta_1)^2, x_1 > 0\}$.

We claim that $\tilde{\mathcal{Z}}_\varepsilon$ is transverse to \dot{W}_0 near 0 for every $\varepsilon > 0$ sufficiently small. It suffices to consider the set $A_0 := \{(2\delta_1)^2 < (x_1 - \delta_1)^2 + x_2^2 < (3\delta_1)^2, x_1 > 0\} \cap \dot{W}_0$. Notice that

$$dH \cdot \tilde{\mathcal{Z}}_\varepsilon = (1 - \tilde{h})dH \cdot \mathcal{Z} + \tilde{h}dH \cdot \mathcal{Z}_\varepsilon + \ell dH \cdot X_{\tilde{h}}.$$

The set $\{(x_1 - \delta_1)^2 + x_2^2 = (3\delta_1)^2, x_1 > 0\} \cap \{y = 0\} \cap \dot{W}_0$ is formed by two points p_1, p_2 projecting to the boundary of the Hill region. Since $dH \cdot \tilde{\mathcal{Z}}_\varepsilon|_{p_i} = dH \cdot \mathcal{Z}_\varepsilon|_{p_i} = \varepsilon dH \cdot X_f, i = 1, 2$, we find a neighborhood $\mathcal{U} \subset \dot{W}_0$ of $\{p_1, p_2\}$ and $\hat{c} > 0$ such that $dH \cdot \tilde{\mathcal{Z}}_\varepsilon|_{\mathcal{U}} > \hat{c}\varepsilon > 0$ for every $\varepsilon > 0$ sufficiently small. Notice that $\hat{c} > 0$ does not depend on ε . Since both \mathcal{Z} and \mathcal{Z}_ε are positively transverse to A_0 , we find $\sigma > 0$, independent of ε , such that $(1 - \tilde{h})dH \cdot \mathcal{Z} + \tilde{h}dH \cdot \mathcal{Z}_\varepsilon > \sigma$ on $A_0 \setminus \mathcal{U}$. For the last term of $dH \cdot \tilde{\mathcal{Z}}_\varepsilon$, we compute

$$\ell dH \cdot X_{\tilde{h}} = h' \cdot [((x_1 - \delta_1)y_1 + x_2y_2)^2 - 2\varepsilon f \cdot ((x_1 - \delta_1)y_1 + x_2y_2)].$$

Recall that f is bounded, and $\varepsilon > 0$ can be taken arbitrarily small. Since $h' \geq 0$, we conclude that $\ell dH \cdot X_{\tilde{h}}|_{A_0 \setminus \mathcal{U}} > -\sigma$ for every $\varepsilon > 0$ sufficiently small. Hence $dH \cdot \tilde{\mathcal{Z}}_\varepsilon > 0$ on A_0 for every $\varepsilon > 0$ sufficiently small, from which the claim follows. This fact also implies that for fixed $\varepsilon > 0$ sufficiently small, $\tilde{\mathcal{Z}}_\varepsilon$ is positively transverse to $W_E \cap \{(x_1 - \delta_1)^2 + x_2^2 < (3\delta_1)^2, x_1 \geq 0\}$, for every $E > 0$ sufficiently small, see (3.8).

The last step is to construct an interpolation between $\tilde{\mathcal{Z}}_\varepsilon$ and

$$(3.9) \quad Y := \frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2} + y_1\partial_{y_1} + y_2\partial_{y_2})$$

near $p_c = 0$. This interpolation depends on the energy $E > 0$. Recall that $\tilde{\mathcal{Z}}_\varepsilon = \mathcal{Z}$ on $(x_1 - \delta_1)^2 + x_2^2 \leq (2\delta_1)^2$. We fix $E > 0$ sufficiently small and construct a Liouville vector field \mathcal{Z}_E transverse to W_E , which coincides with Y for $|x_1|$ sufficiently small. We restrict to the region $\{x_1^2 + x_2^2 < \delta_1^2\}$, where $\tilde{\mathcal{Z}}_\varepsilon$ coincides with \mathcal{Z} .

Let $0 < \delta_E \ll \delta_1$ be a small number to be determined below, and let $\tilde{f}_E = \tilde{f}_E(x_1)$ be a smooth function satisfying $\tilde{f}_E(x_1) = 0$ if $x_1 \leq \delta_E/2$, $\tilde{f}_E(x_1) = 1$ if $x_1 \geq \delta_E$, and $\tilde{f}'_E(x_1) > 0$ if $\delta_E/2 < x_1 < \delta_E$. Let $g(y_1) := -\delta_1 y_1/2$ so that $X_g = \mathcal{Z} - Y = -\frac{\delta_1}{2}\partial_{x_1}$. Define

$$\mathcal{Z}_E := Y + X_g \tilde{f}_E$$

that satisfies $\mathcal{Z}_E = Y$ on $0 < x_1 \leq \delta_E/2$ and $\mathcal{Z}_E = \mathcal{Z}$ on $\delta_E \leq x_1$.

We claim that \mathcal{Z}_E is transverse to W_E on $0 \leq x_1 < \delta_E$ if $E > 0$ is fixed sufficiently small. We compute

$$dH \cdot \mathcal{Z}_E = (1 - \tilde{f}_E)dH \cdot Y + \tilde{f}_E dH \cdot \mathcal{Z} + g dH \cdot X_{\tilde{f}_E}.$$

We see that $dH \cdot Y = \frac{1}{2}(y_1^2 + y_2^2 + ax_1^2 + bx_2^2) + \frac{1}{2}(x_1\partial_{x_1}R + x_2\partial_{x_2}R) > 0$ on $W_E \cap \{|x_1| \leq \delta_E\}$, where $\delta_E := c\sqrt{E}$ for some $c > 0$ sufficiently small independent of $E > 0$. This follows from the fact that $dH \cdot Y|_{W_E \cap \{x_1=0\}} = \frac{1}{2}(y_1^2 + y_2^2 + bx_2^2) + \frac{1}{2}x_2\partial_{x_2}R(0, x_2) > \frac{E}{2} > 0$ for every $E > 0$ fixed sufficiently small. The second and third terms are

non-negative. This follows from (3.8) and the inequality $gdH \cdot X_{\tilde{f}_E} = \delta_1 y_1^2 \tilde{f}'_E / 2 \geq 0$. The claim is proved.

The symmetry of Y with respect to the involution $x_1 \mapsto -x_1$ implies that the same construction can be done on $\{x_1 \leq 0\}$, and also near every p_i to obtain the desired Liouville vector field X_E , transverse to W_E for every $E > 0$ sufficiently small. The Liouville vector field X_0 transverse to \dot{W}_0 is obtained in the above construction of Z_ε by taking $r \rightarrow 0$ near each saddle-center p_i for suitable neighborhoods of the arcs γ_i . \square

3.4. Finite energy planes asymptotic to a Lyapunov orbit. Let us consider as before that for suitable symplectic coordinates near any saddle-center the mechanical Hamiltonian admits the form $H = \frac{1}{2}(y_1^2 + y_2^2 + ax_1^2 + bx_2^2) + R(x)$, where $a < 0, b > 0$ and $R = O(|x|^3)$. For $E > 0$ small, let $A_E := H^{-1}(E) \cap \{|x_1| < c\sqrt{E}\}$, where $c > 0$ is given in Proposition 3.7, and let

$$\lambda = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

In this section, we construct an almost complex structure J_E on $\mathbb{R} \times A_E$ adapted to $\lambda_E := \lambda|_{A_E}$ and admitting a pair of finite energy planes asymptotic to the Lyapunov orbit $P_{2,E}$ through opposite directions.

Let $\hat{H}_E := H(\sqrt{E}\hat{x}, \sqrt{E}\hat{y})/E = \frac{1}{2}(\hat{y}_1^2 + \hat{y}_2^2 + a\hat{x}_1^2 + b\hat{x}_2^2) + R(\sqrt{E}\hat{x}, \sqrt{E}\hat{y})/E$ and notice that $H^{-1}(E) \equiv \hat{H}_E^{-1}(1)$ under the re-scaling $(x, y) = \sqrt{E}(\hat{x}, \hat{y})$. Since \hat{H}_E converges in C_{loc}^∞ to $\hat{H}_0 = \frac{1}{2}(\hat{y}_1^2 + \hat{y}_2^2 + a\hat{x}_1^2 + b\hat{x}_2^2)$ as $E \rightarrow 0^+$, we see that the trajectories on $\hat{H}_E^{-1}(1)$ locally converges to the trajectories on $\hat{H}_0^{-1}(1)$ as $E \rightarrow 0^+$. In coordinates (\hat{x}, \hat{y}) , we consider

$$(3.10) \quad \hat{\lambda} = \frac{1}{2}(\hat{x}_1 d\hat{y}_1 - \hat{y}_1 d\hat{x}_1 + \hat{x}_2 d\hat{y}_2 - \hat{y}_2 d\hat{x}_2)$$

which restricts to a contact form $\hat{\lambda}_E$ on $\hat{A}_E := \hat{H}_E^{-1}(1) \cap \{|\hat{x}_1| < c\}$, for $E \geq 0$ small. Notice that $\hat{\lambda}$ differs from λ by a constant factor depending on E .

Considering new symplectic coordinates $(\hat{x}_1, b^{-1/4}\hat{x}_2, \hat{y}_1, b^{1/4}\hat{y}_2)$ and denoting them again by $(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2)$, we may assume that

$$\hat{H}_0 = \frac{1}{2}(\hat{y}_1^2 + \sqrt{b}\hat{y}_2^2 + a\hat{x}_1^2 + \sqrt{b}\hat{x}_2^2).$$

The 1-form $\hat{\lambda}$ has the same form (3.10) as before. The Reeb vector field on \hat{A}_E is denoted by \hat{R}_E . The Reeb vector field on \hat{A}_0 is given by

$$\hat{R}_0 = -(\hat{y}_1, \sqrt{b}\hat{y}_2, -a\hat{x}_1, -\sqrt{b}\hat{x}_2).$$

Notice that $\hat{\lambda}_0(\hat{R}_0) = 1$. Our strategy is to define an almost complex structure \hat{J}_0 on $\mathbb{R} \times \hat{A}_0$ and find a pair of planes asymptotic to $\hat{P}_{2,0} \subset \hat{H}_0^{-1}(1)$ whose projections to \hat{A}_0 lie in $\{\hat{x}_1 = 0\}$. Then, for any family of almost complex structures \hat{J}_E on $\mathbb{R} \times \hat{A}_E$, $E \geq 0$ small, that smoothly continues \hat{J}_0 , we obtain a pair of corresponding \hat{J}_E -holomorphic planes by automatic transversality [48]. Such planes can now be re-scaled back to the original coordinates (x, y) , giving the desired pair of holomorphic planes asymptotic to $P_{2,E} \subset H^{-1}(E)$.

We use the transverse frame $\{Y_1, Y_2\} = \{j_1 \nabla \hat{H}_0, j_2 \nabla \hat{H}_0\}$, where j_1, j_2 are as in (2.1), adapted to \hat{A}_0 to define an almost complex structure \hat{J}_0 on $\mathbb{R} \times \hat{A}_0$. In that

case, we have

$$Y_1 = (\sqrt{b}\hat{y}_2, -\hat{y}_1, \sqrt{b}\hat{x}_2, -a\hat{x}_1) \quad \text{and} \quad Y_2 = (\sqrt{b}\hat{x}_2, -a\hat{x}_1, -\sqrt{b}\hat{y}_2, \hat{y}_1).$$

Since $\text{span}\{Y_1, Y_2\}$ and $\ker \hat{\lambda}_0$ are transverse to the Reeb vector field \hat{R}_0 , we can project Y_1 and Y_2 to $\ker \hat{\lambda}_0$ along \hat{R}_0 to obtain $\hat{Y}_1 = Y_1 - \hat{\lambda}_0(Y_1)\hat{R}_0$ and $\hat{Y}_2 = Y_2 - \hat{\lambda}_0(Y_2)\hat{R}_0$, both contained in the contact structure $\ker \hat{\lambda}_0$. Then we take the compatible almost complex structure \hat{J}_0 on $\mathbb{R} \times \hat{A}_0$ determined by $\hat{J}_0 \cdot \partial_a = \hat{R}_0$ and

$$(3.11) \quad \hat{J}_0 \cdot \hat{Y}_2 := \hat{Y}_1.$$

Notice that $d\hat{\lambda}_0(\hat{Y}_2, \hat{Y}_1) > 0$. We seek for a pair of \hat{J}_0 -holomorphic planes, both asymptotic to

$$\hat{P}_{2,0} = \{\hat{x}_1 = \hat{y}_1 = 0, \sqrt{b}\hat{y}_2^2 + \sqrt{b}\hat{x}_2^2 = 2\},$$

which approach $\hat{P}_{2,0}$ through opposite directions, that is through different signs of \hat{y}_1 . Taking the local behavior of the flow in \hat{A}_0 into account, we shall construct planes which project onto the hemispheres of the 2-sphere $\{\hat{x}_1 = 0\} \cap \hat{H}_0^{-1}(1)$ whose equator is $\hat{P}_{2,0}$. We consider cylinders

$$\tilde{u} = (d, u): \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \rightarrow \mathbb{R} \times \hat{A}_0,$$

of the form

$$(3.12) \quad d = d(s) \quad \text{and} \quad u(s, t) = (0, g(s) \cos t, f(s), g(s) \sin t),$$

where $d(s)$, $f(s)$ and $g(s)$ will be determined below. Notice that

$$f(s)^2 + \sqrt{b}g(s)^2 = 2, \forall s,$$

since $u(s, t) \in \hat{H}_0^{-1}(1), \forall (s, t)$. Also, \tilde{u} projects to $\hat{x}_1 = 0$. Let us denote by $\pi: T\hat{A}_0 \rightarrow \ker \hat{\lambda}_0$ the projection along \hat{R}_0 . The first condition we need to impose on \tilde{u} to obtain a \hat{J}_0 -holomorphic cylinder is the first equation in (2.11). Using the ansatz (3.12), we compute

$$\begin{aligned} \pi u_s &= u_s - \hat{\lambda}_0(u_s)\hat{R}_0 = (0, g'(s) \cos t, f'(s), g'(s) \sin t) \\ \pi u_t &= u_t - \hat{\lambda}_0(u_t)\hat{R}_0 = (0, -g(s) \sin t, 0, g(s) \cos t) \\ &\quad - \frac{g(s)^2}{2} \left(-f(s), -\sqrt{b}g(s) \sin t, 0, \sqrt{b}g(s) \cos t \right). \end{aligned}$$

We also compute along $u(s, t)$

$$\begin{aligned} \hat{Y}_1 &= (\sqrt{b}g(s) \sin t, -f(s), \sqrt{b}g(s) \cos t, 0) \\ &\quad - \frac{1}{2}(1 - \sqrt{b})g(s)f(s) \sin t \left(-f(s), -\sqrt{b}g(s) \sin t, 0, \sqrt{b}g(s) \cos t \right) \\ \hat{Y}_2 &= \left(\sqrt{b}g(s) \cos t, 0, -\sqrt{b}g(s) \sin t, f(s) \right) \\ &\quad - \frac{1}{2}(1 - \sqrt{b})g(s)f(s) \cos t \left(-f(s), -\sqrt{b}g(s) \sin t, 0, \sqrt{b}g(s) \cos t \right). \end{aligned}$$

It follows from the computation above that

$$\begin{aligned} \cos t \cdot \hat{Y}_1 - \sin t \cdot \hat{Y}_2 &= \frac{\sqrt{b}g(s)}{f'(s)} \pi u_s \\ \sin t \cdot \hat{Y}_1 + \cos t \cdot \hat{Y}_2 &= \frac{f(s)^2 + bg(s)^2}{g(s)f(s)} \pi u_t. \end{aligned}$$

Imposing that $\hat{J}_0 \cdot \pi u_s = \pi u_t$, see (2.11), together with $f(s)^2 + \sqrt{b}g(s)^2 = 2$, and (3.11), we end up with the following differential equation

$$(3.13) \quad f' = -\frac{(2-f^2)f}{f^2 + \sqrt{b}(2-f^2)}.$$

Given any initial condition $f(0) \in (0, \sqrt{2})$, we integrate (3.13) to determine $f(s)$ satisfying $f(s) \rightarrow 0^+$ as $s \rightarrow +\infty$ and $f(s) \rightarrow \sqrt{2}^-$ as $s \rightarrow -\infty$. The value of $g(s) = \sqrt{(2-f(s)^2)/\sqrt{b}}$ is then determined. Notice that $g(s)^2 \rightarrow 2/\sqrt{b}$ as $s \rightarrow +\infty$. We also define

$$d'(s) := \hat{\lambda}_0(u_t) = \frac{g(s)^2}{2} \Rightarrow d(s) = \int_0^s \frac{g(\tau)^2}{2} d\tau,$$

see (2.11). Hence $s = +\infty$ is a positive puncture and $s = -\infty$ is a removable puncture. After removing the puncture at $-\infty$, we obtain a finite energy plane $\tilde{u}_1 = (d_1, u_1): \mathbb{C} \rightarrow \mathbb{R} \times \hat{A}_0$ asymptotic to $\hat{P}_{2,0}$.

Choosing an initial condition $f(0) \in (-\sqrt{2}, 0)$, we proceed in the same way to obtain a finite energy plane $\tilde{u}_2 = (d_2, u_2): \mathbb{C} \rightarrow \mathbb{R} \times \hat{A}_0$ asymptotic to $\hat{P}_{2,0}$ through the opposite direction.

Such planes are automatically transverse, and their Fredholm index is 1, see [48]. Hence if \hat{J}_E is a smooth family of almost complex structures on \hat{A}_E extending \hat{J}_0 on \hat{A}_0 , we obtain two families of \hat{J}_E -holomorphic planes $\hat{u}_{i,E} = (\hat{d}_{i,E}, \hat{u}_{i,E})$, $i = 1, 2$, asymptotic to the Lyapunov orbit $\hat{P}_{2,E} \subset \hat{A}_E$ through opposite directions, and whose projections to $\hat{H}_E^{-1}(1)$ lie in $\{|\hat{x}_1| < c\}$ for any $E > 0$ sufficiently small. We can bring such planes back to the original coordinates $(x, y) = (\sqrt{E}\hat{x}, \sqrt{E}\hat{y})$ in the following way: on the contact structure, we consider the almost complex structure J_E which coincides with \hat{J}_E under the re-scaling, and define $u_{i,E}(s, t) = \sqrt{E}\hat{u}_{i,E}(s, t)$. Since λ gets multiplied by $1/E$ in coordinates (x, y) , the new real-valued function $d_{i,E}$ is defined as $d_{i,E}(s) := E\hat{d}_{i,E}(s)$. The new planes $\tilde{u}_{i,E} = (d_{i,E}, u_{i,E})$ are finite energy J_E -holomorphic planes asymptotic to the Lyapunov orbit $P_{2,E} \subset H^{-1}(E)$ through opposite directions. Moreover, $u_{i,E}(\mathbb{C})$ converges in the Hausdorff topology to the saddle-center at $0 \in \mathbb{R}^4$ as $E \rightarrow 0^+$. This procedure can be done for every saddle-center. Using an auxiliary metric on W_E , the locally defined J_E extends to an almost complex structure on $\mathbb{R} \times W_E$. We have proved the following proposition.

Proposition 3.8. *Under the conditions of Proposition 3.7, the following holds. For every $E > 0$ sufficiently small, there exists an almost complex structure J_E on $\mathbb{R} \times W_E$ adapted to λ_E so that the Lyapunov orbit near each p_i is the asymptotic limit of a pair of J_E -holomorphic planes $\tilde{u}_{i,E} = (d_{i,E}, u_{i,E})$, $i = 1, 2$, through opposite directions, whose projections to W_E are arbitrarily close to p_i as $E \rightarrow 0$.*

3.5. Proof of Theorem 1.2. We complete the proof of Theorem 1.2 by applying the results of the previous sections. We can assume that the potential $V(x) = V_\epsilon(x)$ is as in Proposition 3.2, where $\epsilon > 0$ is sufficiently small. Here, the potential V is changed away from the disk-like compact set $K_0 \subset \{V \leq 0\}$ given by the projection of the singular sphere-like subset $S_0 \subset H^{-1}(0)$. The Hill region $\{V \leq 0\}$ becomes the union of K_0 and l disjoint disk-like compact domains K_i , $i = 1, \dots, l$, touching K_0 at the saddles $v_1, \dots, v_l \in \partial K_0$. We know from Proposition 3.3 that if $\epsilon > 0$ is

fixed sufficiently small, the following conditions hold. If $E > 0$ is sufficiently small, then the sphere-like hypersurface $W_E = H^{-1}(E)$, whose projection to the x -plane contains $K_0 \cup \bigcup_{i=1}^l K_i$, is weakly convex, the only index-2 orbits are the Lyapunov orbits around p_1, \dots, p_l and every index-3 orbit is not linked with any Lyapunov orbit. Since H is a mechanical system, W_E admits a contact form λ_E whose Reeb flow parametrizes the Hamiltonian flow. Hence, we can apply the main result in [10], see Theorem 1.5, to obtain a weakly convex foliation \mathcal{F}_E whose binding orbits are the Lyapunov orbits and l index-3 orbits in each domain bounded by the rigid planes. In particular, W_E is the union of $l + 1$ compact subsets bounded by the pairs of rigid planes. We know that the foliation \mathcal{F}_E given in Theorem 1.5 is the projection to W_E of a finite energy foliation in $\mathbb{R} \times W_E$ associated with an almost complex structure J_E . By Propositions 3.7 and 3.8, we can choose λ_E and J_E (perhaps after a C^∞ -small perturbation of J_E) so that the rigid planes asymptotic to the Lyapunov orbits project arbitrarily close to the saddles v_1, \dots, v_l as $E \rightarrow 0^+$, see Remark 1.6. In particular, W_E contains a compact subset S_E bounded by the closure of all rigid planes so that its projection to the x -plane lies in the domain of the initial potential. In particular, S_E lies inside the initial energy surface $H^{-1}(E)$, and thus \mathcal{F}_E restricted to S_E is the desired weakly convex foliation. This finishes the proof of Theorem 1.2. \square

4. APPLICATIONS

We apply Theorem 1.2 to the Hénon-Heiles potential for energies slightly above $1/6$. We also discuss the existence of weakly convex foliations for decoupled systems, including frozen Hill's lunar problem with centrifugal force, the Stark problem, the Euler problem of two centers, and a chemical reaction model.

4.1. The Hénon-Heiles system. In 1964, Hénon and Heiles [21] proposed the following potential to study the motion of a star in galaxy with an axis of symmetry

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1^2 x_2 - \frac{1}{3}x_2^3.$$

The dynamics of V is very rich, and we refer to the survey [6], and [36, 5, 7, 40, 3] for a discussion on periodic orbits, multiple horseshoes, non-integrability, etc.

The potential V is invariant under the rotation $(x_1, x_2) \mapsto e^{2\pi i/3}(x_1, x_2)$ and under the reflection $(x_1, x_2) \mapsto (-x_1, x_2)$, which means that V exhibits a triangular symmetry (\mathbb{Z}_3 -symmetry under $\frac{2\pi}{3}$ -rotation) and also a reflection symmetry across the vertical axis $x_1 = 0$. If $0 < E < 1/6$, the energy surface $H^{-1}(E)$ contains a strictly convex sphere-like component S_E , see [42, 43], which admits a disk-like global surface of section. If $E = 1/6$, the energy surface becomes singular and the Hill region $\Omega_{1/6} = \{V \leq 1/6\}$ contains a compact subset $K_{1/6}$ bounded by the triangle $T \subset V^{-1}(1/6)$ whose vertices are the saddle points $v_1 = (0, 1)$, $v_2 = (-\sqrt{3}/2, -1/2)$ and $v_3 = (\sqrt{3}/2, -1/2)$, see Figure 4.1. The compact set $K_{1/6}$ is the projection of a singular sphere-like subset $S_{1/6} \subset H^{-1}(1/6)$ that contains three singularities p_1, p_2 and p_3 of saddle-center type, projecting to v_1, v_2 and v_3 , respectively.

Proposition 4.1. *The set $S_{1/6} \setminus \{p_1, p_2, p_3\}$ is dynamically convex and the quaternion frame is positive.*

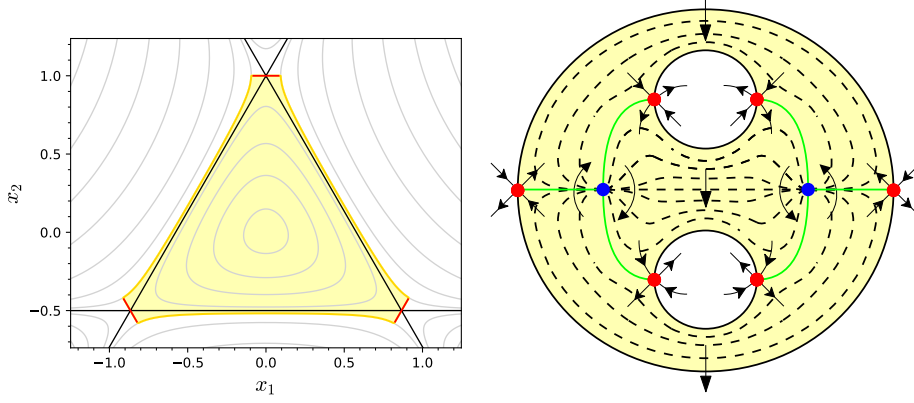


FIGURE 4.1. The Hénon-Heiles system for energies $E > 0$ slightly above the critical value $1/6$. The yellowish region on the left is the projection $K_E \subset \mathbb{R}^2$ of the compact subset $S_E \subset H^{-1}(E)$ to the Hill region. On the right, a cross section of S_E indicates the weakly convex foliation with $l = 3$.

Proof. Notice that $S_{1/6}$ is the Hausdorff limit of S_E as $E \rightarrow 1/6^-$. Since S_E is strictly convex for every $E < 1/6$, see [42, 43], $S_{1/6}$ bounds a convex domain of \mathbb{R}^4 . Let $P \subset S_{1/6} \setminus \{p_1, p_2, p_3\}$ be a periodic orbit. We shall construct a Hamiltonian $\hat{H}: \mathbb{R}^4 \rightarrow \mathbb{R}$ whose energy surface $\hat{H}^{-1}(1/6)$ is regular, star-shaped, and contains a neighborhood of P in S_E . In particular, we formally use \hat{H} to compute the index of P . We obtain \hat{H} by modifying H near p_1, p_2 and p_3 . First, we modify it near $p_1 = (0, 1)$ and then use the \mathbb{Z}_3 -symmetry to modify H near p_2 and p_3 .

Let $\delta > 0$ be small, and let $f_2: \mathbb{R} \rightarrow [0, +\infty)$ be a smooth nondecreasing function satisfying $f_2(t) = 0, \forall t \leq 1 - \delta$, and $f_2(t) = \delta, \forall t \geq 1$. Let

$$\hat{H}(x_1, x_2, y_1, y_2) := H(x_1, x_2, y_1, y_2) + f_2(x_2) = \frac{|y|^2}{2} + V(x_1, x_2) + f_2(x_2).$$

Observe that \hat{H} coincides with H in $\{x_2 < 1 - \delta\}$ and \hat{H} has no critical point in $\{0 < x_2 < 1\}$. Since $f_2 \geq 0$ and $\hat{H}(x_1, 1, 0, 0) = \frac{3}{2}x_1^2 + \frac{1}{6} + \delta > 1/6, \forall x_1 \in \mathbb{R}$, we see that $\hat{H}^{-1}(1/6)$ contains a singular sphere-like component $\hat{S}_{1/6}$ whose projection $\hat{K}_{1/6}$ to the x -plane lies in $K_{1/6}$ away from the vertex $(0, 1) \in T$. Let $Z := \frac{1}{2}(x_1, x_2, y_1, y_2)$ be the radial vector field. A direct computation gives

$$\nabla \hat{H} \cdot Z = \frac{1}{6} + \frac{1}{2}x_1^2x_2 - \frac{1}{6}x_2^3 + \frac{1}{2}f_2'(x_2)x_2 > 0 \quad \text{on } \hat{S}_{1/6} \cap \{0 \leq x_2 < 1\}.$$

We can restrict \hat{H} to a small neighborhood of $\hat{S}_{1/6}$ and modify it near p_2 and p_3 in the same manner so that \hat{H} becomes \mathbb{Z}_3 -symmetric under $(x, y) \mapsto e^{2\pi i/3}(x, y)$, and $\hat{S}_{1/6} = \hat{H}^{-1}(1/6)$ is a regular star-shaped hypersurface. If $\delta > 0$ is sufficiently small, then $\hat{S}_{1/6}$ contains a neighborhood of P in S_E .

Next, we show that the Hessian $\hat{\mathcal{H}}$ of \hat{H} is positive-definite on $T\hat{S}_{1/6}$ along P , except perhaps at precisely two points projecting to $\partial K_{1/6}$ in the case P is a brake orbit. Notice that $f = 0$ near P . Taking $\delta > 0$ sufficiently small, it is enough to

show that the Hessian \mathcal{H} of H is positive-definite on $TS_{1/6}$ near P in the region projecting to the interior of $K_{1/6}$. From [42], this condition is equivalent to

$G := 2(1/6 - V)(V_{x_1x_1}V_{x_2x_2} - V_{x_1x_2}^2) + V_{x_1x_1}V_{x_2}^2 + V_{x_2x_2}V_{x_1}^2 - 2V_{x_1}V_{x_2}V_{x_1x_2} > 0$ on $K_{1/6} \setminus \partial K_{1/6}$. A direct computation gives

$$G = 2(1/6 - V)(1 - x_1^2 - x_2^2).$$

Since $K_{1/6} \setminus \partial K_{1/6} \subset \{x_1^2 + x_2^2 < 1\}$, we conclude that $G > 0$ on $K_{1/6} \setminus \partial K_{1/6}$, and thus \mathcal{H} is positive-definite on $TS_{1/6}$ near P in the region projecting to $K_{1/6} \setminus \partial K_{1/6}$. If P is a brake orbit, then this condition holds along P except at two points projecting to $\partial K_{1/6}$.

Summarizing the construction above, we find a Hamiltonian \hat{H} whose energy surface $\hat{H}^{-1}(1/6)$ is a star-shaped hypersurface $\hat{S}_{1/6}$ containing a neighborhood of P in $S_{1/6}$. Moreover, the Hessian $\hat{\mathcal{H}}$ of \hat{H} is positive-definite along P except possibly at two points projecting to $\partial K_{1/6}$. The Liouville form λ_0 restricts to a contact form on $\hat{S}_{1/6}$, whose Reeb flow parametrizes the Hamiltonian flow on \hat{S}_E , and the (generalized) Conley-Zehnder index of P is $\mu(P)$.

Theorem 3.4 in [26] states that if $S = H^{-1}(1) \subset \mathbb{R}^4$ for a Hamiltonian $H: \mathbb{R}^4 \rightarrow \mathbb{R}$ is a strictly convex hypersurface and $P' \subset S$ is a periodic orbit of the Reeb flow of $\lambda_0|_S$, then $\mu(P') \geq 3$. Investigating its proof, however, we see that it is enough that S is star-shaped, the Hessian of H is ≥ 0 along P' and positive at some point of P' . Such conditions hold for $P \subset \hat{S}_{1/6} = \hat{H}^{-1}(1/6)$, and we conclude that $\mu(P) \geq 3$.

Finally, we show that the quaternion frame is positive on $S_{1/6} \setminus \{p_1, p_2, p_3\}$. Recall from (2.4) that the transverse linearized flow is determined by (2.3), and the argument $\theta(t)$ of a solution $\alpha = \alpha_1(t) + i\alpha_2(t)$ satisfies

$$\dot{\theta} = (\kappa_{11} + \kappa_{33}) \cos^2 \theta + 2\kappa_{12} \cos \theta \sin \theta + (\kappa_{22} + \kappa_{33}) \sin^2 \theta.$$

We know that for every point of $S_{1/6} \setminus \{p_1, p_2, p_3\}$ projecting to $K \setminus \partial K$, \mathcal{H} is positive-definite on $TS_{1/6}$. This implies $\dot{\theta} > 0$, see Lemma 2.1. For those points projecting to $\partial K \setminus \{v_1, v_2, v_3\}$ we know from (2.8) that $\kappa_{11} = \kappa_{33} = 1$, $\kappa_{12} = 0$, and

$$\kappa_{22} = \frac{V_{x_1x_1}V_{x_2}^2 - 2V_{x_1x_2}V_{x_1}V_{x_2} + V_{x_2x_2}V_{x_1}^2}{V_{x_1}^2 + V_{x_2}^2} = 0,$$

which implies that $\dot{\theta} = 1$. we conclude that the quaternion frame is everywhere positive on $S_{1/6} \setminus \{p_1, p_2, p_3\}$. \square

Proposition 4.1 and Theorem 1.2 imply the following theorem.

Theorem 4.2. *For every E slightly above $1/6$, there exists a compact subset $S_E \subset H^{-1}(E)$ admitting a weakly convex foliation as in Theorem 1.2 with $l = 3$, see Figure 4.1. The binding orbits are the three Lyapunov orbits near the saddle-centers and an index-3 orbit projecting near $K_{1/6}$. In particular, S_E contains infinitely many periodic orbits, and each Lyapunov orbit admits infinitely many homoclinic orbits or heteroclinic orbits to other Lyapunov orbits.*

4.2. Decoupled mechanical systems. We present examples of decoupled Hamiltonian systems that admit weakly convex transverse foliations. Although Theorem 1.2 applies to such systems, the foliations in some cases are obtained from gradient flow lines. We discuss the frozen Hill's lunar problem with centrifugal force, the Stark problem, the Euler problem of two centers, and a chemical reaction model.

Let $H = H_1(x_1, y_1) + H_2(x_2, y_2)$ be the sum of two one-degree-of-freedom mechanical Hamiltonians

$$(4.1) \quad H_1(x_1, y_1) = \frac{y_1^2}{2} + V_1(x_1) \quad \text{and} \quad H_2(x_2, y_2) = \frac{y_2^2}{2} + V_2(x_2),$$

where V_1, V_2 are smooth one-dimensional potentials with nondegenerate critical points. The trajectories of $H^{-1}(E)$ project to trajectories of $H_1^{-1}(E_1)$ and $H_2^{-1}(E_2)$, where $E = E_1 + E_2$. The following proposition implies that H is weakly convex, i.e., all of its non-trivial periodic orbits have index ≥ 2 .

Proposition 4.3. *Let H satisfy the conditions above, and let $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{R}^4$ be a simple non-trivial periodic orbit. Then the following statements hold:*

- (i) *If $\gamma_1(t) \equiv (\bar{x}_1, 0)$ is an equilibrium of H_1 corresponding to a nondegenerate minimum of V_1 , and $\gamma_2(t)$ is a non-trivial periodic orbit of H_2 , then $\mu(\gamma) \geq 3$. In particular, its rotation number is > 1 .*
- (ii) *If $\gamma_1(t) \equiv (\bar{x}_1, 0)$ is an equilibrium point of H_1 corresponding to a nondegenerate maximum of V_1 , and $\gamma_2(t)$ is a non-trivial periodic orbit of H_2 , then γ is a hyperbolic orbit and $\mu(\gamma) = 2$. In particular, its rotation number is 1.*
- (iii) *If γ_1 and γ_2 are non-trivial periodic orbits of H_1 and H_2 , respectively, then $\rho(\gamma)$ is an integer ≥ 2 . In particular, $\mu(\gamma) \geq 3$.*

The same conclusions hold in (i) and (ii) for H_1 interchanged with H_2 .

Remark 4.4. *Proposition 4.3 easily generalizes for potentials V_1 and V_2 with degenerate critical points. In particular, the dynamics of any decoupled Hamiltonian $H = (y_1^2 + y_2^2)/2 + V_1(x_1) + V_2(x_2)$ restricted to a regular energy surface is weakly convex.*

Proof of proposition 4.3. The conclusions can be obtained by summing up the rotation numbers on the symplectic planes x_1y_1 and x_2y_2 . For clarity, however, we stick to the geometric definition of $\mu(\gamma)$ and compute it using the quaternion frame $\{X_1, X_2\}$.

First, consider cases (i) and (ii). We have $X_1|_\gamma = g \cdot (y_2\partial_{x_1} + V_2'(x_2)\partial_{y_1})$ and $X_2|_\gamma = g \cdot (V_2'(x_2)\partial_{x_1} - y_2\partial_{y_1})$, where $g = (y_2^2 + V_2'(x_2)^2)^{-1/2}$. Notice that the transverse plane $\Pi_1 := \text{span}\{\partial_{x_1}, \partial_{y_1}\}|_\gamma$ is invariant by the flow. Also, the frame $\{X_1, X_2\}|_\gamma$ has opposite orientation with respect to the frame $\{\partial_{x_1}, \partial_{y_1}\}|_\gamma$.

Since γ is simple, γ_2 is also simple. The winding number of $\gamma_2' = (y_2, -V_2') \in \mathbb{R}^2$ is $+1$ in the clockwise direction. Hence, the winding number of X_1 and X_2 on Π_1 is $+1$ in the counterclockwise direction. Since the linearized flow on Π_1 is given by

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -V_1''(\bar{x}_1) & 0 \end{pmatrix} z, \quad z = (\alpha_1 \ \alpha_2)^T \equiv \alpha_1\partial_{x_1} + \alpha_2\partial_{y_1},$$

we see that if $V_1''(\bar{x}_1) > 0$, then the variation in the argument of every linearized solution on the frame $\{\partial_{x_1}, \partial_{y_1}\}|_\gamma$ is < 0 and thus is greater than 2π on $\{X_1, X_2\}|_\gamma$. Hence, the index of γ is ≥ 3 . If $V_1''(\bar{x}_1) < 0$, then the variations in the argument of all non-trivial solutions on $\{\partial_{x_1}, \partial_{y_1}\}|_\gamma$ give an interval containing 0 in its interior, and thus, an interval containing 2π in its interior on the frame $\{X_1, X_2\}|_\gamma$. Hence, γ is an index-2 hyperbolic orbit.

In case (iii), we use a more direct proof. Since H is decoupled, we have $\rho(\gamma) = \rho(\gamma_1) + \rho(\gamma_2)$, where $\rho(\gamma_i)$ is the rotation number of γ_i in each symplectic factor. Since γ_1 and γ_2 are non-trivial periodic orbits, we have $\rho(\gamma_i) \geq 1, i = 1, 2$. Their

velocity vectors are invariant under the flow in each factor and thus $\rho(\gamma_i)$ is an integer ≥ 1 . Hence $\rho(\gamma)$ is an integer ≥ 2 . In particular, $\mu(\gamma) \geq 3$. \square

Proposition 4.3 tells us that an index-2 orbit of H has the form $\{(\bar{x}_1, 0)\} \times \gamma_2$ or $\gamma_1 \times \{(\bar{x}_2, 0)\}$, where \bar{x}_i is a maximum of V_i and γ_j is a non-trivial periodic orbit of $H_j, j \neq i$.

Notice that every critical point of

$$V(x_1, x_2) := V_1(x_1) + V_2(x_2)$$

has the form (\bar{x}_1, \bar{x}_2) , where \bar{x}_i is a critical point of V_i . Assume that 0 is a critical value of H satisfying the conditions of Theorem 1.2, i.e., the Hill region $\{V \leq 0\}$ has a compact disk-like subset $K_0 \subset \mathbb{R}^2$ whose boundary $\partial K_0 \subset V^{-1}(0)$ contains precisely $l \geq 1$ critical points v_1, \dots, v_l of V , which are all saddles of V , and $V|_{K_0 \setminus \partial K_0} < 0$. Let $S_0 \subset H^{-1}(0)$ be the sphere-like singular subset projecting to K_0 , and let $p_i = (v_i, 0) \in \mathbb{R}^4$ the corresponding saddle-center of H .

Suppose that the saddle $v_i = (\bar{x}_{i,1}, \bar{x}_{i,2})$ is such that $\bar{x}_{i,1}$ is a maximum of V_1 and $\bar{x}_{i,2}$ is a minimum of V_2 . Then $p_i = (v_i, 0) \in \mathbb{R}^4$ is a saddle-center and the Lyapunov orbit near p_i , for $E > 0$ sufficiently small, has the form $\{(\bar{x}_{i,1}, 0)\} \times \gamma_{i,2,E}$, where $\gamma_{i,2,E}(t)$ is a non-trivial periodic orbit of H_2 near $(\bar{x}_{i,2}, 0) \in \mathbb{R}^2$. In particular, its projection to the x -plane is vertical. A similar statement holds if $\bar{x}_{i,1}$ is a minimum of V_1 and $\bar{x}_{i,2}$ is a maximum of V_2 , and we obtain a Lyapunov orbit $\gamma_{i,1,E} \times \{(\bar{x}_{i,2}, 0)\}$ whose projection to the x -plane is horizontal.

For every $E > 0$ small, we denote by K_E the disk-like compact subset of the Hill region $\{V \leq E\}$ bounded by simple arcs of $V^{-1}(E)$ near ∂K_0 and the horizontal and vertical projections of the l Lyapunov orbits near the saddle-centers. Denote by S_E the subset of $H^{-1}(E)$ projecting to K_E . Let $\pi_i(x_1, x_2) := x_i, i = 1, 2$, be the projection on each factor.

Proposition 4.5. *If $\pi_i(K_0 \setminus \partial K_0) \subset \mathbb{R}$ contains no maximum of $V_i, i=1,2$, then*

- (i) $S_0 \setminus \{p_1, \dots, p_l\}$ is dynamically convex.
- (ii) For every $E > 0$ sufficiently small, every periodic orbit on S_E , which is not a Lyapunov orbit around p_i, \dots, p_l , has index ≥ 3 .

Remark 4.6. *The dynamical convexity in Proposition 4.5 also holds if V_1 and V_2 have degenerate critical points.*

Proof of Proposition 4.5. Assume by contradiction that $\gamma = \{(\bar{x}_1, 0)\} \times \gamma_2$ is an index-2 periodic orbit in $S_0 \setminus \{p_1, \dots, p_l\}$, where \bar{x}_1 is a maximum of V_1 and γ_2 is a non-trivial periodic orbit of H_2 . Since $V(\bar{x}_1, \pi_2(\gamma_2(t))) = V_1(\bar{x}_1) + V_2(\pi_2(\gamma_2(t))) \leq 0, \forall t$, we see that $V(\bar{x}_1, \pi_2(\gamma_2))$ attains a minimum < 0 and thus $(\bar{x}_1, \pi_2(\gamma_2))$ contains points in the interior of K_0 . Hence \bar{x}_1 is in the interior of $\pi_1(K_0 \setminus \partial K_0)$, a contradiction. The same holds if $\gamma = \gamma_1(t) \times \{(\bar{x}_2, 0)\}$, where \bar{x}_2 is a maximum of V_2 and γ_1 is a non-trivial periodic orbit of H_1 . This proves (i).

The proof of (ii) follows the same argument as in (i) with the remark that every critical point of V_1 or V_2 is non-degenerate and thus isolated, and thus no index-2 periodic orbit on S_E exists other than the Lyapunov orbits around p_1, \dots, p_l . \square

We shall use Proposition 4.5 to obtain a weakly convex foliation on $H^{-1}(E)$.

Theorem 4.7. *Assume that conditions in Proposition 4.5 hold. Then, for every $E > 0$ sufficiently small, S_E admits a weakly convex foliation whose binding is formed by the Lyapunov orbits near p_1, \dots, p_l and an index-3 orbit in $S_E \setminus \partial S_E$.*

4.3. Frozen Hill's lunar problem with centrifugal force. This problem is a mechanical system introduced in [8] whose Hamiltonian in canonical coordinates ($q = q_1 + iq_2, p = p_1 + ip_2$) is given by

$$H_0(q, p) = \frac{|p|^2}{2} - \frac{1}{|q|} - \frac{3}{2}q_1^2 - \frac{1}{2}(q_1^2 + q_2^2).$$

Using Levi-Civita coordinates ($q = v^2, p = \frac{u}{2v}$), we obtain the regularized Hamiltonian

$$K(v, u) = 4|v|^2(H_0 + c) = \frac{|u|^2}{2} - 2|v|^2(3(v_1^2 - v_2^2)^2 + |v|^4 - 2c) - 4,$$

and $K^{-1}(0)$ doubly covers the regularized $H_0^{-1}(-c)$. Let us assume that $c > 0$. Defining $u = c^{3/4}y$ and $v = c^{1/4}x$, we obtain the parameter-free Hamiltonian

$$(4.2) \quad H(x, y) = \frac{1}{c^{3/2}}(K(v, u) + 4) = \frac{|y|^2}{2} + 4x_1^2 - 8x_1^6 + 4x_2^2 - 8x_2^6,$$

and $H^{-1}(4/c^{3/2})$ still doubly covers the regularized $H_0^{-1}(-c)$.

The potentials $V_1(x_1) = 4x_1^2 - 8x_1^6$ and $V_2(x_2) = 4x_2^2 - 8x_2^6$ have a minimum at 0 and two nondegenerate maxima at $\pm 6^{-1/4}$. Hence $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ has four saddle-points $v_{1,2} = (\pm 6^{-1/4}, 0)$ and $v_{3,4} = (0, \pm 6^{-1/4})$ at level $e_* = \frac{4}{3}\sqrt{\frac{2}{3}}$. The sphere-like singular subset $S_{e_*} \subset H^{-1}(e_*)$ contains four saddle-center equilibrium points p_i projecting to v_i , $i = 1, 2, 3, 4$. The projection of S_{e_*} to the x -plane is the disk-like compact domain K_{e_*} with \mathbb{Z}_4 -symmetry under the $\frac{\pi}{2}$ -rotation, see Figure 4.2. The x_1 and x_2 -projections of K_{e_*} lie in the interval $(-6^{1/4}, 6^{1/4})$, and hence contain no maximum point of V_1 and V_2 , respectively. For $E - e_* > 0$ small, denote by $K_E \subset \mathbb{R}^2$ the disk-like subset of the Hill region $\{V \leq E\}$ near K_{e_*} as in section 4.2. Denote by S_E the subset of $H^{-1}(E)$ projecting to K_E . By Theorem 4.7, we obtain the following application.

Proposition 4.8. *For every $E - e_* > 0$ sufficiently small, the energy surface $H^{-1}(E)$ admits a weakly convex foliation on S_E with $l = 4$. The binding orbits are the Lyapunov orbits around p_1, \dots, p_4 and an index-3 orbit projecting to K_E .*

4.4. The Stark problem. In [8], Cieliebak, Frauenfelder, and van Koert introduced a large class of Hamiltonian systems called Stark systems, which model the motion of an electron attracted by a proton subject to an external electric field. A planar Stark system is a mechanical system whose potential has the form $U = -\frac{1}{|q|} + U_0(q)$, where U_0 is smooth near $q = 0$. We assume that there exists $a > 0$ such that $U_0(\nu^{2a}q) = \nu U_0(q)$ for every q and $\nu > 0$. The frozen Hill's lunar problem studied in the previous section is a Stark system with $a = 1/4$. As before, the Hamiltonian $H_0 = |p|^2/2 + U(q)$ can be regularized using adapted Levi-Civita coordinates $q = c^{1/2}x^2, p = c^{1/2}\frac{y}{2x}$, $c > 0$, and becomes

$$H(x, y) = \frac{|y|^2}{2} + V(x), \quad \text{where} \quad V(x) = 4|x|^2(1 + U_0(x^2)).$$

The energy surface $H^{-1}(4/c^{3/2})$ doubly covers the regularized $H_0^{-1}(-c)$.

The problem studied by Stark [45], known as the Stark problem, is the case of a constant electric field, that is $U_0(x) = -\varepsilon x_1, \varepsilon > 0$ ($a = 1/2$). Many researchers have extensively investigated this problem in a variety of contexts. We refer the

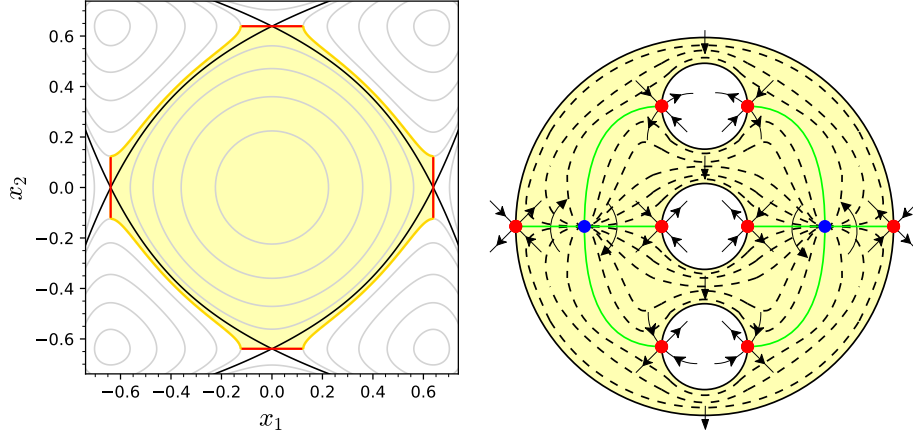


FIGURE 4.2. The frozen Hill's lunar problem with centrifugal force for energies above the critical value. The projection K_E of $S_E \subset H^{-1}(E)$ to the Hill region (left). S_E is foliated by a weakly convex foliation with $l = 4$ (right).

reader to [31, 33] and [4, 22], where the Stark problem is studied in orbital and quantum mechanics, respectively. We also refer to [18] in which the author studies the Stark problem in the flavor of symplectic geometry.

The potential V of the Stark problem is given by

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) = 4(x_1^2 - \varepsilon x_1^4) + 4(x_2^2 + \varepsilon x_2^4).$$

Notice that V admits a minimum at $(0, 0) \in V^{-1}(0)$ and two saddles at $v_{1,2} = (\pm(2\varepsilon)^{-1/2}, 0)$ with critical value $e_* = \varepsilon^{-1} > 0$. The sub-level set $\{V \leq e_*\}$ contains a disk-like compact domain $K_{e_*} \subset \{V \leq e_*\}$ whose boundary contains the saddles v_1 and v_2 . The compact K_{e_*} is the projection of a singular sphere-like subset of $H^{-1}(e_*)$ with two saddle-centers at $p_i = (v_i, 0), i = 1, 2$. For each $E - e_* > 0$, denote by S_E the subset of $H^{-1}(E)$ whose projection to the x -plane is the disk-like region K_E bounded by the projections of the Lyapunov orbits around p_1, p_2 and arcs in $V^{-1}(E)$.

The hypotheses of Proposition 4.5 are trivially satisfied. However, since V_2 has a unique critical point (a nondegenerate global minimum), it is easier to construct the transverse foliation on $H^{-1}(E)$ using the gradient flow lines of H_1 . For $0 < E < e_*$, $H^{-1}(E)$ contains a regular sphere-like hypersurface S_E admitting an open book decomposition whose pages are disk-like global surfaces of section. The binding orbit is the one projecting to $x_1 = y_1 = 0$, and the planes are the pre-images of the gradient flow lines of H_1 under the projection to the $x_1 y_1$ -plane, see Figure 4.3. The singular sphere-like subset S_{e_*} has two singularities at p_1, p_2 . A similar open book exists for S_{e_*} . For $E > e_*$, $H^{-1}(E)$ admits a weakly convex foliation whose Lyapunov orbits are part of the binding, as in the following proposition.

Proposition 4.9. *For every $E > e_*$, the energy surface $H^{-1}(E)$ admits a weakly convex foliation on S_E . The binding orbits are the two Lyapunov orbits around*

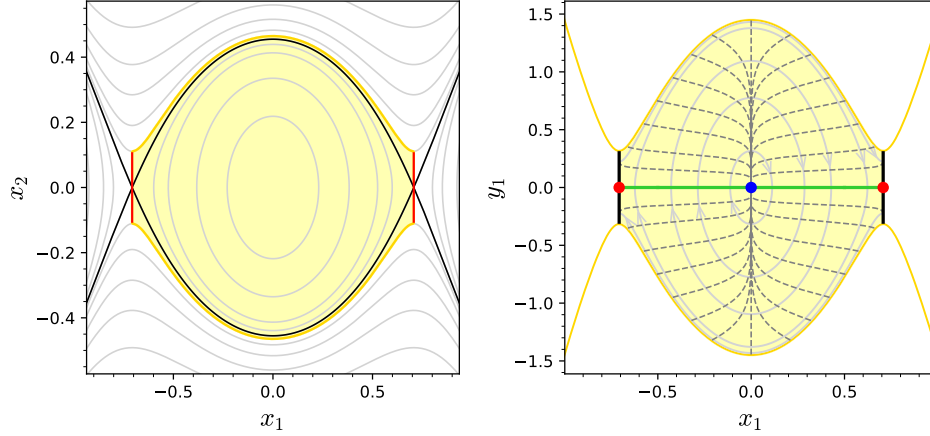


FIGURE 4.3. The Stark problem for energies above the critical value. The energy surface $H^{-1}(E)$ contains a sphere-like hypersurface S_E admitting a weakly convex foliation with $l = 2$, see Figure 1.1. The projection K_E of S_E to the Hill region (left). The binding is formed by the Lyapunov orbits and the orbit projecting to $(x_1, y_1) = (0, 0)$; its index is ≥ 3 . The regular leaves project to the gradient flow lines of H_1 (right).

p_1, p_2 and an index-3 orbit projecting to K_E , and the regular leaves project to the gradient flow lines of H_1 .

Remark 4.10. In general, for a decoupled potential $V = V_1(x_1) + V_2(x_2)$, where V_2 is a coercive function with a single critical point (a global minimum), a transverse foliation whose regular leaves are planes and cylinders can be obtained from the gradient flow lines of $H_1 = y_1^2/2 + V_1(x_1)$. This follows from the fact that the trajectories of $H_2 = y_2^2/2 + V_2(x_2)$ are embedded circles around the minimum, and the inverse image of the gradient flow lines of H_1 project to (x_2, y_2) as annuli between those circles. For certain energy surfaces, they may determine open book decompositions whose pages are global surfaces of section. For instance, see [32].

4.5. The Euler problem of two centers. The Euler problem of two fixed centers in the plane describes the motion of a massless body (the satellite) under the influence of two fixed bodies (the primaries) which attract the satellite according to Newton's law of gravitation. See [39, 46]. Fix $0 < \mu < 1$ and let $E = (-1, 0) \in \mathbb{R}^2$ and $S = (1, 0) \in \mathbb{R}^2$ denote the primaries with respective mass μ and $1 - \mu$. The potential U_μ of central force fields centered at S and E is given by

$$U_\mu(x_1, x_2) = -\frac{\mu}{\sqrt{(x_1 + 1)^2 + x_2^2}} - \frac{1 - \mu}{\sqrt{(x_1 - 1)^2 + x_2^2}},$$

and the mechanical Hamiltonian is $H_\mu = \frac{y_1^2 + y_2^2}{2} + U_\mu(x_1, x_2)$. Euler was the first to notice in 1760 that this problem is completely integrable [14, 15, 16]. In coordinates $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, one easily checks the existence of a unique rest point, given by

$p_c = (\bar{x}_1, 0, 0, 0)$, where

$$-1 < \bar{x}_1 = \frac{2\mu - 1}{1 + 2\sqrt{\mu - \mu^2}} < 1.$$

This critical point is a saddle-center and the corresponding critical value is equal to

$$(4.3) \quad c_{\text{crit}} := H_\mu(p_c) = -\frac{1}{2} - \sqrt{\mu - \mu^2}.$$

We change coordinates (x_1, x_2) to elliptic coordinates $(u, v) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$, where

$$x_1 = \cosh u \cos v, \quad x_2 = \sinh u \sin v,$$

and momenta (y_1, y_2) to (p_u, p_v) satisfying

$$y_1 = \frac{p_u \sinh u \cos v - p_v \cosh u \sin v}{\cosh^2 u - \cos^2 v},$$

$$y_2 = \frac{p_u \cosh u \sin v + p_v \sinh u \cos v}{\cosh^2 u - \cos^2 v}.$$

The change of coordinates $(x, y, p_x, p_y) \mapsto (u, v, p_u, p_v)$ is symplectic up to a constant factor, and we obtain the new Hamiltonian

$$H_\mu = \frac{1}{\cosh^2 u - \cos^2 v} \left(\frac{p_u^2 + p_v^2}{2} - \mu(\cosh u - \cos v) - (1 - \mu)(\cosh u + \cos v) \right).$$

The collisions to S and E correspond to $(u, v) = (0, 0)$ and $(u, v) = (0, \pi)$, respectively, where the denominator in the expression above vanishes. In order to regularize such collisions, we fix $c \in \mathbb{R}$ and consider the regularized Hamiltonian

$$K_{\mu,c} := (H_\mu - c)(\cosh^2 u - \cos^2 v)$$

$$= \frac{p_u^2}{2} - \cosh u - c \cosh^2 u + \frac{p_v^2}{2} + (2\mu - 1) \cos v + c \cos^2 v,$$

where the energy c is now seen as a parameter.

Except for the collision points, the dynamics on $K_{\mu,c}^{-1}(0)$ corresponds to the dynamics on $H_\mu^{-1}(c)$ via a double covering map which identifies antipodal points $(u, v, p_u, p_v) \sim -(u, v, p_u, p_v)$.

Denote $\Sigma_{\mu,c} := K_{\mu,c}^{-1}(0)$. We consider $\Sigma_{\mu,c}$ lying inside $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{R}$ with coordinates (u, v, p_u, p_v) . The Hamiltonian $K := K_{\mu,c}$ is decoupled, that is

$$(4.4) \quad K = K_1(u, p_u) + K_2(v, p_v),$$

where $K_1(u, p_u) := \frac{p_u^2}{2} - \cosh u - c \cosh^2 u$ and $K_2(v, p_v) := \frac{p_v^2}{2} + (2\mu - 1) \cos v + c \cos^2 v$ are first integrals of motion. Denote by $V_1(u) = -\cosh u - c \cosh^2 u$ and $V_2(v) = (2\mu - 1) \cos v + c \cos^2 v$ the respective potentials of K_1 and K_2 .

We consider negative energies $c < 0$, so that the motions are bounded. If $c < c_{\text{crit}}$, then $\Sigma_{\mu,c}$ contains two dynamically convex sphere-like components admitting disk-like global surfaces of section. If $c_{\text{crit}} < c < 0$, then $\Sigma_{\mu,c}$ is diffeomorphic to $S^1 \times S^2$ corresponding to the connected sum of the lower components. We consider the energy values $c_1 < c_2$ given by

$$-1 < c_{\text{crit}} < c_1 := -\frac{1}{2} < c_2 := -\left| \mu - \frac{1}{2} \right| \leq 0.$$

Notice that $c_2 = 0$ only for $\mu = 1/2$.

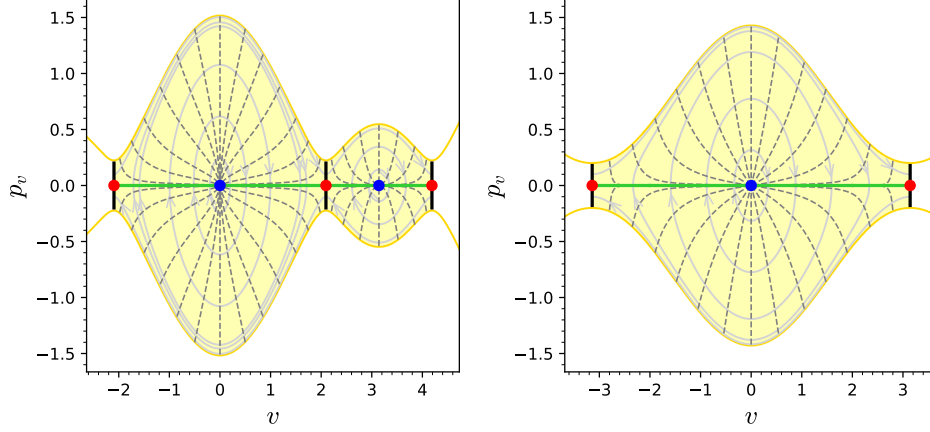


FIGURE 4.4. The Euler problem for $\mu = 1/4$ and $c_{\text{crit}} < c = -1/2 < c_2$ (left), and $c_2 < c = -1/5 < 0$ (right). Each chamber admits a weakly convex foliation with $l = 2$. The regular leaves project to the gradient flow lines of K_2 .

For $c \leq c_1$, K_1 has a unique critical point at $(0, 0) \in K_1^{-1}(-1 - c)$, which is a global minimum. The dynamics of K_1 is that of a center about $(0, 0)$. For $c_1 < c < 0$, K_1 has three critical points, a saddle at $(0, 0) \in K_1^{-1}(-1 - c)$ and two symmetric minima on the u -axis.

For $c < c_2$, K_2 has 4 critical points in $\mathbb{R}/2\pi\mathbb{Z} \times \{0\} \subset \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, two local minima at $(0, 0)$ and $(\pi, 0)$, and two saddles with the same critical value $-\frac{(2\mu-1)^2}{4c}$. The saddles correspond to index-2 hyperbolic periodic orbits, and the minima correspond to elliptic periodic orbits with index ≥ 3 . For $c_2 \leq c < 0$, K_2 has only 2 critical points at $(0, 0)$ and $(\pi, 0)$, a minimum and a saddle, depending on the sign of $2\mu - 1$. The critical value of the saddle is $|2\mu - 1| + c$. These orbits correspond to an index-2 hyperbolic orbit and an elliptic orbit with index ≥ 3 . Their projections to the up_u -plane are simple orbits enclosing the origin.

Lifting the gradient flow lines of K_2 to the energy surface $\Sigma_{\mu,c} = K_{\mu,c}^{-1}(0) \simeq S^1 \times S^2$, we obtain two different types of transverse foliations:

- I. If $c_{\text{crit}} < c < c_2$, then $\Sigma_{\mu,c}$ admits two subsets S_E, S'_E admitting a weakly convex foliation with $l = 2$. The index-2 orbits project to the continuation of the Lyapunov orbit under the 2 : 1 regularization map. The other binding orbits project to collision-brake orbits in $x_2 = 0$.
- II. If $c_2 \leq c < 0$, the Lyapunov orbit collapses to one of the collision-brake orbits, which becomes hyperbolic after bifurcation. $\Sigma_{\mu,c}$ has only one component that admits a weakly convex foliation with $l = 2$. The two binding orbits project to collision-brake orbits under the double-covering map.

The bifurcation in the up_u -plane at level c_1 is transverse to the foliation and thus does not alter its type.

4.6. **Chemical reaction model.** In [37], the authors study a chemical reaction described by a mechanical system whose potential is

$$(4.5) \quad V(x_1, x_2) = V_1(x_1) + V_2(x_2) = -\frac{1}{2}\alpha x_1^2 + \frac{1}{4}\beta x_1^4 + \frac{1}{2}x_2^2,$$

where $\alpha\beta > 0$. The potential V has precisely three critical points at $v_0 = (0, 0)$ and $v_{1,2} = (\pm(\alpha/\beta)^{1/2}, 0)$. The corresponding critical values are 0 and $-\frac{\alpha^2}{4\beta}$, respectively. We split it into two cases:

- I. $\alpha, \beta > 0$: the critical level $H^{-1}(e_*)$, $e_* = 0$, contains two singular sphere-like subsets S_0, S'_0 with a singularity at the saddle-center $(v_0, 0) \in \mathbb{R}^4$. Their projections to the x -plane are compact disk-like domains touching each other at v_0 . For $E > 0$, the sphere-like hypersurface $H^{-1}(E)$ admits a weakly convex foliation whose leaves project to the gradient flow lines of H_1 . This foliation is called a 3 – 2 – 3 foliation, and the Lyapunov orbit at $x_1 = 0$ is one of the binding orbits.
- II. $\alpha, \beta < 0$: the critical level $H^{-1}(e_*)$, $e_* = -\alpha^2/(4\beta)$, contains a singular sphere-like subset with two singularities at $(v_1, 0)$ and $(v_2, 0)$. Its projection to the x -plane is a compact disk-like domain with two singularities at v_1, v_2 . For $E - e_* > 0$, the energy surface also admits a weakly convex foliation whose leaves project to the flow lines of H_1 . The binding includes the Lyapunov orbits at $x_1 = \pm(\alpha/\beta)^{1/2}$.

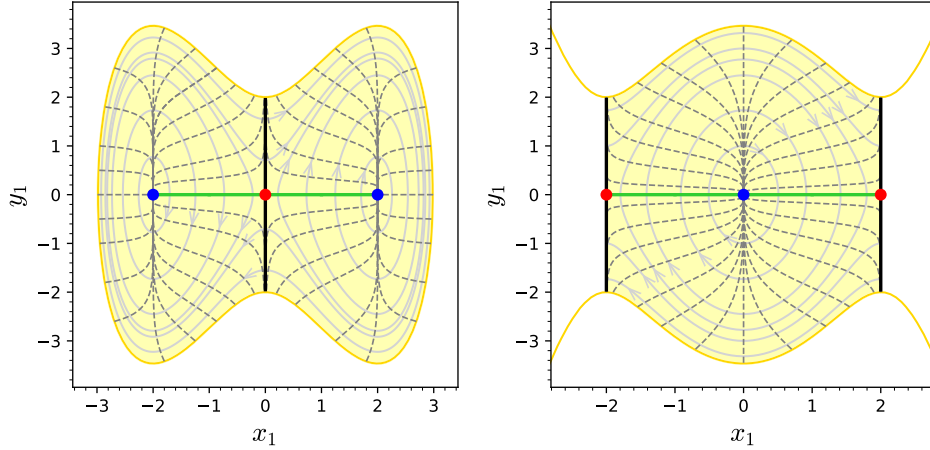


FIGURE 4.5. The chemical reaction model. Each leaf of the foliation is obtained as the pre-image of a gradient flow line of V_1 . On the left (case I), $H^{-1}(E)$ admits a 3 – 2 – 3 foliation, i.e., both sides $S_E = H^{-1}(E) \cap \{x_1 \geq 0\}$ and $S'_E := H^{-1}(E) \cap \{x_1 \leq 0\}$ admit a weakly convex foliation with $l = 1$. On the right (case II), $S_E = H^{-1}(E) \cap \{|x_1| \leq (\alpha/\beta)^{1/2}\}$ admits a weakly convex foliation with $l = 2$.

ACKNOWLEDGEMENT

NdP was partially supported by CAPES/MATH-AMSUD 88881.878892/2023-01. SK was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. RS-2025-16070003). A part of this work was done during SK's visit to the Mathematisches Forschungsinstitut Oberwolfach (MFO) as an Oberwolfach Leibniz Fellow in 2020. SK cordially thanks the MFO for its excellent support and stimulating working atmosphere. PS acknowledges the support of the NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai and the 2022 National Foreign Experts Program. PS was partially supported by FAPESP (2016/25053-8) and CNPq (306106/2016-7). PS was partially supported by the National Natural Science Foundation of China (grant number W2431007). PS thanks the support of the Shenzhen International Center for Mathematics - SUSTech. AS thanks the Instituto de Matemática Pura e Aplicada (IMPA) for the post-doc position. Part of this work was conducted during visits to the Southern University of Science and Technology (SUSTech) and the Kongju National University (KNU). AS thanks both institutes for their hospitality.

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NAIARA V. DE PAULO, UNIVERSIDADE FEDERAL DE SANTA CATARINA, DEPARTAMENTO DE MATEMÁTICA, RUA MARECHAL RONDON, 880, CEP 89065-200, BAIRRO SALTO DO NORTE, BLUMENAU SC, BRAZIL

Email address: `naiara.vergian@ufsc.br`

SEONGCHAN KIM, DEPARTMENT OF MATHEMATICS EDUCATION, KONGJU NATIONAL UNIVERSITY, GONGJU 32588, REPUBLIC OF KOREA

Email address: `seongchankim@kongju.ac.kr`

PEDRO A. S. SALOMÃO, SHENZHEN INTERNATIONAL CENTER FOR MATHEMATICS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHENZHEN, CHINA

Email address: `psalomao@sustech.edu.cn`

A. SCHNEIDER, UNIVERSIDADE ESTADUAL DO CENTRO-OESTE, RUA CAMARGO VARELA DE SÁ, 03, GUARAPUAVA PR, BRAZIL, 85040-080

Email address: `alexandro@unicentro.br`